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FRONT-LIKE ENTIRE SOLUTIONS FOR A DELAYED NONLOCAL DISPERSAL EQUATION WITH CONVOLUTION TYPE BISTABLE NONLINEARITY

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ABSTRACT. This paper is concerned with front-like entire solutions of a delayed nonlocal dispersal equation with convolution type bistable nonlinearity. Here, a solution defined for all $(x,t) \in \mathbb{R}^2$ is an entire solution. It is known that the equation has an increasing traveling wavefront with nonzero wave speed under some reasonable conditions. We first give the asymptotic behavior of traveling wavefronts at infinity. Then, by the comparison argument and sub-super-solutions method, we construct new types of entire solutions other than traveling wavefronts and equilibrium solutions of the equation, which behave like two increasing traveling wavefronts propagating from both sides of the x-axis and annihilate at a finite time. Finally, various qualitative properties of the entire solutions are also investigated.

1. Introduction. In this paper, we study the entire solutions of the following delayed nonlocal dispersal equation:

$$\frac{\partial u}{\partial t} = J * u - u - du + \int_{\mathbb{R}} K(y) b(u(x-y, t-\tau))\, dy. \tag{1.1}$$

This model represents the population dynamics of a single species with age-structure and dispersal, see the first author of this paper **[31]** and Huang et al. **[12]** for more details. Here, $u(x,t)$ represents the total mature population of the species (after the maturation age $\tau > 0$) at location x, time t, $d > 0$ the coefficient of death for the mature species

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and $b(\cdot)$ the birth function. $J * u - u$ is a nonlocal dispersal operator and represents transportation due to long range dispersion mechanisms, where $J * u$ is a spatial convolution defined by

$$(J * u)(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,dy.$$

As stated in [**3, 7**], if $J(x-y)$ is the probability distribution of jumping from location y to location x, then the rate at which individuals arrive to location x from all other places is

$$(J * u)(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,dy,$$

and the rate at which they leave location x to travel to all other places is

$$u(x,t) = \int_{\mathbb{R}} J(x-y)u(x,t)\,dy.$$

The kernel function $K(x-y)$ accounts for the probability that an individual born at location y, time $t-\tau$, will be at location x at time t. Throughout this paper, we assume that:

(H) Both kernels $J \in C^1(\mathbb{R})$ and $K \in C^2(\mathbb{R})$ satisfy $J(x) = J(-x) \geq 0$ and $K(x) = K(-x) \geq 0$ for $x \in \mathbb{R}$;

$$\int_{\mathbb{R}} J(x)\,dx = 1, \qquad \int_{\mathbb{R}} K(x)\,dx = 1;$$

J, K are compactly supported.

(B) Birth function $b \in C^2(\mathbb{R})$:

(B1) $b(0) = d\alpha - b(\alpha) = d - b(1) = 0$ for some $0 < \alpha < 1$.

(B2) $b'(u) > 0$ for $u \in (0,1)$, $b(u) < du$ for $0 < u < \alpha$ and $b(u) > du$ for $\alpha < u < 1$.

(B3) $0 < \max\{b'(0), b'(1)\} < d < b'(\alpha)$ (bistable nonlinearity).

(B4) $\int_0^1 [b(u) - du]\,du > 0$ (unbalanced case).

A typical birth function which has been widely used in the mathematical biology literature is $b(u) = pu^2 e^{-\gamma u}$, with constants $p > 0$ and $\gamma > 0$, and satisfies the above assumptions for a wide range of parameters p and γ, see [**31**].

We mention two special cases of (1.1) covered by the following analysis. If $K(x) = \delta(x)$, where $\delta(\cdot)$ is the Dirac delta function, then equation (1.1) reduces to the following nonlocal dispersal equation with local nonlinearity term [**20**]:

$$\frac{\partial u}{\partial t} = J * u - u - du(x,t) + b(u(x,t-\tau)). \tag{1.2}$$

Furthermore, while taking $\tau = 0$ and $f(u) = -du + b(u)$, (1.2) becomes the more general nonlocal dispersal equation [**1, 2, 6, 7**]:

$$\frac{\partial u}{\partial t} = J * u - u + f(u). \tag{1.3}$$

In recent years, nonlocal dispersal equations (1.1)–(1.3) have attracted significant attention due to their important applications in many scientific disciplines, such as material science [**1**], biology [**6**], epidemiology [**16**] and neural networks [**36**]. For such equations, the most attractive object is the spatial dynamics, including asymptotic behavior, spreading speeds, traveling wave solutions and entire solutions, see [**1, 2, 3, 4, 5, 6, 7, 14, 19, 20, 22, 30, 33**], and the references cited therein.

In this paper, we are interested in front-like entire solutions of (1.1) which are obtained by the interaction of traveling wavefronts. It is well known that the traveling wavefront is an important class of solutions for dynamical models in biology since it can explain spatial spread or invasion of the species. Mathematically, a *traveling wave solution* of (1.1) connecting $\{e_1, e_2\} \subset \{0, \alpha, 1\}$ is a special translation invariant solution of the form $u(x,t) = U(\xi)$, $\xi = x + ct$, U the wave profiles that propagate through the one-dimensional spatial domain at a constant velocity c. If $U(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wavefront.

Traveling wave solutions of (1.1) have been investigated and are quite well understood, see [**12, 31, 32, 34, 35**]. The first author [**31**] investigated the existence and uniqueness (up to translations) of traveling wave solutions when equation (1.1) is monostable. Huang et al. [**12**] further proved that the planar traveling wave fronts with $c \geq c^*$ of (1.1) in $\mathbb{R}^N$ are globally asymptotically stable. The existence of traveling wave solutions with speed $c > c^*$ of (1.1) is also obtained in [**31**] by introducing two auxiliary monotone nondecreasing birth functions when equation (1.1) is crossing-monostable. In [**34**], the authors further obtained the existence of traveling wave solutions with speed $c = c^*$

and showed that the slowest wave speed c^* coincides with the spreading speed of (1.1). At the same time, they proved that the traveling wave solution with a large speed is exponentially stable by the weighted energy method. In a recent paper, the authors **[32]** established the existence of non-monotone traveling wave solutions and proved the uniqueness of traveling wave solutions, including those which are monotone and non-monotone. When equation (1.1) is bistable, the author **[35]** showed that there exists a unique constant $c \in \mathbb{R}$ such that (1.1) admits a traveling wavefront with speed c connecting equilibria $u = 0$ and $u = 1$. Furthermore, they proved that the traveling wavefronts of (1.1) are unique and globally exponentially stable with phase shift.

