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SEMI-COSIMPLICIAL OBJECTS AND SPREADABILITY

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ABSTRACT. To a semi-cosimplicial object (SCO) in a category, we associate a system of partial shifts on the inductive limit. We show how to produce an SCO from an action of the infinite braid monoid $\mathbb{B}_\infty^+$ and provide examples. In categories of (noncommutative) probability spaces, SCOs correspond to spreadable sequences of random variables; hence, SCOs can be considered as the algebraic structure underlying spreadability.

1. Introduction. Distributional symmetries have been intensely studied in probability theory in recent decades, see **[11]** for an inspiring overview. More recently, it has emerged that distributional symmetries are also crucial for the further development of noncommutative probability theory and that an important role is played by a specific distributional symmetry (or invariance principle) which is called *spreadability*, i.e., the invariance of distribution when passing from a sequence of random variables to a subsequence. See **[13]** for the beginning of the story. It may be argued that these symmetries become more transparent from an algebraic point-of-view if probability theory is interpreted as a study of associative algebras and their states, and thus the point of view of noncommutative probability theory is a natural one.

In this paper, we deepen these connections to algebra by including concepts from category theory and homological algebra. Not only do we obtain a better idea of the meaning of spreadability, we also derive the natural level of generality for constructing further examples. Unlike other probabilistic symmetries which are based on group actions,

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spreadability has a homological flavor. Making this explicit is one of the main targets of this paper.

For this purpose, we need to study semi-cosimplicial objects (SCOs for short). We briefly recollect the relevant concepts, see for example [**19,** Chapter 8.1], for more details. Some of the most fundamental ideas of algebraic topology and homological algebra relate to simplices, and they can be based on the simplicial category Δ. The objects of Δ are finite ordered sets, usually written as

$$[n] := \{0, 1, \dots, n\}, \quad n \in \mathbb{N}_0,$$

and the morphisms are all non-decreasing maps between these objects. An interesting subcategory Δ_S, called the *semi-simplicial category*, is obtained by considering the same objects but only (strictly) increasing maps as morphisms. Other names in use for this important category Δ_S are 'restricted simplicial' [**2**] and 'incomplete simplicial' [**15**], see [**19,** subsection 8.1.10] for historical remarks regarding the terms). In this paper, only the semi-simplicial category Δ_S is relevant, and thus, we give further definitions only in this context. We remark, however, that it is always interesting to ask whether constructions actually can be extended to the simplicial category Δ in some way.

A covariant functor F from the semi-simplicial category Δ_S to another category $\mathcal{C}$ is called a *semi-cosimplicial object* (SCO) in $\mathcal{C}$. We work out a more explicit description of an SCO by noting that the morphisms of Δ_S are generated by the face maps

$$\delta^k \colon [n-1] \longrightarrow [n], \qquad m \longmapsto \begin{cases} m & \text{if } m < k, \\ m+1 & \text{if } m \geq k. \end{cases}$$

Here, $k = 0, \dots, n$ and $n \in \mathbb{N}$. Following the usual convention, we omit the index n in the notation of the δ^k and leave the domain and codomain to the context. The δ^k satisfy the cosimplicial identities

$$\delta^j \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j,$$

and these cosimplicial identities provide a presentation of the category Δ_S. The functor F takes $[n]$ to $F[n]$ and δ^k to

$$F(\delta^k) : F[n-1] \longrightarrow F[n].$$

Simplifying the notation, we can then write F^n for $F[n]$ and δ^k for $F(\delta^k)$ and obtain the explicit definition of an SCO to be used in the sequel.

Definition 1.1. A *semi-cosimplicial object* (SCO) in the category $\mathcal{C}$ is a sequence $(F^n)_{n\in\mathbb{N}_0}$ of objects in $\mathcal{C}$, together with morphisms (coface operators)

$$\delta^k : F^{n-1} \longrightarrow F^n, \quad k = 0, \ldots, n,$$

satisfying the cosimplicial identities

$$\delta^j \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j.$$

If there is an additional object F^{-1} in $\mathcal{C}$ together with a morphism

$$\delta^0 : F^{-1} \longrightarrow F^0$$

satisfying the cosimplicial identities, then we have an *augmented semi-cosimplicial object.*

The reader is referred to **[19, 18, 20]**, for example, for more information regarding (semi co-)simplicial objects and for a development of the rich and far developed theory.

In this paper, we proceed as follows. In Section 2, we develop some category theory, which provides a general framework. In particular, we show that, by forming an inductive limit from a given SCO, we obtain, in addition, a sequence of adapted endomorphisms with properties reflecting the SCO. We call this an SCO-*system* of partial shifts, and we study some of its properties. The most basic example of an SCO-system of partial shifts (providing a good guide for intuition) appears on the set $\mathbb{N}_0$ of nonnegative integers, and it consists of the sequence of maps

$$\alpha_k \colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$

(with $k \in \mathbb{N}_0$) given by $\alpha_k(m) := m$ if $m < k$ and $\alpha_k(m) := m + 1$ if $m \geq k$ (missing the position k). Note the close similarity to the face maps described above.

We should emphasize that SCO-systems of partial shifts are nothing but a convenient tool for handling SCOs; in particular, they are a useful bridge to the probabilistic contexts studied later, but with the techniques of Section 2, it is possible to give formulations directly in terms of the SCO, if this is preferred. We also include in Section 2 some examples of semi-cosimplicial groups which can be constructed in an elementary way.

In Section 3, we study a method of constructing SCOs from actions of the infinite braid monoid $\mathbb{B}_\infty^+$. This generalizes the idea of braidability in **[7]** (which is discussed briefly later in this introduction) and gives a somewhat simplified manner of thinking about this concept. To our knowledge, this is also a new method of creating cosimplicial identities as needed for an SCO, and it generates a wealth of nontrivial examples worthy of further study. For instance, there is the corresponding standard semi-cosimplicial cohomology theory, which we mention briefly in Remark 3.5 but do not investigate further in this paper.

In Section 4, we first recall the definition of spreadability in various categories of (noncommutative) probability spaces. Then, we state Theorem 4.3 which, together with Theorem 4.5, is our main result. It shows that SCOs in these categories induce spreadable sequences of random variables and, conversely, the distribution of a spreadable sequence can always be achieved from an SCO in such a category. We develop only the most basic part of the theory here, but it should suffice to convince the reader that SCOs are the fundamental algebraic structure underlying spreadability.

In order to guide the reader's intuition through the paper let us insert here an example in the category of unital associative algebras (or $*$-algebras) which is fundamental in many ways. Let $\mathcal{B}$ be such an algebra. Then, we can form an SCO $(X^n)_{n\in\mathbb{N}_0}$ with tensor products

$$X^n := \bigotimes_0^n \mathcal{B},$$

together with coface operators

$$\delta^k : \bigotimes_0^{n-1} \mathcal{B} \longrightarrow \bigotimes_0^n \mathcal{B},$$

$$x_0 \otimes \cdots \otimes x_{n-1} \longmapsto x_0 \otimes \cdots \otimes x_{k-1} \otimes \mathbb{1} \otimes x_k \otimes \cdots \otimes x_{n-1}.$$

The cosimplicial identities can easily be directly checked in this case. Alternatively, the reader who has studied Section 3 is invited to verify that this is a special case of the theory presented in Theorem 3.1. In fact, it comes from the braid group representation factoring through the representation of the symmetric group, which permutes the tensor products.

We can also illustrate the theory of SCO-systems of partial shifts from Section 2 with this example. There is an inductive system of tensor products of the algebra $\mathcal{B}$ with itself, with inclusions

$$x \longmapsto x \otimes \mathbb{1}$$

and inductive limit

$$\mathcal{A} := \bigotimes_{0}^{\infty} \mathcal{B}.$$

Hence, on $\mathcal{A}$, we have the tensor shift

$$\alpha_0 : \mathcal{A} \longrightarrow \mathcal{A},$$

and we obtain the canonically associated SCO-system of partial shifts by considering the sequence of algebra homomorphisms

$$\alpha_k \colon \mathcal{A} \longrightarrow \mathcal{A} \quad (\text{with } k \in \mathbb{N}_0),$$

given by

$$\alpha_k(x) := \begin{cases} x & \text{if } x \in \bigotimes_0^{k-1} \mathcal{B}, \\ \alpha_0(x) & \text{if } x \in \bigotimes_k^{\infty} \mathcal{B}. \end{cases}$$

By additionally choosing a unital linear functional invariant under all of these partial shifts, we finally arrive in our example at the theory of spreadability in (noncommutative) probability spaces. The basic example is to choose any unital linear functional $\varphi_{\mathcal{B}}$ on $\mathcal{B}$ and then to construct the infinite tensor product

$$\varphi := \bigotimes_{0}^{\infty} \varphi_{\mathcal{B}}.$$

Convex combinations of such products provide more examples. The reader familiar with the notion of spreadability, which is reviewed in Section 4, will have no difficulty verifying that under these circumstances the embeddings of the noncommutative probability space $(\mathcal{B}, \varphi_{\mathcal{B}})$ into the different positions of the tensor product provide an example of a spreadable sequence.

In order to make some finer distinctions we refer in this paper to the version of spreadability based on $*$-algebras and $*$-homomorphisms, most relevant for the probabilistic point of view, as $*$-spreadable. An interesting example, Example 4.4 is provided where, in contrast to the tensor product example given above, no simplification based on the more traditional idea of exchangeability is possible, and the full strength of the results in Section 3 is needed, namely, we construct spreadable sequences of operators and, in particular, of projections in the tower associated to a subfactor, in the theory of von Neumann algebras, see [**9**]. Spreadability follows for all values of the Jones index but $*$-spreadability only appears if the index is small, i.e., less than or equal to 4.

Applications of spreadability are not the topic of this paper; however, we finish this introduction with a short review of the literature, including some of our motivations and background as to why, in particular, the study of $*$-spreadability is important and relevant for a probabilist. In fact, in the $*$-algebra setting, where we use states instead of unital linear functionals, the above tensor product example is the basis of what probabilists call *exchangeability*, an important special case of spreadability. It is intimately connected to the representation of the symmetric group mentioned above.

Clearly, the category of $*$-probability spaces includes classical probability in the sense that we can consider a commutative $*$-algebra of complex functions on a classical probability space and a positive functional induced by a probability measure. In this case, it is always possible to find a version of the random variables where spreadability and $*$-spreadability amounts to the same thing. In order to avoid technical difficulties in the following discussion, we always assume that we have Lebesgue spaces; thus, we can, for example, represent homomorphisms of the measure algebras by point transformations (modulo sets of measure zero). See [**17,** 1.4C], for more details and further references.

Here, the interest in spreadability comes from the fact that a de Finetti-type theorem can be proved, i.e., we can deduce a form of conditional independence. Using the background and terminology provided in [**11**], we have

Theorem 1.2. *Let $(\xi_n)_{n\in\mathbb{N}_0}$ be a sequence of (classical) random variables (realized by measure-preserving maps between Lebesgue spaces). Further let Σ_∞ denote the σ-algebra generated by the sequence $(\xi_n)_{n\in\mathbb{N}_0}$ and Σ_n the σ-algebra generated by $\xi_0, \ldots, \xi_n$, for all $n \in \mathbb{N}_0$. Then the following are equivalent:*

(a) *(ξ_n) is spreadable.*

(b) *(ξ_n) is exchangeable.*

(c) *(ξ_n) is conditionally* i.i.d. (*independent and identically distributed*).

(d) *For all $n \in \mathbb{N}$, there exist $\Sigma_n - \Sigma_{n-1}$-measurable and measure-preserving maps δ_k, for $k = 0, \ldots, n$, such that (with $0 \le N < n$) we have $\xi_N = \xi_N \circ \delta_k$ for $N < k$ and $\xi_{N+1} = \xi_N \circ \delta_k$ for $N \ge k$.*

(e) *There exist Σ_∞-measurable measure-preserving maps $(\beta_k)_{k\ge 0}$ such that we have*

$$\xi_N = \xi_N \circ \beta_k \quad \text{for } N < k$$

and

$$\xi_{N+1} = \xi_N \circ \beta_k \quad \text{for } N \ge k.$$

We do not further discuss exchangeability in this paper and recommend [**7, 13, 8**] for additional results regarding exchangeability from our point-of-view. The equivalence of (a), (b) and (c) is provided by [**11,** Theorem 1.1], where not only a proof but much further information may be found, see also [**10**]. In fact, among other things, it is shown there how, from spreadability, a very transparent proof of the classical de Finetti theorem (which is the equivalence with (c)) may be obtained via the mean ergodic theorem.

The equivalence of (a) and (d) is precisely the topic of this paper, applied to this specific situation. In fact, it is a special case of the equivalence of (2) (a) and (2) (c) in our Theorem 4.3 (or Theorem 4.5). More explicitly, it follows from spreadability that, omitting ξ_k from $\xi_0, \ldots, \xi_n$, yields the same distribution as does $\xi_0, \ldots, \xi_{n-1}$, and this allows us to define the measure-preserving transformation δ_k. Conversely, if we begin with such a δ_k, then, by $\delta^k p := p \circ \delta_k$ on polynomials p in the random variables $\xi_0, \ldots, \xi_{n-1}$, we obtain an SCO as in Theorem 4.3 (2) (c) and deduce spreadability from that as described there. Note that, in (d), we have again used the convention that the dependence on n of the maps δ_k is suppressed in notation. Equivalently, by the theory developed in Section 2, we have the formulation in (e) involving the measure-preserving maps version of what we call partial shifts.

We now return to the general noncommutative setting. The case of $*$-spreadability for noncommutative random variables in von Neumann algebras arising from actions of the infinite braid group $\mathbb{B}_\infty$ by state-preserving $*$-automorphisms was investigated extensively in **[7]**, and the sequences obtained in this way were called braidable there (here, we call them $*$-*braidable*). The fact that $*$-braidable sequences are $*$-spreadable, obtained in **[7]**, again follows from our Theorem 3.1, together with Theorem 4.5. The converse question, "is every $*$-spreadable sequence necessarily also $*$-braidable?," seems to remain open at the moment. In fact, this open question was one of the motivations for this paper. Our expectation is that the characterization of $*$-spreadability in terms of SCOs achieved in Theorem 4.5 will provide a tool for constructing examples which show that the answer is negative.

It was shown **[13]** that, in the setting of (in general noncommutative) von Neumann algebras and corresponding noncommutative probability spaces, there is still a version of de Finetti's theorem for $*$-spreadable sequences which makes use of a generalized notion of noncommutative stochastic independence. The proof involves refined applications of the mean ergodic theorem, and it is in this context that the idea of partial shifts first appeared (which follows in Section 2 and is derived from SCOs). Moreover, the braidability results in **[7]** show that, in the noncommutative setting, $*$-spreadability is much more general than $*$-exchangeability (which involves representations of the infinite symmetric group while representations of the infinite braid group are sufficient to produce spreadability, as explained above). Hence, there are many indications that, in noncommutative probability theory, the notion of spreadability is actually more fundamental than exchangeability.

Finally, we mention the notion of quantum spreadability, developed in **[3]**, which strengthens the notion of spreadability using the idea of quantum increasing sequence spaces. It has been shown **[3]** that quantum spreadability is equivalent to free independence; hence, it is strong enough to enforce a very specific structure for the noncommutative probability space. In contrast, in this paper, we consider the (weaker) classical notion of spreadability but, in general, we apply it to noncommutative probability spaces as well. Here, we find that this does not enforce a specific structure for the noncommutative probability space but instead yields interesting general results (for example, the de Finetti-type results) for a wide range of such spaces and, as such, is worthy of further study.

We expect that the clear identification of SCOs as the algebraic backbone of spreadability obtained here will lead to the construction of additional examples and to new theoretical developments.

2. Adaptedness, SCOs and partial shifts. We begin by giving a definition of adaptedness in terms of category theory and derive a global formulation for SCOs by the so called SCO-systems of partial shifts. Later on, this helps us describe the connection between (co)simplicial and probability theory in a flexible and convenient way. With slight modifications, here we follow the approach in [**6,** subsection 3.2]. For category theory itself, we follow [**14**]. Consider the category

$$\omega = \{0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots\}$$

and another category $\mathcal{C}$ which allows ω-colimits (which are the same as inductive limits). We briefly recall what this means. Suppose that

$$F\colon \omega \longrightarrow \mathcal{C}$$

is a functor, i.e., we have

$$F_0 \xrightarrow{i_1} F_1 \xrightarrow{i_2} F_2 \xrightarrow{i_3} \cdots$$

with morphisms

$$i_n : F_{n-1} \longrightarrow F_n, \quad n \in \mathbb{N},$$

between $\mathcal{C}$-objects. We also refer to this functor as a filtration (by a slight abuse of terminology, the sequence of objects is also called a filtration). An ω-colimit (or inductive limit) is an object

$$F_\infty = \lim_{\rightarrow} F$$

in $\mathcal{C}$ which, together with canonical arrows,

$$\mu_n : F_n \longrightarrow F_\infty \quad n \geq 0,$$

forms a universal cone, see [**14,** III.3]. Pictorially, this is a commuting diagram:

$$\begin{tikzcd}
F_0 \arrow[r, "i_1"] \arrow[dr, "\mu_0"] \arrow[ddr, "\mu'_0"'] & F_1 \arrow[r, "i_2"] \arrow[d, "\mu_1"'] & F_2 \arrow[r, "i_3"] \arrow[dl, "\mu_2"'] \arrow[ddl, "\mu'_2"] & \dots \\
& F_\infty \arrow[d, "\exists! f"] & & \\
& F' & &
\end{tikzcd}$$

(μ'_1 not drawn). Thus, the $\mathcal{C}$-object F_∞ is determined up to isomorphism by the fact that there are morphisms

$$\mu_n : F_n \longrightarrow F_\infty, \quad n \in \mathbb{N}_0,$$

which satisfy the equations $\mu_n\, i_n = \mu_{n-1}$, $n \in \mathbb{N}$, and are universal with respect to any morphisms $\mu'_n : F_n \to F'$, $n \in \mathbb{N}_0$, which satisfy the equations μ'_n, $i_n = \mu'_{n-1}$, $n \in \mathbb{N}$. In many examples, these morphisms involve inclusions of sets; however, this is not necessarily the case in general.