In addition to traveling wave solutions, another important issue in population dynamics is the interaction between traveling wavefronts. Mathematically, this phenomenon may be described by the so called front-like entire solutions that are defined for all $(x,t) \in \mathbb{R}^2$. In particular, traveling wavefronts are special examples of the entire solutions and consist of two 1-dimensional manifolds, namely,

$$u(x,t;\theta) := U(x+ct+\theta) \quad \text{and} \quad u(x,t;\theta) := U(-x+ct+\theta),$$

where θ varies in $\mathbb{R}$ (note that the wave speed c is unique in the bistable case). From the dynamical point of view, the study of entire solutions can help us fully understand the transient dynamics and structures of the global attractor **[17]**. From the viewpoint of biology, such entire solutions provide some new invasion methods of the species. In recent years, much work has been devoted to the entire solutions for various evolution equations for both bistable and monostable nonlinearities, see e.g., **[10, 11, 17, 28, 30]** for reaction-diffusion equations with and without delays, **[8, 24, 25]** for lattice differential equations with local and global interaction and **[14, 22, 29, 32]** for nonlocal dispersal equations. We also refer the reader to **[9, 18, 23, 27]** for the reaction-diffusion system and **[15]** for the nonlocal dispersal system.

For equation (1.1) with monostable and crossing-monostable nonlinearities, the author **[32]** studied the existence of its entire solutions. Combining a spatially independent solution and traveling wavefronts with different speeds, some new entire solutions have been constructed. However, to the best of our knowledge, for the bistable case, there are no results for the entire solutions other than traveling wave solutions.

In the current paper, we try to establish some results on this subject for (1.1). More precisely, we prove the existence of entire solutions which behave like two traveling wavefronts propagating from both sides of the x-axis and then annihilating in a finite time. As t goes forward, these solutions converge to one of the positive equilibria. The basic idea for the construction of front-like entire solutions is to use traveling fronts to build super- and sub-solutions of (1.1) and then deduce the existence of the entire solutions trapped between these super and sub-solutions.

There are two important points to make.

(i) The key step for constructing entire solutions is having precise information on the asymptotic behavior of traveling wave solutions of (1.1) at infinity. For the monostable case, the authors of the current paper [**32**] obtained asymptotic behavior by the method developed by Carr and Chmaj [**2**], see also Coville, et al., [**5**]. In this paper, we shall still apply this method. Since our equation (1.1) is bistable, the technique details are different from the monostable case.

(ii) In order to establish the existence of entire solutions, we study the solutions $u_n(x,t)$ of the following initial value problem:

(1.4)
$$\begin{cases} \dfrac{\partial u_n}{\partial t} = J*u_n - u_n - du_n + \int_{\mathbb{R}} K(y)u_n(x-y,t-\tau)\,dy & x\in\mathbb{R},\ t>-n, \\ u_n(x,-n+s) = u_{n,0}(x,s) & \\ \quad = \max\{U(x-c(n-s)+h), U(-x-c(n-s)+h')\} & x\in\mathbb{R},\ s\in[-\tau,0]. \end{cases}$$

By constructing appropriate sub- and super-solutions, some new entire solutions are obtained by passing to the limit $n\to+\infty$. Note that, for the nonlocal dispersal equation (1.4), the sequence functions $u_n(x,t)$ are not smooth enough with respect to x, and hence, its convergence is not ensured. In order to obtain a convergent subsequence of $\{u_n(x,t)\}$ we must make $\{u_n(x,t)\}$ possess a property which is similar to a global Lipschitz condition with respect to x, see Lemma 4.9.

The rest of this paper is organized as follows. In Section 2, we state the main results for the entire solutions of (1.1). In Section 3, the asymptotic behavior of traveling wavefronts is obtained. In Section 4, we first establish the existence of entire solutions and then study their qualitative properties.

2. Main results. In this section, we shall present the main results for the entire solutions of (1.1). Before stating the main results, we provide the next definition.

Definition 2.1.

(i) A function $\Phi(x,t)$, $(x,t) \in \mathbb{R}^2$ is called an *entire solution* of (1.1) if, for any $x \in \mathbb{R}$, $\Phi(x,t)$ is differential for all $t \in \mathbb{R}$ and $\Phi(x,t)$ satisfies (1.1) for all $(x,t) \in \mathbb{R}^2$.

(ii) Let $m \in \mathbb{N}$ and $p, p_0 \in \mathbb{R}^m$. We say that a sequence of function $W^p(x,t)$ *converges* to a function $W^{p_0}(x,t)$ in the sense of the topology $\mathcal{T}$ if, for every compact set $S \subset \mathbb{R}^2$, functions $W^p(x,t)$ and $(\partial/\partial t)W^p(x,t)$ converge uniformly in $(x,t) \in S$ to $W^{p_0}(x,t)$ and $(\partial/\partial t)W^{p_0}(x,t)$, respectively, as $p \to p_0$.

In a recent paper **[35]**, we have shown the existence, uniqueness and stability of traveling wavefronts for (1.1) by applying a continuation method and squeezing technique. We recall the main result of **[35]** as follows.

Theorem 2.2. *Assume that* (H) *and* (B) *hold. Then* (1.1) *admits a non-decreasing traveling wavefront* $U(\xi)$ *with speed* $c = c_0 \neq 0$, *which satisfies*

$$\begin{cases} cU'(\xi) = (J * U)(\xi) - U(\xi) - dU(\xi) + \int_{\mathbb{R}} K(y) b(U(\xi - y - c\tau))\, dy, \\ U(-\infty) = 0, \quad U(+\infty) = 1, \end{cases} \tag{2.1}$$

where $\xi = x + ct$ *and* $' = d/d\xi$. *Moreover, the solution* U *of* (2.1) *is strictly monotone increasing, unique up to a translation and globally asymptotically stable.*

Theorem 2.3. *Assume that* (H) *and* (B) *hold. Let* (U, c_0) *be a solution of* (2.1) *with speed* $c_0 > 0$. *Then, for any constants* $\omega_1 \in \mathbb{R}$ *and* $\omega_2 \in \mathbb{R}$, *there exists an entire solution* $\Phi_{\omega_1,\omega_2}(x,t)$ *of* (1.1) *defined for all* $(x,t) \in \mathbb{R}^2$ *such that*

$$\lim_{t \to -\infty} \Big\{ \sup_{x \geq 0} |\Phi_{\omega_1,\omega_2}(x,t) - U(x + c_0 t + \omega_1)| + \sup_{x \leq 0} |\Phi_{\omega_1,\omega_2}(x,t) - U(-x + c_0 t + \omega_2)| \Big\} = 0.$$

Furthermore, the following statements hold:

(i) $0 < \Phi_{\omega_1,\omega_2}(x,t) < 1$ *and* $(\partial/\partial t)\Phi_{\omega_1,\omega_2}(x,t) > 0$ *for* $(x,t) \in \mathbb{R}^2$.