Lemma 2.1. *Given morphisms*

$$\alpha^{(n)} : F_{n-1} \longrightarrow F_n, \quad n \in \mathbb{N},$$

such that

$$\mu_{n+1}\, \alpha^{(n+1)}\, i_n = \mu_n\, \alpha^{(n)} \quad (\textit{for all } n \in \mathbb{N})$$

there exists a unique morphism

$$\alpha : F_\infty \longrightarrow F_\infty$$

such that

$$\alpha\, \mu_{n-1} = \mu_n\, \alpha^{(n)} \quad (\textit{for all } n \in \mathbb{N}).$$

Proof. If we define $\mu'_n := \mu_{n+1}\, \alpha^{(n+1)}$, then

$$\mu'_n\, i_n = \mu_{n+1}\, \alpha^{(n+1)}\, i_n = \mu_n\, \alpha^{(n)} = \mu'_{n-1},$$

and we obtain α from the universal property ($\alpha\mu_{n-1} = \mu'_{n-1}$). $\square$

Definition 2.2. A morphism

$$\alpha : F_\infty \longrightarrow F_\infty$$

given as in Lemma 2.1 is called an *adapted endomorphism*, with respect to the filtration, determined by $(\alpha^{(n)})_{n\in\mathbb{N}}$.

Intuitively, $\alpha^{(n)}$ describes how α acts on (the image of) the $(n-1)$th object in the filtration and adaptedness describes the compatibility of these actions. The terminology is motivated by stochastic processes and their time evolutions, cf., Section 4.

Lemma 2.3. *Let α be an adapted endomorphism, with respect to a filtration. If $\alpha\mu_n = \mu_n$ for some n, then also $\alpha\mu_k = \mu_k$ for all $k \leq n$.*

Proof. If $\alpha\mu_n = \mu_n$ for some n, then

$$\alpha\mu_{n-1} = \mu_n\alpha^{(n)} = \mu_{n+1}\alpha^{(n+1)}i_n = \alpha\mu_n i_n = \mu_n i_n = \mu_{n-1}.$$

By iterating this argument, we obtain the stated result. $\square$

If α satisfies the condition of Lemma 2.3, then we say that α acts trivially on (the image of) the nth object.

Definition 2.4. Let $(\alpha_k)_{k\in\mathbb{N}_0}$ be a sequence of adapted endomorphisms (with respect to a common filtration). If the sequence satisfies

(1) for each $k \in \mathbb{N}$, the endomorphism α_k acts trivially on (the image of) the $(k-1)$th object,
(2) $\alpha_j\alpha_i = \alpha_i\alpha_{j-1}$ if $i, j \in \mathbb{N}_0$ and $i < j$,

then we say that $(\alpha_k)_{k\in\mathbb{N}_0}$ is an SCO-*system of partial shifts* (for this filtration).

Note that, if $(\alpha_k)_{k\in\mathbb{N}_0}$ is an SCO-system of partial shifts, then, for all $\ell \in \mathbb{N}$, the sequence $(\alpha_k)_{k\geq\ell}$ is also an SCO-system of partial shifts if everything is suitably relabeled ($k \mapsto k - \ell$).

Proposition 2.5. *Let* $(\alpha_k)_{k\in\mathbb{N}_0}$ *be an* SCO*-system of partial shifts. Then*

$$\alpha_k(\alpha_0)^N\mu_0 = \begin{cases}(\alpha_0)^N\mu_0 & \text{if } N<k,\\ (\alpha_0)^{N+1}\mu_0 & \text{if } N\ge k.\end{cases}$$

Proof. If $N<k$, then, using properties (2) and (1) of partial shifts,

$$\alpha_k(\alpha_0)^N\mu_0 = (\alpha_0)^N\alpha_{k-N}\mu_0 = (\alpha_0)^N\mu_0.$$

If $N\ge k$, then, using property (2) of partial shifts,

$$\alpha_k(\alpha_0)^N\mu_0 = (\alpha_0)^k\alpha_0(\alpha_0)^{N-k}\mu_0 = (\alpha_0)^{N+1}\mu_0. \qquad \square$$

Proposition 2.5 explains the terminology of partial shifts: we regard α_0 as a full shift while α_k for $k\ge 1$ acts trivially on an initial part and only shifts the remaining part. We explain the origin of this concept in the theory of spreadability further in Section 4. Of course, the property in Proposition 2.5 also reminds us of the origin of coface operators from face maps, i.e., specific strictly increasing functions missing one point, mentioned in Section 1; thus, we have come full circle. Note that Proposition 2.5 applied to the relabeled SCO-systems (as mentioned above) gives additional relationships.

The next theorem, the main result of this section, gives a correspondence between SCOs and SCO-systems of partial shifts. While we usually suppress the covariant functor F corresponding to an SCO in the notation, in this argument, we write it down to make the interplay with the inductive limit construction explicit.

Theorem 2.6.

(a) *Let a covariant functor* F *from the semi-simplicial category* Δ_S *to a category* $\mathcal{C}$ *be given, with the corresponding* SCO *in* $\mathcal{C}$ *described by* $F[n]$, $F(\delta^k)$, $k=0,\dots,n$ *and* $n\in\mathbb{N}_0$. *We restrict to a functor from* ω *to* $\mathcal{C}$, *also denoted by* F, *given by*

$$F[0]\xrightarrow{i_1}F[1]\xrightarrow{i_2}F[2]\xrightarrow{i_3}\cdots$$

where

$$i_n := F(\delta^n): F[n-1]\longrightarrow F[n] \quad \text{for } n\in\mathbb{N}.$$

If there exists an ω*-colimit* F_∞, *then, on* F_∞, *we obtain an* SCO*-system of partial shifts* $(\alpha_k)_{k\in\mathbb{N}_0}$, *where the* α_k *for* $n\in\mathbb{N}$, $k\in\mathbb{N}_0$, *are determined by*

$$\alpha_k^{(n)} := \begin{cases} F(\delta^k) : F[n-1] \longrightarrow F[n] & \text{if } k = 0, \dots, n, \\ F(\delta^n) : F[n-1] \longrightarrow F[n] & \text{if } k > n. \end{cases}$$

We call this the SCO-*system of partial shifts canonically associated to the* SCO.

(b) *Conversely, if* $(\alpha_k)_{k\in\mathbb{N}_0}$ *is an* SCO-*system of partial shifts for a filtration* $(F_n)_{n\in\mathbb{N}_0}$ *such that the* $\mu_n \colon F_n \to F_\infty$ *are monic, then defining* $F[n] := F_n$ *and*

$$F(\delta^k) := \alpha_k^{(n)} : F[n-1] \longrightarrow F[n] \quad \text{for } k = 0, \dots, n,\ n \in \mathbb{N}_0,$$

where the $\alpha_k^{(n)}$ *determine* α_k, *yields an* SCO *and* $(\alpha_k)_{k\in\mathbb{N}_0}$ *is canonically associated to this* SCO.

Proof. In (a), it follows that, for all $i, j \in \mathbb{N}_0$ with $i < j$,

$$\alpha_j^{(n+1)}\alpha_i^{(n)} = \alpha_i^{(n+1)}\alpha_{j-1}^{(n)}.$$

In fact, if $i \le n$ and $j \le n+1$,

$$\alpha_j^{(n+1)}\alpha_i^{(n)} = F(\delta^j)F(\delta^i) = F(\delta^i)F(\delta^{j-1}) = \alpha_i^{(n+1)}\alpha_{j-1}^{(n)},$$

if $i > n$ and $j > n+1$,

$$\alpha_j^{(n+1)}\alpha_i^{(n)} = F(\delta^{n+1})F(\delta^n) = \alpha_i^{(n+1)}\alpha_{j-1}^{(n)},$$

if $i \le n$ and $j > n+1$,

$$\alpha_j^{(n+1)}\alpha_i^{(n)} = F(\delta^{n+1})F(\delta^i) = F(\delta^i)F(\delta^n) = \alpha_i^{(n+1)}\alpha_{j-1}^{(n)}.$$

If the filtration

$$F[0] \xrightarrow{i_1} F[1] \xrightarrow{i_2} \cdots$$

with

$$i_n := F(\delta^n) = \alpha_n^{(n)} : F[n-1] \longrightarrow F[n],$$

for $n \in \mathbb{N}$, yields an ω-colimit F_∞ with morphisms

$$\mu_n : F[n] \longrightarrow F_\infty,$$

for all $n \in \mathbb{N}_0$, satisfying $\mu_{n+1}i_{n+1} = \mu_n$, then

$$i_{n+1}\alpha_k^{(n)} = \alpha_{n+1}^{(n+1)}\alpha_k^{(n)} = \alpha_k^{(n+1)}\alpha_n^{(n)} = \alpha_k^{(n+1)}i_n$$

and, by applying μ_{n+1}, we obtain

$$\mu_n \alpha_k^{(n)} = \mu_{n+1} i_{n+1} \alpha_k^{(n)} = \mu_{n+1} \alpha_k^{(n+1)} i_n.$$

From Lemma 2.1, we obtain, for all $k \in \mathbb{N}_0$, an adapted endomorphism $\alpha_k : F_\infty \to F_\infty$. We verify the properties of an SCO-system of partial shifts. First, for $k \in \mathbb{N}_0$,

$$\alpha_{k+1} \mu_k = \mu_{k+1} \alpha_{k+1}^{(k+1)} = \mu_{k+1} i_{k+1} = \mu_k,$$

which is property (1). Second, for all $n \in \mathbb{N}$ and $i < j$,

$$\begin{aligned}
\alpha_j \alpha_i \mu_{n-1} = \alpha_j \mu_n \alpha_i^{(n)} &= \mu_{n+1} \alpha_j^{(n+1)} \alpha_i^{(n)} \\
&= \mu_{n+1} \alpha_i^{(n+1)} \alpha_{j-1}^{(n)} = \cdots \\
&= \alpha_i \alpha_{j-1} \mu_{n-1}.
\end{aligned}$$

This implies $\alpha_j \alpha_i = \alpha_i \alpha_{j-1}$ if $i < j$, which is property (2). In fact, for $n \in \mathbb{N}$, we can define

$$\mu'_{n-1} := \alpha_j \alpha_i \mu_{n-1} = \alpha_i \alpha_{j-1} \mu_{n-1}$$

and verify that

$$\mu'_n i_n = \alpha_j \alpha_i \mu_n i_n = \alpha_j \alpha_i \mu_{n-1} = \mu'_{n-1}.$$

It follows from the universal property that there is a unique morphism β such that $\mu'_{n-1} = \beta \mu_{n-1}$ for all $n \in \mathbb{N}$. Thus, $\beta = \alpha_j \alpha_i$ but, by a similar argument, $\beta = \alpha_i \alpha_{j-1}$ as well, which proves our claim

Starting with (b), we merely reverse the above argument to obtain, for all $0 \leq i < j \leq n+1$, $n \in \mathbb{N}$,

$$\mu_{n+1} \alpha_j^{(n+1)} \alpha_i^{(n)} = \alpha_j \alpha_i \mu_{n-1} = \alpha_i \alpha_{j-1} \mu_{n-1} = \mu_{n+1} \alpha_i^{(n+1)} \alpha_{j-1}^{(n)}.$$

By assumption, the μ_n are monic, and we obtain

$$\alpha_j^{(n+1)} \alpha_i^{(n)} = \alpha_i^{(n+1)} \alpha_{j-1}^{(n)},$$

which gives the cosimplicial identities for

$$F(\delta^k) := \alpha_k^{(n)} : F[n-1] \longrightarrow F[n], \quad k = 0, \ldots, n,\ n \in \mathbb{N}.$$

Hence, if we apply the construction in (a) to this SCO, then we obtain an SCO-system of partial shifts with the same $\alpha_k^{(n)}$, $k = 0, \ldots, n$, $n \in \mathbb{N}$, as for the original system. However, the remaining $\alpha_k^{(n)}$ with $k > n$,

$n \in \mathbb{N}$, are also the same as for the original system. In fact, from property (1) of partial shifts, we find (for $k > n$, $n \in \mathbb{N}$)

$$\mu_n \alpha_k^{(n)} = \alpha_k \mu_{n-1} = \mu_{n-1} = \mu_n i_n,$$

and, since the μ_n are monic, this implies $\alpha_k^{(n)} = i_n$. We conclude that the original sequence $(\alpha_k)_{k \in \mathbb{N}_0}$ is canonically associated to the SCO from which it is constructed. $\square$

Remark 2.7. Note that an SCO-system of partial shifts $(\alpha_k)_{k \in \mathbb{N}_0}$ canonically associated to an SCO satisfies the stronger local property, i.e., implying adaptedness, that the following diagram is commutative for all $n \in \mathbb{N}$ and $0 \leq k \leq n$:

$$\begin{tikzcd} F[n-1] \arrow[r, "i_n"] \arrow[d, "\alpha_k^{(n)}"'] & F[n] \arrow[d, "\alpha_k^{(n+1)}"] \\ F[n] \arrow[r, "i_{n+1}"'] & F[n+1] \end{tikzcd}$$

Of course, this local property is also satisfied by any SCO-system of partial shifts with filtration

$$F_0 \xrightarrow{i_1} F_1 \xrightarrow{i_2} \cdots,$$

where each $\mu_n : F_n \to F_\infty$ is monic. In this case, we are allowed to switch freely between SCOs and SCO-systems of partial shifts.

In fact, the length of the proof of Theorem 2.6 should not distract the reader from the fact that, in all of the examples in this paper, the correspondence is nothing but a rather direct assembling of all δ^k (for fixed k and between different objects) into a single morphism α_k on the inductive limit. While in homological algebra, SCOs are the natural starting point, in probability theory, it is a common practice to study phenomena by constructing a large universe, i.e., a probability space common to all variables. Therefore, it may be the SCO-system of partial shifts which first comes into view. This was indeed the case in the theory of (noncommutative) spreadability to which we apply our results in Section 4.

Before developing some substantive connections with actions of the braid group and noncommutative probability in subsequent sections, we first give some examples by direct construction.

Example 2.8. It is worth noting that, in the category of sets, to each system of mappings

$$\alpha_k : X \longrightarrow X, \quad k \in \mathbb{N}_0,$$

satisfying $\alpha_j \alpha_i = \alpha_i \alpha_{j-1}$ for $i < j$, there is a canonical choice of filtration for which $(\alpha_k)_{k\in\mathbb{N}_0}$ is an SCO-system of partial shifts, as follows.

Let X be a set and, for each $k \in \mathbb{N}_0$, let

$$\alpha_k : X \longrightarrow X$$

be a mapping. Furthermore, suppose that $\alpha_j \alpha_i = \alpha_i \alpha_{j-1}$ for $i, j \in \mathbb{N}_0$, $i < j$. For each $n \in \mathbb{N}_0$, let

$$X_n := \{x \in X : \alpha_{n+1}(x) = x\}$$

(the fixed point set of α_{n+1}), and let

$$i_n : X_{n-1} \longrightarrow X_n$$

be the inclusion mapping (which is well-defined since, for $x \in X_{n-1}$, we have $\alpha_{n+1}(x) = \alpha_{n+1}\alpha_n(x) = \alpha_n \alpha_n(x) = x$). We will assume that X is equal to the inductive limit

$$X_\infty := \bigcup_{n\in\mathbb{N}_0} X_n$$

(if it is not, then we simply replace X by X_∞ after noting that $\alpha_n(X_\infty) \subset X_\infty$ for all $n \in \mathbb{N}_0$). We claim that $(\alpha_k)_{k\in\mathbb{N}_0}$ is an SCO-system of partial shifts for the filtration

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \cdots .$$

First, we see that each α_k is adapted by defining the mapping $\alpha_k^{(n)} : X_{n-1} \to X_n$ by $\alpha_k^{(n)} := (\alpha_k)|_{X_{n-1}}$ for all $n \in \mathbb{N}$. We see that Definition 2.4 (1) is trivially satisfied and Definition 2.4 (2) is given by assumption.

Through easy modifications of these arguments such a canonical filtration based on fixed points can be obtained in many other categories, for example, in the categories of noncommutative probability spaces considered in Section 4.

Example 2.9. For each $n \in \mathbb{N}_0$, let G_n be a subgroup of the general linear group $\mathrm{GL}(n+1, R)$ over a unital ring R such that $i_{n+1}(G_n) \subset G_{n+1}$. Here, i_{n+1} is the canonical embedding of $\mathrm{GL}(n+1, R)$ in $\mathrm{GL}(n+2, R)$, i.e.,

$$i_{n+1}(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

for all $g \in \mathrm{GL}(n+1, R)$, where each of the two zeros denotes a column or a row of $n+1$ zeros. We view $\mathbb{S}_{n+1}$ as the group of permutations on $\{0, 1, \ldots, n\}$ and let c_k denote the cycle $(k\ k+1 \cdots n)$. Let π_{n+1} be the action of $\mathbb{S}_{n+1}$ on $\mathrm{GL}(n+1, R)$ given by conjugation by permutation matrices, and suppose that G_n is invariant under this action, for all $n \in \mathbb{N}_0$.

We can construct an SCO, say F, in the category of groups by defining

$$F[n] := G_n$$

and

$$F(\delta^k) : F[n-1] \longrightarrow F[n]$$

by

$$F(\delta^k) = \pi_{n+1}(c_k) i_n \quad \text{for } 0 \leq k \leq n.$$

(This means inserting a kth row and column with a 1 at the intersection and 0s elsewhere. From that, the cosimplicial identities are easy to check.) Through this construction, we see, in particular, that

$$\begin{array}{ll} (\mathrm{GL}(n, \mathbb{C}))_n, & (U(n, \mathbb{C}))_n, \\ (\mathrm{SU}(n, \mathbb{C}))_n, & (\mathbb{S}_n)_n \end{array}$$

are semi-cosimplicial groups, and it follows from Theorem 2.6 that we obtain SCO-systems of partial shifts on their inductive limits

$$\begin{array}{ll} \mathrm{GL}(\infty, \mathbb{C}), & U(\infty, \mathbb{C}), \\ \mathrm{SU}(\infty, \mathbb{C}), & \mathbb{S}_\infty, \end{array}$$

respectively.

For the symmetric groups $(\mathbb{S}_n)_{n\in\mathbb{N}}$, we express the structure as a semi-cosimplicial group in a more direct way. In this case, we have

$$F[n] = G_n = \mathbb{S}_{n+1} \quad \text{for } n \in \mathbb{N}_0.$$

We think of

$$F[0] = G_0 = \mathbb{S}_1$$

as the trivial group while, for $n \geq 1$, we have Coxeter generators $\sigma_N := (N-1\ N)$ or star generators

$$\gamma_N := (0\ N), \quad \text{for } N = 1, \ldots, n$$

in both cases. Then, we can check that $F(\delta^0)\sigma_N = \sigma_{N+1}$ for all N while, for $k \geq 1$,

$$F(\delta^k)(\gamma_N) = \begin{cases} \gamma_N & \text{if } N < k, \\ \gamma_{N+1} & \text{if } N \geq k. \end{cases}$$

The formula for $k \geq 1$ may be considered as an instance of Proposition 2.5 for the relabeling $k \mapsto k-1$.

These examples of semi-cosimplicial groups belong to a general scheme of producing SCOs which we develop in its full generality in the next section.