(ii) $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |\Phi_{\omega_1,\omega_2}(x,t) - 1| = 0$ *and* $\lim_{t\to-\infty} \sup_{|x|\le M_1} \Phi_{\omega_1,\omega_2}(x,t) = 0$ *for any* $M_1 \in \mathbb{R}^+$.

(iii) $\lim_{|x|\to+\infty} \sup_{t\ge t_1} |\Phi_{\omega_1,\omega_2}(x,t) - 1| = 0$ *for any* $t_1 \in \mathbb{R}$.

(iv) $\Phi_{\omega_1,\omega_2}(x,t)$ *converges to*

$$\begin{cases} U(-x + c_0 t + \omega_2) & \text{as } \omega_1 \to -\infty \text{ in the sense of topology } \mathcal{T}; \\ U(x + c_0 t + \omega_1) & \text{as } \omega_2 \to -\infty \text{ in the sense of topology } \mathcal{T}. \end{cases}$$

(v) *For any* $(x,t) \in \mathbb{R}^2$, $\Phi_{\omega_1,\omega_2}(x,t)$ *is increasing with respect to* $(\omega_1,\omega_2) \in \mathbb{R}^2$.

(vi) *The entire solution* $\Phi_{\omega_1,\omega_2}(x,t)$ *depends continuously on* $(\omega_1,\omega_2) \in \mathbb{R}^2$ *in the sense of the topology* $\mathcal{T}$.

(vii) *The entire solution* $\Phi_{\omega_1,\omega_2}(x,t)$ *is Lyapunov stable in the following sense. For any given* $\varphi > 0$, *there exists a* $\delta > 0$ *such that, for any* $\varphi(x,\cdot) \in C([-\tau,0],[0,1])$ *and* $\sup_{x\in\mathbb{R}} |\varphi(x,\cdot) - \Phi_{\omega_1,\omega_2}(x + x_0, t_0 + \cdot)| \le \delta$, *there is*

$$|u(x,t;\varphi) - \Phi_{\omega_1,\omega_2}(x + x_0, t + t_0)| < \epsilon$$

for any $x \in \mathbb{R}$ *and* $t \ge 0$, *where* $x_0 \in \mathbb{R}$ *and* $t_0 \in \mathbb{R}$ *are two real constants.*

Theorem 2.4. *Assume that* (H) *and* (B) *hold. Let* (U, c_0) *be a solution of* (2.1) *with speed* $c_0 < 0$. *Then, for any constants* $\omega_1 \in \mathbb{R}$ *and* $\omega_2 \in \mathbb{R}$, *there exists an entire solution* $\Phi_{\omega_1,\omega_2}(x,t)$ *of* (1.1) *defined for all* $t \in \mathbb{R}$ *such that*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\ge 0} |\Phi_{\omega_1,\omega_2}(x,t) - U(-x + c_0 t + \omega_1)| + \sup_{x\le 0} |\Phi_{\omega_1,\omega_2}(x,t) - U(x + c_0 t + \omega_2)| \Big\} = 0.$$

Moreover, (v)–(vii) *in Theorem* 2.3 *still hold. Furthermore, the following hold:*

(i) $0 < \Phi_{\omega_1,\omega_2}(x,t) < 1$ *and* $(\partial/\partial t)\Phi_{\omega_1,\omega_2}(x,t) < 0$ *for* $(x,t) \in \mathbb{R}^2$.

(ii) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\Phi_{\omega_1,\omega_2}(x,t)=0$ *and* $\lim_{t\to-\infty}\inf_{|x|\le M_2}|\Phi_{\omega_1,\omega_2}(x,t)-1|=0$ *for any* $M_2\in\mathbb{R}^+$.

(iii) $\lim_{|x|\to+\infty}\sup_{t\ge t_2}|\Phi_{\omega_1,\omega_2}(x,t)|=0$ *for any* $t_2\in\mathbb{R}$.

(iv) $\Phi_{\omega_1,\omega_2}(x,t)$ *converges to*

$$\begin{cases} U(x+c_0t+\omega_2) & \text{as } \omega_1\to+\infty \text{ in the sense of topology } \mathcal{T};\\ U(-x+c_0t+\omega_1) & \text{as } \omega_2\to+\infty \text{ in the sense of topology } \mathcal{T}.\end{cases}$$

Remark 2.5. Theorem 2.4 is a consequence of Theorem 2.3. Indeed, when $c_0<0$, we set $\widetilde{c}_0=-c_0>0$, and $\widetilde{U}(x+\widetilde{c}_0t)=1-U(-x+c_0t)=1-U(-(x+\widetilde{c}_0t))$. Then, by (2.1), we can see that

$$\begin{cases}\widetilde{c}_0\widetilde{U}'(\xi)=(J*\widetilde{U})(\xi)-\widetilde{U}(\xi)-d\widetilde{U}(\xi)+\int_{\mathbb{R}}K(y)g(\widetilde{U}(\xi-y-\widetilde{c}_0\tau))\,dy,\\ \widetilde{U}(-\infty)=0,\quad \widetilde{U}(+\infty)=1,\end{cases}\tag{2.2}$$

where $g(u)=d-b(1-u)$. Hence, for any $\theta\in\mathbb{R}$, $\widetilde{U}(x+\widetilde{c}_0+\theta)$ is a traveling wave solution of

$$\frac{\partial u}{\partial t}=J*u-u-du+\int_{\mathbb{R}}K(y)g(u(x-y,t-\tau))\,dy.$$

Note that $g'(u)=b'(1-u)$ for $u\in[0,1]$. It is easy to see that g satisfies assumption (B) by replacing α with $1-\alpha$. Applying Theorem 2.3 to (2.2), we obtain an entire solution $\widetilde{\Phi}_{\omega_1,\omega_2}(x,t)$ of (2.2) which satisfies Theorem 2.3 (i)–(iv) and

$$\lim_{t\to-\infty}\Big\{\sup_{x\ge0}|\widetilde{\Phi}_{\omega_1,\omega_2}(x,t)-\widetilde{U}(x+\widetilde{c}_0t+\omega_1)| + \sup_{x\le0}|\widetilde{\Phi}_{\omega_1,\omega_2}(x,t)-\widetilde{U}(-x+\widetilde{c}_0t+\omega_2)|\Big\}=0.$$

3. Asymptotic behavior of traveling waves. In order to construct two-front entire solutions, it is crucial to have precise information on the asymptotic behavior of wave tails. In this section, we shall study the asymptotic behavior of traveling wave fronts of (1.1) at infinity with the help of Ikehara's theorem.

We first define two complex functions $\mathcal{P}_0(\lambda)$ and $\mathcal{P}_1(\lambda)$ by

$$\mathcal{P}_0(\lambda) = \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1 - c\lambda - d + b'(0)e^{-\lambda c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda y}dy,$$
$$\mathcal{P}_1(\lambda) = \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1 - c\lambda - d + b'(1)e^{-\lambda c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda y}dy,$$

where $\lambda \in \mathbb{C}$. Then, the next result holds.

Lemma 3.1. *Assume that* (H) *and* (B) *hold. The equation* $\mathcal{P}_i(\lambda) = 0$ *has two real roots* $\lambda_{i1} < 0$ *and* $\lambda_{i2} > 0$ *such that*

$$\mathcal{P}_i(\lambda)\begin{cases} > 0 & \textit{for } 0 < \lambda < \lambda_{i1}, \\ < 0 & \textit{for } \lambda_{i1} < \lambda < \lambda_{i2}, \\ > 0 & \textit{for } \lambda > \lambda_{i2}, \end{cases}$$

where $i = 0, 1$.

Proof. Since, for $\lambda \in \mathbb{R}$,

$$\mathcal{P}_0(0) = -d + b'(0) < 0, \qquad \mathcal{P}_1(0) = -d + b'(1) < 0,$$

and

$$\frac{\partial^2}{\partial\lambda^2}\mathcal{P}_0(\lambda) = \int_{\mathbb{R}} y^2 J(y)e^{-\lambda y}dy + b'(0)e^{-\lambda c\tau}\int_{\mathbb{R}} (y+c\tau)^2 K(y)e^{-\lambda y}dy > 0,$$
$$\frac{\partial^2}{\partial\lambda^2}\mathcal{P}_1(\lambda) = \int_{\mathbb{R}} y^2 J(y)e^{-\lambda y}dy + b'(1)e^{-\lambda c\tau}\int_{\mathbb{R}} (y+c\tau)^2 K(y)e^{-\lambda y}dy > 0,$$

it is easy to see that the conclusion holds. □

In order to give precise exponential decay, we need the following form of Ikehara's theorem **[2]**. The proof of Ikehara's theorem may be found, e.g., in **[26]**.

Lemma 3.2. *For a positive non-decreasing function* $u(\xi)$, *we define*

$$\mathcal{F}(\lambda) := \int_{-\infty}^{0} e^{-\lambda\xi}u(\xi)\,d\xi.$$

If $\mathcal{F}$ has the representation $\mathcal{F}(\lambda) = \mathcal{H}(\lambda)/(\vartheta - \lambda)^{k+1}$, where $k > -1$, $\vartheta > 0$, and $\mathcal{H}(\lambda)$ is analytic in the strip $0 < \mathrm{Re}\lambda \leq \vartheta$, then

$$\lim_{\xi \to -\infty} \frac{u(\xi)}{\xi^k e^{\vartheta \xi}} = \frac{\mathcal{H}(\vartheta)}{\Gamma(\vartheta + 1)},$$

where Γ is the Gamma function.

Lemma 3.3. *Assume that $U(\xi)$ is an increasing solution of (2.1) with speed $c \neq 0$. Then, there exists a constant $\gamma_0 > 0$ such that*

$$\sup_{\xi \in \mathbb{R}} U(\xi) e^{-\gamma_0 \xi} < \infty.$$

Proof. Let $\epsilon_1 = d - b'(0) > 0$, $\epsilon_2 = d + b'(0) > 0$ and $M = (1/2)\max_{0 \leq u \leq 1} |b''(u)|$. In view of $\epsilon_1 > 0$ and $\lim_{\xi \to -\infty} U(\xi) = 0$, there exists a $\xi_0 < 0$ large enough such that, for any $\xi < \xi_0$,

(3.1)
$$\frac{\epsilon_1}{4}\left[U(\xi) + \int_{\mathbb{R}} K(y) U(\xi - y - c\tau)\, dy\right] \geq M \int_{\mathbb{R}} K(y) U^2(\xi - y - c\tau)\, dy.$$

Thus, by Taylor's expansion, we have

$$\begin{aligned}
&\int_{\mathbb{R}} K(y) b(U(\xi - y - c\tau))\, dy \\
&\quad = \int_{\mathbb{R}} K(y)\left[b(0) + b'(0)U(\xi - y - c\tau) + \frac{1}{2} b''(\varpi) U^2(\xi - y - c\tau)\right] dy \\
&\quad \leq b'(0) \int_{\mathbb{R}} K(y) U(\xi - y - c\tau)\, dy + M \int_{\mathbb{R}} K(y) U^2(\xi - y - c\tau)\, dy,
\end{aligned}$$

where ϖ is a some function between 0 and $U(\xi - y - c\tau)$.

We first show that, for $\xi < \xi_0$, $U(\xi)$ and $\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy$ are integrable on $(-\infty, \xi]$. It can be seen from (2.1) and (3.1) that, for any $\xi < \xi_0$,

(3.2)
$$\begin{aligned}
&- cU'(\xi) + (J * U)(\xi) - U(\xi) \\
&\quad = dU(\xi) - \int_{\mathbb{R}} K(y) b(U(\xi - y - c\tau))\, dy \\
&\quad \geq dU(\xi) - b'(0) \int_{\mathbb{R}} K(y) U(\xi - y - c\tau)\, dy
\end{aligned}$$

$$
\begin{aligned}
&- M \int_{\mathbb{R}} K(y)U^2(\xi - y - c\tau)\, dy \\
\geq\ & \frac{\epsilon_1}{4}U(\xi) - \frac{\epsilon_2}{2}\left[\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy - U(\xi)\right] \\
&+ \frac{\epsilon_1}{4}\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy \\
&+ \frac{\epsilon_1}{4}\left[\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy + U(\xi)\right] \\
&- M \int_{\mathbb{R}} K(y)U^2(\xi - y - c\tau)\, dy \\
\geq\ & \frac{\epsilon_1}{4}U(\xi) - \frac{\epsilon_2}{2}\left[\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy - U(\xi)\right] \\
&+ \frac{\epsilon_1}{4}\int_{\mathbb{R}} K(y)U(\xi - y - c\tau)\, dy.
\end{aligned}
$$

By Lebesgue's dominated convergence theorem and Fubini's theorem, we obtain, as $\eta \to -\infty$,