3. Semi-cosimplicial objects from actions of the braid monoid $\mathbb{B}_\infty^+$. The braid groups $\mathbb{B}_n$ were introduced by Artin [**1**], see [**12**] for a recent overview. For $n \geq 2$, $\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the relations

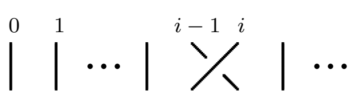
$$\text{(B1)} \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } |i-j| = 1;$$

$$\text{(B2)} \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1.$$

The inclusions

$$\mathbb{B}_2 \subset \mathbb{B}_3 \subset \cdots \subset \mathbb{B}_\infty$$

are apparent, where $\mathbb{B}_\infty$ denotes the inductive limit. The Artin generator σ_i will be presented as a geometric braid as follows:



Our convention in drawing diagrams of braids is that reading formulae from left to right corresponds to top-down compositions in the diagram.

It turns out that, for the following arguments, we do not need inverses of the Artin generators. Hence, we consider $\mathbb{B}_\infty^+$, the monoid generated by $(\sigma_i)_{i\in\mathbb{N}}$.

Suppose that $\mathbb{B}_\infty^+$ acts on a set X; we simply write $gx \in X$ for the result of $g \in \mathbb{B}_\infty^+$ acting on $x \in X$. We define for $n \in \mathbb{Z}$, $n \geq -1$,

$$X^n := \{x \in X \colon \sigma_k x = x \text{ if } k \geq n+2\},$$

which gives an increasing sequence

$$X^{-1} \subset X^0 \subset X^1 \subset \cdots$$

of subsets of the set X.

Theorem 3.1. *$(X^n)_{n\geq -1}$ is an augmented semi-cosimplicial set (an augmented* SCO *in the category of sets), with the coface operators δ^k given by*

$$\begin{aligned} \delta^k \colon \quad X^{n-1} &\longrightarrow X^n \quad (k = 0, \ldots, n,\ n \in \mathbb{N}_0) \\ x &\longmapsto \sigma_{k+1} \ldots \sigma_{n+1}\, x. \end{aligned}$$

Note that $\sigma_{n+1}x = x$, for $x \in X^{n-1}$; thus, if $x \in X^{n-1}$, then for $k < n$, we can also write $\delta^k x = \sigma_{k+1} \cdots \sigma_n x$ and, for $k = n$, we have $\delta^n x = x$. Hence,

$$\delta^n : X^{n-1} \longrightarrow X^n$$

is only the inclusion map; in particular, this applies to the augmentation

$$\delta^0 : X^{-1} \longrightarrow X^0.$$

Proof. We use a double induction argument to prove

$$\delta^j \delta^i = \delta^i \delta^{j-1} \colon X^{n-1} \longrightarrow X^{n+1}$$

for all $n \in \mathbb{N}_0$ and $i = 0, \ldots, n$, $j = 1, \ldots, n+1$ such that $i < j$. Fix $n \in \mathbb{N}_0$. First, suppose that $j = n+1$. If $i = n$, then, for $x \in X^{n-1}$,

$$\delta^j \delta^i x = \delta^{n+1} \delta^n x = x = \delta^n \delta^n x = \delta^i \delta^{j-1} x.$$

If, for all $x \in X^{n-1}$, the equation $\delta^j \delta^i x = \delta^i \delta^{j-1} x$ is valid for $j = n+1$ and for some i with $1 \leq i \leq n$, then,

$$\delta^{i-1}\delta^{j-1}x = \sigma_i \delta^i \delta^{j-1} x = \sigma_i \delta^j \delta^i x = \delta^j \sigma_i \delta^i x = \delta^j \delta^{i-1} x.$$

We conclude by induction that, for all $x \in X^{n-1}$ and $j = n+1$, the equation $\delta^j \delta^i x = \delta^i \delta^{j-1} x$ is valid for all $0 \leq i \leq n$.

Now, suppose that, for all $x \in X^{n-1}$ and some j with $2 \leq j \leq n+1$, we have $\delta^j \delta^i x = \delta^i \delta^{j-1} x$ for all $0 \leq i < j$. Then, for $i < j-1$,

$$\begin{aligned}
\delta^{j-1}\delta^i x &= \sigma_j \delta^j \delta^i x = \sigma_j \delta^i \delta^{j-1} x \\
&= \sigma_j \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j \sigma_{j+1} \cdots \sigma_{n+1} \sigma_j \cdots \sigma_{n+1} x \\
&= \sigma_{i+1} \cdots \sigma_j \sigma_{j-1} \sigma_j \sigma_{j+1} \cdots \sigma_{n+1} \sigma_j \cdots \sigma_{n+1} x \\
&= \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j \sigma_{j-1} \sigma_{j+1} \cdots \sigma_{n+1} \sigma_j \cdots \sigma_{n+1} x \\
&= \sigma_{i+1} \cdots \sigma_{n+1} \sigma_{j-1} \sigma_j \cdots \sigma_{n+1} x = \delta^i \delta^{j-2} x.
\end{aligned}$$

(Here, $\cdots$ always stands for σs with subscripts increasing by steps of 1, including the case where we have the same σ to the left and to the right of $\cdots$.) By an induction argument for j, this proves the theorem. $\square$

We remark that the theorem and the proof are still valid if the X^n are replaced by any subsets $\widetilde{X}^n \subset X^n$ such that

$$\delta^k(\widetilde{X}^{n-1}) \subset \widetilde{X}^n$$

is always satisfied.

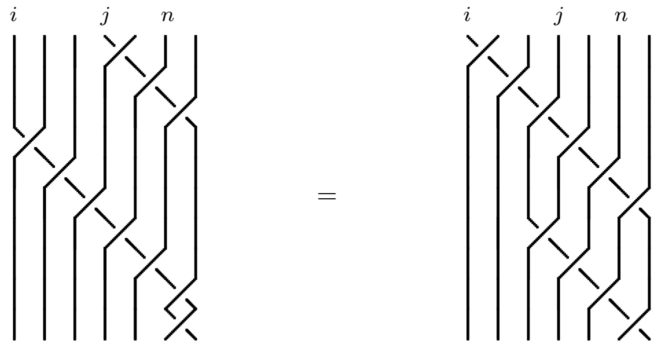
An alternative proof may be based on checking the following braid equalities:

$$(\sigma_{j+1} \cdots \sigma_{n+1})(\sigma_{i+1} \cdots \sigma_{n+1})\sigma_{n+1} = (\sigma_{i+1} \cdots \sigma_{n+1})(\sigma_j \cdots \sigma_{n+1})$$

(for $0 \leq i < j \leq n$) in $\mathbb{B}_\infty^+$, illustrated in the following diagram, together with $\sigma_{n+1} x = x$ for $x \in X^{n-1}$.

Note by looking at the diagram that the i- and j-strands are not entangled with the other strands (which are always above them), but they are entangled with each other.

Combining Theorem 3.1 with Theorem 2.6 provides us with an SCO-system of partial shifts $(\alpha_k)_{k \in \mathbb{N}_0}$ canonically associated to an action of $\mathbb{B}_\infty^+$. For an application of Proposition 2.5 to such a situation in



Example 4.4, we provide the following simplified formulae for powers of these partial shifts.

Lemma 3.2. *If $x \in X^n \subset X$ (with $n \in \mathbb{N}_0$), then, for all $N \geq 1$,*

$$(\alpha_n)^N(x) = \sigma_{n+N} \ldots \sigma_{n+1} x.$$

Proof. From Theorem 3.1, we have

$$(\alpha_n)^N(x) = (\sigma_{n+1} \cdots \sigma_{n+N})(\sigma_{n+1} \cdots \sigma_{n+N-1}) \cdots \sigma_{n+1} x$$

(understood to be $\sigma_{n+1}x$ if $N = 1$). This simplifies as shown above, as can be seen with an induction proof using the braid relations together with $x \in X^n$. $\square$

We are mainly interested in situations where we have a (left) $\mathbb{B}_\infty^+$-module V, in which case Theorem 3.1 yields (at least) an augmented semi-cosimplicial abelian group. We give examples in a probabilistic setting in Section 4. However, here we will give a few direct applications of Theorem 3.1.

Example 3.3. It follows from Theorem 3.1 that the sequence $(\mathbb{B}_n)_n$ is a semi-cosimplicial group with the conjugation action of braids on themselves. In this case, we choose $X := \mathbb{B}_\infty$ and $X^n := \mathbb{B}_{n+1}$ for all $n \in \mathbb{N}_0$ (and we define $X^0 = \mathbb{B}_1$ to be the trivial group). In fact, as required in our definition of X^n, we have

$$\mathbb{B}_{n+1} = \{x \in \mathbb{B}_\infty \colon \sigma_k x \sigma_k^{-1} = x \quad \text{for all } k \geq n+2\},$$

see for example, [**7,** Proposition 4.12] for a proof. Then, for $x \in X^{n-1} = \mathbb{B}_n$, with $n \in \mathbb{N}$, we have

$$\delta^k(x) := \sigma_{k+1} \cdots \sigma_{n+1} x \sigma_{n+1}^{-1} \cdots \sigma_{k+1}^{-1}.$$

From the braid relations, we can check that $\delta^0(\sigma_N) = \sigma_{N+1}$ for all N. If we use the so-called square roots of free generators $\gamma_1, \dots, \gamma_n$ as generators for $\mathbb{B}_{n+1}$, defined by

$$\gamma_N := (\sigma_1 \cdots \sigma_{N-1}) \sigma_N (\sigma_{N-1}^{-1} \cdots \sigma_1^{-1})$$

then we have, for $k \geq 1$, a direct way of describing the coface operators by

$$\delta^k(\gamma_N) = \begin{cases} \gamma_N & \text{if } N < k, \\ \gamma_{N+1} & \text{if } N \geq k. \end{cases}$$

The formula for $k \geq 1$ may be considered as an instance of Proposition 2.5 for the relabeling

$$k \longmapsto k - 1.$$

Again, this can be checked by direct computation using the braid relations. Alternatively, a detailed study of the so-called square roots of free generators of the braid groups and of the corresponding partial shifts may be found in [**7,** Section 4]. It is not accidental that this looks very similar to our Example 2.9 with the sequence of symmetric groups considered as a semi-cosimplicial group. In fact, it is instructive to check that we can go from the braid groups example to the symmetric groups example via the natural quotient map. More generally, because the braid groups have symmetric groups as quotients, we can always produce examples of SCOs from Theorem 3.1 by actions of symmetric groups (interpreting them as actions of braid groups). The semi-cosimplicial groups produced in Example 2.9 are all of this type.