(3.3)

$$
\begin{aligned}
\int_\eta^\xi [(J * U)(s) - U(s)]\, ds &= \int_\eta^\xi \int_{\mathbb{R}} J(y)(U(s-y) - U(s))\, dy\, ds \\
&= -\int_\eta^\xi \int_{\mathbb{R}} J(y) \int_0^1 yU'(s - \theta y)\, d\theta\, dy\, ds \\
&= -\int_{\mathbb{R}} J(y)y \int_0^1 [U(\xi - \theta y) - U(\eta - \theta y)]\, d\theta\, dy \\
&\longrightarrow -\int_{\mathbb{R}} J(y)y \int_0^1 U(\xi - \theta y)\, d\theta\, dy,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4) \qquad & \int_\eta^\xi \left[\int_{\mathbb{R}} K(y)U(s - y - c\tau)\, dy - U(s)\right] ds \\
&= \int_\eta^\xi \int_{\mathbb{R}} K(y)(U(s - y - c\tau) - U(s))\, dy\, ds \\
&= -\int_\eta^\xi \int_{\mathbb{R}} (y + c\tau)K(y) \int_0^1 U'(s - \theta(y + c\tau))\, d\theta\, dy\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}} (y+c\tau)K(y) \int_0^1 \int_\eta^\xi U'(s-\theta(y+c\tau))\, ds\, d\theta\, dy \\
&= -\int_{\mathbb{R}} (y+c\tau)K(y) \\
&\quad \cdot \int_0^1 [U(\xi-\theta(y+c\tau)) - U(\eta-\theta(y+c\tau))]\, d\theta\, dy \\
&\longrightarrow -\int_{\mathbb{R}} (y+c\tau)K(y) \int_0^1 U(\xi-\theta(y+c\tau))\, d\theta\, dy.
\end{aligned}
$$

Integrating (3.2) from $-\infty$ to ξ, by (3.3) and (3.4) we have that, for any $\xi \leq \xi_0$,

$$
\begin{aligned}
&-cU(\xi) - \int_{\mathbb{R}} J(y)y \int_0^1 U(\xi-\theta y)\, d\theta\, dy \\
&\quad - \frac{\epsilon_2}{2} \int_{\mathbb{R}} (y+c\tau)K(y) \int_0^1 U(\xi-\theta(y+c\tau))\, d\theta\, dy \\
&\geq \frac{\epsilon_1}{4} \int_{-\infty}^{\xi} U(z)\, dz \\
&\quad + \frac{\epsilon_1}{4} \int_{-\infty}^{\xi} \int_{\mathbb{R}} K(y)U(z-y-c\tau)\, dy\, dz.
\end{aligned}
\tag{3.5}
$$

It is clear that $|J(y)yU(\xi+\theta y)| \leq |J(y)y| \in L^1(\mathbb{R}\times[0,1])$ and $|K(y)yU(\xi+\theta y| \leq |K(y)y| \in L^1(\mathbb{R}\times[0,1])$ are due to the assumptions on J and K. Hence, (3.5) implies that $U(\xi)$ and $\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\, dy$ are integrable on $(-\infty, \xi]$ for $\xi < \xi_0$.

Now, we are ready to prove the conclusion of Lemma 3.3. Since $U(\xi)$ is increasing, for any $y \in \mathbb{R}$, we have

$$
yJ(y)U(\xi-y) \leq yJ(y) \int_0^1 U(\xi-\theta y)\, d\theta,
$$

$$
(y+c\tau)K(y)U(\xi-(y+c\tau)) \leq (y+c\tau)K(y) \int_0^1 U(\xi-\theta(y+c\tau))\, d\theta.
$$

Since J and K are compactly supported, for simplicity, we assume that there exists an $X_0 \in \mathbb{R}^+$ such that $J(x) = 0$ and $K(x) = 0$ for any

$x \in \mathbb{R}$ with $|x| > X_0$. Thus, by (3.5), we obtain

$$\frac{\epsilon_1}{4}\int_{-\infty}^{\xi} U(z)\,dz \le |c|U(\xi) - U(\xi+X_0)\int_{-\infty}^{0} yJ(y)\,dy - \frac{\epsilon_2}{2}U(\xi+X_0+|c|\tau)\int_{-\infty}^{-c\tau}(y+c\tau)K(y)\,dy.$$

Furthermore, we have

$$\frac{\epsilon_1}{4}\int_{-\infty}^{\xi} U(z)\,dz \le \left[|c| - \int_{-\infty}^{0} yJ(y)\,dy - \frac{\epsilon_2}{2}\int_{-\infty}^{-c\tau}(y+c\tau)K(y)\,dy\right]U(\xi+X_0+|c|\tau)). \tag{3.6}$$

It is easy to see that, for any $r > 0$ and $\xi \le \xi_0$,

$$\int_{-\infty}^{\xi} U(z)\,dz = \int_{-\infty}^{0} U(s+\xi)\,ds \ge \int_{-r}^{0} U(s+\xi)\,ds \ge rU(\xi - r). \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$\frac{\epsilon_1}{4}rU(\xi-r) \le \left[|c| - \int_{-\infty}^{0} yJ(y)\,dy - \frac{\epsilon_2}{2}\int_{-\infty}^{-c\tau}(y+c\tau)K(y)\,dy\right]U(\xi+X_0+|c|\tau).$$

Thus, there exists an $r_0 > 0$ sufficiently large and some $\theta \in (0,1)$ such that

$$U(\xi - r_0) \le \theta U(\xi + X_0 + |c|\tau)$$

for any $\xi \le \xi_0$, which may be written as

$$U(\xi - X_0 - |c|\tau - r_0) \le \theta U(\xi)$$

for any $\xi \le \xi_0 + X_0 + |c|\tau$. Let

$$\mathscr{A}(\xi) = U(\xi)e^{-\gamma_0\xi}, \quad \text{where } \gamma_0 = \frac{1}{X_0+|c|\tau+r_0}\ln\frac{1}{\theta} > 0.$$

Then, we obtain

$$\mathscr{A}(\xi - X_0 - |c|\tau - r_0) = U(\xi - X_0 - |c|\tau - r_0)e^{-\gamma_0(\xi - X_0 - |c|\tau - r_0)}$$

$$= \frac{1}{\theta} U(\xi - X_0 - |c|\tau - r_0) e^{-\gamma_0 \xi} \leq U(\xi) e^{-\gamma_0 \xi}$$
$$= \mathscr{A}(\xi).$$

This, together with $\lim_{\xi \to +\infty} U(\xi) e^{-\gamma_0 \xi} = 0$, implies that

$$\sup_{\xi \in \mathbb{R}} U(\xi) e^{-\gamma_0 \xi} < +\infty.$$

The proof is complete. □

Theorem 3.4. *Assume that $U(\xi)$ is an increasing solution of* (2.1) *with speed $c \neq 0$. Then,*

$$\lim_{\xi \to -\infty} e^{-\lambda_{02} \xi} U(\xi) = a_0, \qquad \lim_{\xi \to -\infty} e^{-\lambda_{02} \xi} U'(\xi) = a_0 \lambda_{02},$$

where a_0 is a positive constant.

Proof. For λ with $0 < \mathrm{Re}\lambda < \gamma_0$, we can define the two-sided Laplace transform

$$\ell(\lambda) = \int_{\mathbb{R}} e^{-\lambda \xi} U(\xi) \, d\xi.$$

It is easy to see that (2.1) can be rewritten as

(3.8)
$$(J * U)(\xi) - U(\xi) - cU'(\xi) - dU(\xi) + b'(0) \int_{\mathbb{R}} K(y) U(\xi - y - c\tau) \, dy$$
$$= b'(0) \int_{\mathbb{R}} K(y) U(\xi - y - c\tau) \, dy - \int_{\mathbb{R}} K(y) b(U(\xi - y - c\tau)) \, dy.$$

Note that

$$\int_{-\infty}^{+\infty} e^{-\lambda \xi} (J * U(\xi)) \, d\xi = \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} \int_{-\infty}^{+\infty} U(\xi - y) e^{-\lambda(\xi - y)} \, d\xi \, dy$$
$$= \ell(\lambda) \int_{\mathbb{R}} J(y) e^{-\lambda y} dy,$$

and

$$\begin{aligned}&\int_{-\infty}^{+\infty} e^{-\lambda\xi}\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy\,d\xi\\&\quad=\int_{-\infty}^{+\infty} K(y)e^{-\lambda(y+c\tau)}\int_{-\infty}^{+\infty} U(\xi-y-c\tau)e^{-\lambda(\xi-y-c\tau)}\,d\xi\,dy\\&\quad=\ell(\lambda)\int_{\mathbb{R}} K(y)e^{-\lambda(y+c\tau)}dy.\end{aligned}$$

Taking the Laplace transform to (3.8) yields

$$\begin{aligned}\mathcal{P}_0(\lambda)\ell(\lambda)=\int_{-\infty}^{+\infty} e^{-\lambda\xi}\bigg[&b'(0)\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy\\&-\int_{\mathbb{R}} K(y)b(U(\xi-y-c\tau))\,dy\bigg]\,d\xi.\end{aligned}\tag{3.9}$$

By $\lim_{\xi\to-\infty} U(\xi)=0$ and Taylor's expansion, we have

$$\begin{aligned}&\left|b'(0)\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy-\int_{\mathbb{R}} K(y)b(U(\xi-y-c\tau))\,dy\right|\\&\quad\le\max_{0\le u\le 1}|b''(u)|\int_{\mathbb{R}} K(y)U^2(\xi-y-c\tau)\,dy\\&\quad\le U(\xi+X_0+|c|\tau)\max_{0\le u\le 1}|b''(u)|\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy\end{aligned}$$

Hence, the right-hand side of equality (3.9) is defined for λ such that $0<\mathrm{Re}\lambda<2\gamma_0$. Now, we use a property of Laplace transforms **[26,** page 58]. Since $U(\xi)>0$, there exists a constant ϑ such that $\ell(\lambda)$ is analytic for $0<\mathrm{Re}\lambda<\vartheta$ and $\ell(\lambda)$ has a singularity at $\lambda=\vartheta$. Thus, $\ell(\lambda)$ is defined for $0<\mathrm{Re}\lambda<\lambda_{02}$.

We rewrite (3.9) as follows:

$$\begin{aligned}\int_{-\infty}^{0} U(\xi)e^{-\lambda\xi}d\xi=&-\int_{0}^{+\infty} U(\xi)e^{-\lambda\xi}\,d\xi\\&+\frac{1}{\mathcal{P}_0(\lambda)}\int_{\mathbb{R}} e^{-\lambda\xi}\bigg[b'(0)\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy\\&\qquad-\int_{\mathbb{R}} K(y)b(U(\xi-y-c\tau))\,dy\bigg]\,d\xi.\end{aligned}$$

Clearly, $\int_0^{+\infty} U(\xi)e^{-\lambda\xi}d\xi$ is analytic for $\mathrm{Re}\lambda > 0$, and the equation $\mathcal{P}_0(\lambda) = 0$ does not have any zeros with $\mathrm{Re}\lambda = \lambda_{02}$ other than $\lambda = \lambda_{02}$. In fact, let $\lambda = \lambda_{02} + i\beta$. Then, $\mathcal{P}_0(\lambda) = 0$ implies that

$$\begin{aligned}\int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}\cos(\beta y)\,dy - 1 - c\lambda_{02} - d + b'(0)&\\ \cdot \int_{\mathbb{R}} K(y)e^{-\lambda_{02}(y+c\tau)}[\cos\beta y\cos\beta c\tau - \sin\beta y\sin\beta c\tau]\,dy = 0&\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}\sin(\beta y)\,dy + c\beta + b'(0)\\ &\quad\cdot \int_{\mathbb{R}} K(y)e^{-\lambda_{02}(y+c\tau)}[\sin\beta y\cos\beta c\tau + \cos\beta y\sin\beta c\tau]\,dy = 0.\end{aligned}$$

Since $\mathcal{P}_0(\lambda_{02}) = 0$, (3.10) can be rewritten as

$$\begin{aligned}&-2\int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}\sin^2\left(\frac{\beta y}{2}\right)dy\\ &\quad= b'(0)\int_{\mathbb{R}} K(y)e^{-\lambda_{02}(y+c\tau)}\\ &\quad\cdot\left[2\sin^2\left(\frac{\beta c\tau}{2}\right) + 2\sin^2\left(\frac{\beta y}{2}\right)\right.\\ &\qquad\left. - 4\sin^2\left(\frac{\beta c\tau}{2}\right)\sin^2\left(\frac{\beta y}{2}\right) + \sin\beta y\sin\beta c\tau\right]dy.\end{aligned}$$

By computation, we obtain

$$\begin{aligned}&2\sin^2\left(\frac{\beta c\tau}{2}\right)+2\sin^2\left(\frac{\beta y}{2}\right)-4\sin^2\left(\frac{\beta c\tau}{2}\right)\sin^2\left(\frac{\beta y}{2}\right)+\sin\beta y\sin\beta c\tau\\ &= 2\sin^2\left(\frac{\beta c\tau}{2}\right)\cos^2\left(\frac{\beta y}{2}\right)+2\cos^2\left(\frac{\beta c\tau}{2}\right)\sin^2\left(\frac{\beta y}{2}\right)+\sin\beta y\sin\beta c\tau\\ &\geq 4\left|\sin\left(\frac{\beta c\tau}{2}\right)\cos\left(\frac{\beta c\tau}{2}\right)\sin\left(\frac{\beta y}{2}\right)\cos\left(\frac{\beta y}{2}\right)\right| + \sin\beta y\sin\beta c\tau\\ &= |\sin\beta y\sin\beta c\tau| + \sin\beta y\sin\beta c\tau \geq 0,\end{aligned}$$

which implies that $\beta = 0$.