Example 3.4. For another class of examples, we can consider solutions of the Yang-Baxter equations. For illustration, we choose the most basic setting: If Y is a set and r is a function from $Y \times Y$ to itself, then r is called a *set-theoretic solution* of the Yang-Baxter equation if, on $Y \times Y \times Y$, it satisfies $r^{12} r^{23} r^{12} = r^{23} r^{12} r^{23}$, where the superscript indicates on which copies r acts. See, for example, [**5**] for a recent investigation into such solutions. Clearly, this defines an action of

$\mathbb{B}_\infty^+$ on an infinite Cartesian product X of copies of Y, where σ_k is represented by $r^{k-1,k}$.

Remark 3.5. Finally, we mention here that, on $\mathbb{B}_\infty^+$-modules, we obtain among other things a version of the standard semi-cosimplicial cohomology theory which is always defined for SCOs in a module category. In fact, for all $n \in \mathbb{N}_0$, the differential

$$d^n := \sum_{k=0}^{n} (-1)^k \delta^k : V^{n-1} \longrightarrow V^n$$

satisfies $d^{n+1}d^n = 0$ and gives rise to the cohomology groups

$$H^n := \ker(d^{n+1}) / \operatorname{im}(d^n).$$

We shall perform a few direct computations for the SCOs produced from Theorem 3.1. On V^{-1}, we have

$$d^0 = \delta^0 : x \longmapsto x.$$

In addition,

$$d^1 = \delta^0 - \delta^1 : V^0 \longrightarrow V^1;$$

hence, $d^1 x = \sigma_1 x - x$. It follows that both $\operatorname{im}(d^0)$ and $\ker(d^1)$ are equal to the fixed point set of σ_1; thus, H^0 is trivial. Further,

$$d^2 = \delta^0 - \delta^1 + \delta^2 : V^1 \longrightarrow V^2;$$

thus,

$$d^2 x = \sigma_1\sigma_2 x - \sigma_2 x + x$$

for $x \in V^1$, and we find

$$H^1 = \{x \in V^1 : (\sigma_2 - \sigma_1\sigma_2)x = x\} / \{x \in V^1 : x = \sigma_1 y - y \text{ for } y \in V^0\}.$$

We are unaware of any interpretation of these cohomology groups; however, it may be interesting to investigate when these groups are nontrivial and if they can play a role in the study of braid group representations.

We remark that other connections between the simplicial category and braid groups are investigated in the literature, see for example, **[20]**. However, it is unclear at the moment how these investigations are related to our results above.

4. Semi-cosimplicial objects and spreadability in noncommutative probability. In this section, we develop our theory within various categories of noncommutative probability spaces. We begin with a very general situation and then, by specializing, make contact with settings that have a genuine probabilistic interpretation, as discussed in Section 1. We refer to [**16**] for further motivation to study these categories.

First, consider a category with objects $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C}$$

is a linear functional with $\varphi(\mathbb{1}) = 1$, i.e., unital, and with morphisms

$$\alpha : (\mathcal{A}, \varphi) \longrightarrow (\mathcal{B}, \psi),$$

where α is an algebra homomorphism satisfying $\alpha(\mathbb{1}) = \mathbb{1}$, i.e., unital, and $\psi \circ \alpha = \varphi$. We call this the *category of noncommutative probability spaces* (as in [**16**]). We mention here that there is no particular difficulty working out the following theory in a non-unital setting, but, for definiteness, we will concentrate on this standard version.

Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. If $\mathcal{B}$ is a unital associative algebra and

$$\iota : \mathcal{B} \longrightarrow \mathcal{A}$$

a unital algebra homomorphism, then we may think of it as a morphism

$$\iota : (\mathcal{B}, \varphi_B) \longrightarrow (\mathcal{A}, \varphi)$$

with $\varphi_B := \varphi \circ \iota$. This is called a (*noncommutative*) *random variable*. A sequence $(\iota_N)_{N=0}^{\infty}$ of such random variables is called a (*noncommutative*) *random process*, and any expression of the form

$$\varphi(\iota_{N_1}(b_1), \ldots, \iota_{N_k}(b_k)), \quad b_i \in \mathcal{B},$$

(repetitions allowed), is called a *moment* of the process. If the variables do not commute, we cannot speak of a joint distribution in the classical sense; however, there is the following replacement for it.

Let $\mathcal{A}^f := *_{N=0}^{\infty} \mathcal{B}$ be the (unital) free product of infinitely many copies of $\mathcal{B}$. See for example, [**4**] for further uses of this construction in noncommutative probability. Let

$$\lambda_N : \mathcal{B} \longrightarrow \mathcal{A}^f$$

denote the canonical unital homomorphisms arising from this construction. The universal property ensures that there exists a unique unital homomorphism

$$\pi : \mathcal{A}^f \longrightarrow \mathcal{A}$$

such that $\pi \circ \lambda_N = \iota_N$ for all $N \in \mathbb{N}_0$. The unital linear functional φ^f on $\mathcal{A}^f$ defined by $\varphi^f := \varphi \circ \pi$ is called the *distribution* of the random process. Another way to think of a distribution is as a collection of all moments.

Remark 4.1. In the literature, it is often a sequence of elements $(x_N)_{N\in\mathbb{N}_0}$ in a noncommutative probability space which is called a noncommutative random process. This is merely a special case of our setting where $\mathcal{B}$ has a single generator and its image under ι_N is called x_N. The more flexible setting chosen here allows us to include multivariable processes without much additional effort.

Processes with the same distribution are called *stochastically equivalent*. Therefore, if we are satisfied with stochastically equivalent versions, it is possible to restrict our attention to free products: the processes (ι_n) and (λ_n) given above have the same distribution if we endow $\mathcal{A}^f$ with the functional φ^f.

Definition 4.2. A sequence $(\iota_N)_{N\in\mathbb{N}_0}$ of unital homomorphisms from $\mathcal{B}$ to the noncommutative probability space $(\mathcal{A}, \varphi)$ is called *spreadable* if its distribution is unchanged when passing to a subsequence, i.e., if, for all $N \in \mathbb{N}_0$, ι_N is replaced by $\iota_{i(N)}$ such that $N_1 < N_2$ implies $i(N_1) < i(N_2)$.

Spreadability is a distributional symmetry (or invariance principle). It only depends on the distribution, and a sequence $(\iota_N)_{N\in\mathbb{N}_0}$ is spreadable if and only if the corresponding canonical sequence $(\lambda_N)_{N\in\mathbb{N}_0}$ is spreadable (where $\mathcal{A}^f$ is equipped with the functional φ^f defined above). As indicated above, an equivalent description may be given in terms of moments where the definition of spreadability reduces to a system of equalities for numbers.

This leads us to the main theorem. Informally stated, any appearance of SCOs in the category of noncommutative probability spaces always induces spreadability and, conversely, for any spreadable sequence, we may always find an SCO which reproduces its distribution. Hence, SCOs can be interpreted as the fundamental algebraic structure underlying spreadability.

Theorem 4.3.

(1) *Let an* SCO *be given in the category of noncommutative probability spaces with filtration* $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}_0}$ *and inductive limit* $(\mathcal{A}_\infty, \varphi_\infty)$*. Let*

$$\iota_0 := \mu_0 \colon \mathcal{A}_0 \longrightarrow \mathcal{A}_\infty$$

and

$$\iota_N := (\alpha_0)^N \iota_0 \quad \text{for } N \in \mathbb{N}_0.$$

Then, $(\iota_N)_{N\in\mathbb{N}_0}$ *is spreadable.* (*Here,* α_0 *is what we call the full shift among the partial shifts associated to the* SCO*, see Section* 2.)