Since $U(\xi)$ is increasing,

$$\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}U(\xi) = \frac{H(\lambda_{02})}{\Gamma(\lambda_{02}+1)},$$

by Lemma 3.2. We then let $\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}U(\xi) = a_0$.

We now prove that

$$\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}U'(\xi) = a_0\lambda_{02}.$$

By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}J*U(\xi) &= \lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}\int_{-\infty}^{+\infty} J(y)U(\xi-y)\,dy\\ &= \int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}y}\Big[\lim_{\xi\to-\infty} e^{-\lambda_{02}(\xi-y)}U(\xi-y)\Big]dy\\ &= a_0\int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}y}dy\end{aligned}$$

and

$$\begin{aligned}&\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}\int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy\\ &\quad= \int_{-\infty}^{+\infty} K(y)e^{-\lambda_{02}(y+c\tau)}\Big[\lim_{\xi\to-\infty} e^{-\lambda_{02}(\xi-y-c\tau)}U(\xi-y-c\tau)\Big]\,dy\\ &\quad= a_0\int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}(y+c\tau)}dy.\end{aligned}$$

Since, as $\xi\to-\infty$,

$$\begin{aligned}\int_{\mathbb{R}} K(y)b(U(\xi-y-c\tau))\,dy = b'(0)&\int_{\mathbb{R}} K(y)U(\xi-y-c\tau))\,dy\\ &+O(1)\int_{\mathbb{R}} K(y)U^2(\xi-y-c\tau))\,dy,\end{aligned}$$

then

$$\begin{aligned}&\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}\int_{\mathbb{R}} K(y)b(U(\xi-y-c\tau))\,dy\\ &\qquad\qquad= a_0b'(0)\int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}(y+c\tau)}dy.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{\xi\to-\infty} e^{-\lambda_{02}\xi}U'(\xi)\\
&\quad=\frac{a_0}{c}\left[\int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}y}dy-1-d+b'(0)\int_{-\infty}^{+\infty} J(y)e^{-\lambda_{02}(y+c\tau)}dy\right]\\
&\quad=a_0\lambda_{02}.
\end{aligned}$$

The proof is complete. □

Corollary 3.5. *Assume that $U(\xi)$ is described as in Theorem* 3.4. *Then, there exists a $\varrho_1=\varrho_1(U)$ such that*

$$\lim_{\xi\to-\infty}\frac{U(\xi+\varrho_1)}{e^{\lambda_{02}\xi}}=1, \tag{3.11}$$

$$\lim_{\xi\to-\infty}\frac{U'(\xi)}{U(\xi)}=\lambda_{02}. \tag{3.12}$$

Proof. Equation (3.11) follows from Theorem 3.4. By (2.1), we have

$$\begin{aligned}
\lim_{\xi\to-\infty}\frac{cU'(\xi)}{U(\xi)} &= \lim_{\xi\to-\infty}\int_{\mathbb{R}} J(y)\frac{U(\xi-y)}{U(\xi)}\,dy-1-d\\
&\quad+\lim_{\xi\to-\infty}\int_{\mathbb{R}} K(y)\frac{b(U(\xi-y-c\tau))}{U(\xi)}\,dy\\
&= \lim_{\xi\to-\infty}\int_{\mathbb{R}} J(y)\frac{U(\xi-y)}{U(\xi)}\,dy-1-d\\
&\quad+b'(0)\lim_{\xi\to-\infty}\int_{\mathbb{R}} K(y)\frac{U(\xi-y-c\tau)}{U(\xi)}\,dy\\
&=\int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}dy-1-d\\
&\quad+b'(0)\int_{\mathbb{R}} K(y)e^{-\lambda_{02}(y+c\tau)}\,dy\\
&=c\lambda_{02},
\end{aligned}$$

which implies that (3.12) holds. The proof is complete. □

Similarly, we can obtain the asymptotic behavior of traveling wavefronts at positive infinity.

Theorem 3.6. *Assume that $U(\xi)$ is an increasing solution of* (2.1) *with speed $c \neq 0$. Then,*

$$\lim_{\xi\to+\infty} e^{-\lambda_{11}\xi}(1-U(\xi)) = a_1, \qquad \lim_{\xi\to+\infty} e^{-\lambda_{11}\xi}U'(\xi) = -a_1\lambda_{11},$$

where a_1 is a positive constant.

Corollary 3.7. *Assume that $U(\xi)$ is as described in Theorem* 3.6. *Then there exists a $\varrho_2 = \varrho_2(U)$ such that*

$$\lim_{\xi\to+\infty} \frac{1-U(\xi+\varrho_2)}{e^{\lambda_{11}\xi}} = 1, \qquad \lim_{\xi\to+\infty} \frac{U'(\xi)}{1-U(\xi)} = -\lambda_{11}.$$

According to Theorems 3.4 and 3.6, there exist positive constants K_0, k, η, γ and δ such that

$$ke^{\lambda_{02}\xi} \leq U(\xi) \leq K_0 e^{\lambda_{02}\xi}, \qquad \text{for all } \xi \leq 0, \tag{3.13}$$

$$ke^{\lambda_{02}\xi} \leq (K*U)(\xi) \leq K_0 e^{\lambda_{02}\xi}, \qquad \text{for all } \xi \leq 0, \tag{3.14}$$

$$\eta k e^{\lambda_{02}\xi} \leq \eta U(\xi) \leq U'(\xi), \qquad \text{for all } \xi \leq 0, \tag{3.15}$$

$$\eta k e^{\lambda_{02}\xi} \leq \eta (K*U)(\xi) \leq U'(\xi), \qquad \text{for all } \xi \leq 0, \tag{3.16}$$

$$\gamma e^{\lambda_{11}\xi} \leq 1-U(\xi) \leq \delta e^{\lambda_{11}\xi}, \qquad \text{for all } \xi \geq 0, \tag{3.17}$$

$$\gamma e^{\lambda_{11}\xi} \leq 1-(K*U)(\xi) \leq \delta e^{\lambda_{11}\xi}, \qquad \text{for all } \xi \geq 0, \tag{3.18}$$

$$\eta\gamma e^{\lambda_{11}\xi} \leq \eta(1-U(\xi)) \leq U'(\xi), \qquad \text{for all } \xi \geq 0 \tag{3.19}$$

$$\eta\gamma e^{\lambda_{11}\xi} \leq \eta(1-(K*U)(\xi)) \leq U'(\xi), \qquad \text{for all } \xi \geq 0, \tag{3.20}$$

where $(K*U)(\xi) = \int_{\mathbb{R}} K(y)U(\xi-y-c\tau)\,dy$.