(2) *Let* $(\iota_N)_{N\in\mathbb{N}_0}$ *be a sequence of unital homomorphisms from the unital algebra* $\mathcal{B}$ *to the noncommutative probability space* $(\mathcal{A}, \varphi)$*, and let* $(\mathcal{A}^f, \varphi^f)$ *be the corresponding* (*unital*) *free product equipped with the distribution* (*as described above, with* λ_N*,* $N \in \mathbb{N}_0$*, denoting the canonical embeddings, etc.*)*. Consider the following statements* (a), (b) *and* (c):

(a) $(\iota_N)_{N\in\mathbb{N}_0}$ *is spreadable.*

(b) *Let* $\mathcal{A}^f_n$ *be generated by* $\lambda_0(\mathcal{B}), \ldots, \lambda_n(\mathcal{B})$ (*as a unital algebra*)*, for all* $n \in \mathbb{N}_0$*. The sequence* $(\mathcal{A}^f_n, \varphi^f_n)_{n\in\mathbb{N}_0}$ *is an* SCO *in the category of noncommutative probability spaces with coface operators given by*

$$\delta^k : (\mathcal{A}^f_{n-1}, \varphi^f_{n-1}) \longrightarrow (\mathcal{A}^f_n, \varphi^f_n),$$

for $k = 0, \ldots, n$ (*with* φ^f_n *the restriction of* φ^f *to* $\mathcal{A}^f_n$)*, determined* (*for* $b \in \mathcal{B}$) *by*

$$\lambda_N(b) \longmapsto \begin{cases} \lambda_N(b) & \text{if } N < k, \\ \lambda_{N+1}(b) & \text{if } N \geq k. \end{cases}$$

(c) *Let* $\mathcal{A}_n$ *be generated by* $\iota_0(\mathcal{B}), \ldots, \iota_n(\mathcal{B})$ (*as a unital algebra*)*, for all* $n \in \mathbb{N}_0$*. The sequence* $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}_0}$ *is an* SCO *in the category of* *noncommutative probability spaces with coface operators given by*

$$\delta^k : (\mathcal{A}_{n-1}, \varphi_{n-1}) \longrightarrow (\mathcal{A}_n, \varphi_n),$$

for $k = 0, \dots, n$ (*with* φ_n *the restriction of* φ *to* $\mathcal{A}_n$), *determined* (*for* $b \in \mathcal{B}$) *by*

$$\iota_N(b) \longmapsto \begin{cases} \iota_N(b) & \textit{if } N < k, \\ \iota_{N+1}(b) & \textit{if } N \geq k. \end{cases}$$

Then, (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c).

Proof. We begin by proving (1). Note that, in the category of noncommutative probability spaces, we can form inductive limits, and hence, we can go from SCOs to SCO-systems of partial shifts (as established in Section 2) whenever convenient. (The same applies to the $*$-setting studied later.)

Let $\varphi(q)$ be a moment of a subsequence $(\iota_{i(N)})_{N \in \mathbb{N}_0}$. We define p to be the finite product obtained by replacing each factor $\iota_{i(N)}$ in q by ι_N. Suppose that the subscripts N appearing in this way are

$$N_1 < N_2 < \cdots < N_R.$$

Let $M_1 := N_1$ and, for $2 \leq r \leq R$, define

$$M_r := N_r + [i(N_{r-1}) - N_{r-1}].$$

Then, $N_r \leq M_r \leq i(N_r)$ and, using the properties of partial shifts stated in Proposition 2.5, we can verify that

$$\alpha_{M_R}^{i(N_R)-M_R} \cdots \alpha_{M_1}^{i(N_1)-M_1}(p) = q.$$

In fact, successively applying the partial shifts results in replacement of the factors of p with the corresponding factors in q. Since the partial shifts preserve the functional φ, the proof is complete.

Now we prove the equivalence of (2) (a) and (2) (b). The formula in (b) always determines an algebra homomorphism δ^k between the free products $\mathcal{A}_{n-1}^f$ and $\mathcal{A}_n^f$, for $k = 0, \dots, n$ and $n \in \mathbb{N}$. It is easily verified that these δ^k satisfy the cosimplicial identities. (Therefore, this is always an SCO in the category of algebras.)

If $(\iota_N)_{N\in\mathbb{N}_0}$ is spreadable, then $\varphi_n^f \circ \delta^k = \varphi_{n-1}^f$ since we may always consider the subsequence which misses the kth position. Hence, (a) implies (b).

Conversely, given (b), the morphism δ^k (for $0 \leq k \leq n-1$) maps any polynomial in

$$\lambda_0(b_0), \ldots, \lambda_{n-1}(b_{n-1}), \quad b_i \in \mathcal{B},$$

in $\mathcal{A}_{n-1}^f$ into the corresponding polynomial in

$$\lambda_0(b_0), \ldots, \lambda_{k-1}(b_{k-1}), \lambda_{k+1}(b_k), \ldots, \lambda_n(b_{n-1})$$

in $\mathcal{A}_n^f$. (Note that

$$\delta^n : \mathcal{A}_{n-1}^f \longrightarrow \mathcal{A}_n^f$$

is merely the embedding of $\mathcal{A}_{n-1}^f$ into $\mathcal{A}_n^f$. Compare with our construction of the inductive limit from an SCO in Section 2.) If

$$i : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$

is any strictly increasing function, then, by an induction argument, we can always find a composition of coface operators which sends a polynomial p in

$$\lambda_{N_1}(b_1), \ldots, \lambda_{N_R}(b_R)$$

into the corresponding polynomial q in

$$\lambda_{i(N_1)}(b_1), \ldots, \lambda_{i(N_R)}(b_R).$$

(We have seen an explicit formula for this, using partial shifts, in the proof of (1).) Hence, (b) implies (a).

Finally, we find that the implication from (2) (c) to 2 (a) is, in fact, a special case of (1). Here, we identify each $\mathcal{A}_n$ with its image in the inductive limit and omit each morphism μ_n, which is merely an embedding of $\mathcal{A}_n$ into $\mathcal{A}_\infty \subset \mathcal{A}$. $\square$

We may now use our construction of SCOs from representations of braid monoids in Section 3 to produce many examples of spreadable sequences. This includes exchangeability which comes from representations of the symmetric groups. As an example, reconsider the tensor product presented in Section 1. The general case of exchangeability is characterized from this point of view in [**7,** Theorem 1.9]. We can also begin with the semi-cosimplicial groups constructed in Sections 2 and 3 and obtain SCOs in the category of noncommutative probability spaces based on the corresponding group algebras. For $\mathbb{B}_\infty$, the group von Neumann algebra is studied from this point of view in [**7,** Section 5]. A study of other groups is postponed to future work.

Example 4.4. Instead, we illustrate our theory here with an interesting example of spreadable sequences from the theory of subfactors in von Neumann algebras. For this, we follow [**9**], in particular subsection 4.4, where more details may be found. Note that, to get a better fit with our previous notation, we use a different numbering of the tower from that used in [**9**].

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite factors with finite Jones index $\beta = [\mathcal{M} : \mathcal{N}]$. Then, with $\mathcal{M}_{-1} := \mathcal{N}$, $\mathcal{M}_0 := \mathcal{M}$, Jones' basic construction yields a tower

$$\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots ,$$

and its weak closure with respect to the Markov trace tr yields the finite factor $\mathcal{M}_\infty$. (In order to interpret the finite factor $\mathcal{M}_\infty$ as an inductive limit, we need a category of von Neumann algebras and von Neumann algebraic noncommutative probability spaces, but, for the following arguments, we can also stay in the category of unital associative algebras and linear functionals introduced above where the inductive limit is merely the union of all M_k.) The algebra $\mathcal{M}_\infty$ is generated (as a von Neumann algebra) by $\mathcal{M}$ together with a sequence of orthogonal projections $(e_n)_{n\in\mathbb{N}}$, called the Temperley-Lieb projections, satisfying $e_n \in \mathcal{M}_n$ and

$$e_n e_{n\pm1} e_n = \beta^{-1} e_n, \qquad e_n e_m = e_m e_n \quad \text{if } |n-m| \geq 2$$

(for all n, m). Then, with $\beta = 2 + q + q^{-1}$, and defining

$$g_n := q e_n - (\mathbb{1} - e_n),$$

it turns out that the g_n satisfy the braid relations. Therefore, this determines a representation

$$\mathbb{B}_\infty \ni \sigma \longmapsto g \in \mathcal{M}_\infty$$

by invertible elements $g \in \mathcal{M}_\infty$ which, in particular, maps the Artin generator σ_n to g_n, for all n. Hence, we can define an action of $\mathbb{B}_\infty$ on $\mathcal{M}_\infty$ by $\sigma x := g x g^{-1}$ for $\sigma \in \mathbb{B}_\infty$ and $x \in \mathcal{M}_\infty$. Clearly, the Markov trace is invariant for this action (since it is a trace). Further, note that $\mathcal{M} = \mathcal{M}_0$ commutes with e_n, and hence, with g_n for $n \geq 2$. From Theorem 3.1 (and Theorem 2.6) we obtain an SCO (and an SCO-system of partial shifts), and hence, from Theorem 4.3 we obtain spreadable sequences. Explicitly, with Lemma 3.2, we conclude that the sequence $(\iota_n)_{n\in\mathbb{N}_0}$ of noncommutative random variables

$$\iota_n \colon \mathcal{M} \longrightarrow \mathcal{M}_\infty$$

given by $\iota_0 := \mathrm{id}$ and, for $n \geq 1$,

$$\iota_n := \mathrm{Ad}(g_n \cdots g_1)$$

is spreadable. In particular, for any $x \in \mathcal{M}$, the sequence $(x_N)_{N\in\mathbb{N}_0}$ given by

$$x_0 := x, x_1 := g_1 x g_1^{-1}, \ldots, x_N := g_N \cdots g_1 x g_1^{-1} \cdots g_N^{-1}, \ldots$$

is spreadable, always with respect to the Markov trace.

With the following modification, we can find further spreadable sequences. For any $m \in \mathbb{N}_0$, consider the m-shifted action of $\mathbb{B}_\infty$ on $\mathcal{M}_\infty$, determined by

$$\sigma_n \longmapsto g_{n+m}, \quad \text{for all } n \in \mathbb{N}.$$

It follows that the sequence $(\iota_n)_{n\in\mathbb{N}_0}$ of noncommutative random variables

$$\iota_n \colon \mathcal{M}_m \longrightarrow \mathcal{M}_\infty,$$

given by $\iota_0 := \mathrm{id}$ and, for $n \geq 1$,

$$\iota_n := \mathrm{Ad}(g_{m+n} \cdots g_{m+1})$$

is spreadable. In particular for any $x \in \mathcal{M}_m$, the sequence $(x_N)_{N\in\mathbb{N}_0}$, given by

$$x_0 := x, \qquad x_1 := g_{m+1} x g_{m+1}^{-1}, \ldots,$$
$$x_N := g_{m+N} \cdots g_{m+1} x g_{m+1}^{-1} \cdots g_{m+N}^{-1}, \ldots$$

is spreadable, with respect to the Markov trace, and, for all $m \in \mathbb{N}$, we find a spreadable sequence of projections $(e_{m,N})_{N\in\mathbb{N}_0}$, given by

$$e_{m,0} := e_m, \qquad e_{m,1} := g_{m+1} e_m g_{m+1}^{-1}, \ldots,$$
$$e_{m,N} := g_{m+N} \cdots g_{m+1} e_m g_{m+1}^{-1} \cdots g_{m+N}^{-1}, \ldots.$$

Note that, in general, the braid group representations are not unitary, and hence, the coface operators and partial shifts in these arguments are algebra homomorphisms, but not necessarily $*$-homomorphisms. This implies that the projections $e_{m,N}$ may be non-orthogonal projections. We comment further on this at the end of the section.

Now, in the final part of this paper, we turn to $*$-algebras and to the probabilistic setting from where the notion of spreadability originally comes. Again, this general setting can also be found in [**16**]. If, in a noncommutative probability space $(\mathcal{A}, \varphi)$, $\mathcal{A}$ is a (unital) $*$-algebra and φ is a unital positive linear functional (i.e., a state, which is positive in the sense that $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$) then we call $(\mathcal{A}, \varphi)$ a noncommutative $*$-*probability space*. We obtain the corresponding category by requiring morphisms

$$\alpha\colon (\mathcal{A}, \varphi) \longrightarrow (\mathcal{B}, \psi)$$

to be unital $*$-homomorphisms such that $\psi \circ \alpha = \varphi$. There is no need to repeat the definitions of random variables, moments, distributions and spreadability since they remain the same; however, we now refer to the new category of noncommutative $*$-probability spaces. In particular, to define $*$-spreadability we need $*$-homomorphisms in Definition 4.2. In practice, this can make a big difference. For example, in the situation of a random process specified by a sequence $(x_N)_{N\in\mathbb{N}_0}$ of elements in a noncommutative probability space, see Remark 4.1, if we work in the category of noncommutative $*$-probability spaces, we must take into account not only the elements x_N themselves but also their adjoints x_N^*. In order to make the difference clear, we talk about $*$-moments, $*$-distributions and $*$-spreadability but the reader should be aware that, in the literature, exclusively working in this setting, the latter is usually merely called spreadability.

It may be immediately verified that we can transfer our previous arguments to the category of noncommutative $*$-probability spaces and, in this way, successfully deal with $*$-spreadability. For convenience, we repeat Theorem 4.3 explicitly in the $*$-setting and add a useful simplification which is available for faithful states. Recall that a positive functional φ is called faithful if $\varphi(a^*a) = 0$ for $a \in \mathcal{A}$ implies $a = 0$.

Theorem 4.5.

(1) *Let an* SCO *be given in the category of noncommutative* $*$*-probability spaces with filtration* $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}_0}$ *and inductive limit* $(\mathcal{A}_\infty, \varphi_\infty)$*. Let*

$$\iota_0 := \mu_0 \colon \mathcal{A}_0 \longrightarrow \mathcal{A}_\infty$$

and

$$\iota_N := (\alpha_0)^N \iota_0 \quad \textit{for } N \in \mathbb{N}_0.$$

Then, $(\iota_N)_{N\in\mathbb{N}_0}$ *is* $*$*-spreadable.* (*Here* α_0 *is what we call the full shift among the partial shifts associated to the* SCO.)

(2) *Let* $(\iota_N)_{N\in\mathbb{N}_0}$ *be a sequence of unital* $*$*-homomorphisms from the unital* $*$*-algebra* $\mathcal{B}$ *to the noncommutative* $*$*-probability space* $(\mathcal{A}, \varphi)$*, and let* $(\mathcal{A}^f, \varphi^f)$ *be the corresponding* (*unital*) *free product equipped with the* $*$*-distribution* (*with* λ_N*,* $N \in \mathbb{N}_0$*, denoting the canonical embeddings, etc.*)*. Consider the following statements* (a), (b), (c):

(a) $(\iota_N)_{N\in\mathbb{N}_0}$ *is* $*$*-spreadable.*

(b) *Let* $\mathcal{A}_n^f$ *be generated by* $\lambda_0(\mathcal{B}), \ldots, \lambda_n(\mathcal{B})$ (*as a unital* $*$*-algebra*)*, for all* $n \in \mathbb{N}_0$*. The sequence* $(\mathcal{A}_n^f, \varphi_n^f)_{n\in\mathbb{N}_0}$ *is an* SCO *in the category of noncommutative* $*$*-probability spaces with coface operators given by*

$$\delta^k : (\mathcal{A}_{n-1}^f, \varphi_{n-1}^f) \longrightarrow (\mathcal{A}_n^f, \varphi_n^f),$$

for $k = 0, \ldots, n$*, with* φ_n^f *the restriction of* φ^f *to* $\mathcal{A}_n^f$*, determined* (*for* $b \in \mathcal{B}$) *by*

$$\lambda_N(b) \longmapsto \begin{cases} \lambda_N(b) & \textit{if } N < k, \\ \lambda_{N+1}(b) & \textit{if } N \geq k. \end{cases}$$

(c) *Let* $\mathcal{A}_n$ *be generated by* $\iota_0(\mathcal{B}), \ldots, \iota_n(\mathcal{B})$ (*as a unital* $*$*-algebra*)*, for all* $n \in \mathbb{N}_0$*. The sequence* $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}_0}$ *is an* SCO *in the category of noncommutative* $*$*-probability spaces with coface operators given by*

$$\delta^k : (\mathcal{A}_{n-1}, \varphi_{n-1}) \longrightarrow (\mathcal{A}_n, \varphi_n),$$

for $k = 0, \ldots, n$*, with* φ_n *the restriction of* φ *to* $\mathcal{A}_n$*, determined* (*for* $b \in \mathcal{B}$) *by*

$$\iota_N(b) \longmapsto \begin{cases} \iota_N(b) & \textit{if } N < k, \\ \iota_{N+1}(b) & \textit{if } N \geq k. \end{cases}$$

Then (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c)*. Moreover, if* φ *is faithful, then* (a) $\implies$ (c)*.*

Proof. All except the final statement regarding the faithful case can be proved by checking that the proof of Theorem 4.3 can be adapted to the category of noncommutative $*$-probability spaces. Now, assume that φ is faithful and that $(\iota_N)_{N\in\mathbb{N}_0}$ is $*$-spreadable. If we try to define coface operators δ^k by the formulae in (c), we note that these formulae guarantee the cosimplicial identities on the generators, and hence, on $\mathcal{A}_{n-1}$ if we can extend these formulae to morphisms, necessarily in a unique way, by the $*$-homomorphism property. Thus, we only need check that extending the formulae given in (c) for δ^k as morphisms is well defined. Indeed, if p is any noncommutative polynomial in $\iota_0(b_0), \ldots, \iota_{n-1}(b_{n-1})$, $b_i \in \mathcal{B}$, then, by $*$-spreadability, we find that $\varphi(\delta^k p) = \varphi(p)$ and

$$\varphi(p^* p) = \varphi((\delta^k p^*)(\delta^k p));$$

thus, $p = 0$ implies $\delta^k p = 0$ (since φ is faithful). This shows that δ^k is well defined as a morphism in the category of noncommutative $*$-probability spaces. □

Note that, all that is needed for the converse direction (a) $\Rightarrow$ (c) is the well-definedness of the morphisms δ^k. The assumption that φ is faithful is merely a convenient sufficient condition to enforce that.

We conclude by revisiting the spreadable sequences in towers of von Neumann algebras studied in Example 4.4. This example shows that some care must be taken in distinguishing spreadability and $*$-spreadability. As noted in [**9,** Example 4.2.10], when the index β is mall, i.e., $\beta \le 4$, the representations of $\mathbb{B}_\infty$ are unitary. This implies that the corresponding SCO-systems of partial shifts are given by $*$-endomorphisms, and hence, we are in the setting of Theorem 4.5. We conclude that, when the index is small, these sequences are actually $*$-spreadable. Of course, $*$-spreadability fits better into the category of von Neumann algebras as specific $*$-algebras, and the whole theory of $*$-braidability developed in [**7**] is now applicable in this situation. It is less clear how to make good use of spreadability when the index β is big, i.e., $\beta > 4$, when we cannot expect $*$-spreadability and the de Finetti-type results of [**7, 13**] (also briefly discussed in Section 1) to be available.

The reader may refer back to the discussion in Section 1 for a more comprehensive view of the importance of $*$-spreadability in (noncommutative) probability theory. Many additional examples may be found in [**7**].

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