4. Entire solutions.

4.1. Initial value problem. In this subsection, we consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = J*u-u-du+\int_{\mathbb{R}} K(y)b(u(x-y,t-\tau))\,dy, & x\in\mathbb{R},\ t>0, \\ u(x,s)=\varphi(x,s), & x\in\mathbb{R},\ s\in[-\tau,0]. \end{cases} \tag{4.1}$$

We shall give some existence and comparison theorems for solutions, super- and sub-solutions of (4.1) which will be used in the sequel.

We make the following extension for the function b. Define $\widehat{b}: \mathbb{R} \to \mathbb{R}$ by

$$\widehat{b}(u) = \begin{cases} b(u), & u \in (-\infty, 1], \\ b(1) + b'(1)(u-1), & u \in (1, +\infty). \end{cases} \tag{4.2}$$

In the remainder of paper, we replace the function $b(\cdot)$ by $\widehat{b}(\cdot)$ and continue to denote $\widehat{b}(\cdot)$ by $b(\cdot)$. We note that this replacement does not affect the main results of this paper since the definition of $b(\cdot)$ on $[0,1]$ does not change. Obviously, $b \in C^2[0,2]$, and

$$|b'(u_1) - b'(u_2)| = \max_{w \in [0,1]} |b''(w)||u_1 - u_2|, \quad u_1, u_2 \in [0,2]. \tag{4.3}$$

Let X be the Banach space defined by

$$X = \{\varphi(x) | \varphi(x) : \mathbb{R} \to \mathbb{R} \text{ is uniformly continuous and bounded}\}$$

with the usual supremum norm $|\cdot|_X$. Let

$$X^+ = \{\varphi(x) \in X : \varphi(x) \geq 0,\ x \in \mathbb{R}\}.$$

It is easily seen that X^+ is a closed cone of X, and its induced partial ordering makes X into a Banach lattice. For simplicity, we denote $X_{[0,k]} = \{\varphi \in X : 0 \leq \varphi(x) \leq k,\ x \in \mathbb{R}\}$.

$$\int_{\mathbb{R}} J(x-y)[u(y) - u(x)]\, dy : X \longrightarrow X$$

is a bounded linear operator with respect to the norm $|\cdot|_X$. Then, for $t > 0$,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]\, dy, \\ u(x,0) = \varphi(x) \in X, \end{cases} \tag{4.4}$$

generates a strongly continuous semigroup $T(t)$ on X and $T(t)X^+ \subset X^+$, that is, $T(t)u(x) \gg 0$ if $u(x) \geq 0$ has nonempty support and $t > 0$. Moreover, the mild solution of (4.4) can be given by $u(x,t) = T(t)\varphi(x)$. For more details, we refer to Pan et al. **[20]**. The theory of the operator semigroup may be seen in Pazy **[21]**. Ignat and Rossi **[13]** introduced a linear semigroup and gave the fundamental solution of (4.4), that is,

$S(x,t) = e^{-t}\delta_0(x) + K(x,t)$, where $\delta_0(x)$ is the delta measure and

$$K(x,t) = \int_{\mathbb{R}} (e^{t(\widehat{J}(\xi)-1)} - e^{-t}) e^{ix\cdot\xi} d\xi,$$

$\widehat{J}$ represents the Fourier transform to J. In addition, the solution of (4.4) can be written as $T(t)\varphi(x) = (S * \varphi)(x,t)$.

Let $\mathcal{C} = C([-\tau,0],X)$ be the Banach space of all continuous functions from $[-\tau,0]$ into X. For any $\varphi \in \mathcal{C}_{[0,2]} = \{\varphi \in \mathcal{C} : \varphi(x,s) \in [0,2],\ x \in \mathbb{R},\ s \in [-\tau,0]\}$, define

$$F(\varphi)(x) = -d\varphi(x,0) + \int_{\mathbb{R}} K(x-y)b(\varphi(y,-\tau))\, dy, \quad x \in \mathbb{R}.$$

Since $b \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, we can verify that $F(\varphi) \in X$ and $F : \mathcal{C}_{[0,2]} \to X$ are globally Lipschitz continuous.

Lemma 4.1. *Assume that* $\varphi(\cdot,s) \in X_{[0,2]}$, $s \in [-\tau,0]$. *Then,* (4.1) *has a unique mild solution* $u(x,t)$ *defined for all* $(x,t) \in \mathbb{R} \times (0,+\infty)$ *which can also be formulated by the following integral equation*:

$$u(x,t) = T(t)\varphi(x,0) + \int_0^t T(t-s)F(u_s)(x)\, ds.$$

Moreover, $u(\cdot,t) \in X_{[0,2]}$ *for all* $t > 0$, *and* $u(\cdot,t)$ *is a classical solution of* (4.1) *for* $(x,t) \in \mathbb{R} \times [0,+\infty)$.

Definition 4.2. A continuous function $v : \mathbb{R} \times [-\tau,l) \to [0,u^*]$ with $l > 0$ is called a super-solution (sub-solution) of (1.1) if

$$v(x,t) \geq (\leq) T(t-s)v(x,s) + \int_s^t T(t-r)F(v_r)(x)\, dr \tag{4.5}$$

for all $0 \leq s < t < l$. If v is both a super- and a sub-solution on $[0,l)$, then it is said to be a mild solution of (1.1).

Remark 4.3. Assume that $v : \mathbb{R} \times [-\tau,l) \to [0,u^*]$ with $l > 0$, v is C in $x \in \mathbb{R}$, C^1 in $t \in [0,l)$, and satisfies the inequality:

$$\frac{\partial v}{\partial t} \geq (\leq) J * v - v - dv + \int_{\mathbb{R}} K(y)b(v(x-y,t-\tau))\, dy, \quad x \in \mathbb{R},\ t > 0.$$

Then, by the positivity of $T(t) : X^+ \to X^+$, it follows that (4.5) holds, and hence, v is a super-solution (sub-solution) of (1.1) on $[0, l)$.

We now establish the comparison theorem.

Lemma 4.4. *For any pair of super- and sub-solutions $u^+(x,t)$ and $u^-(x,t)$, respectively, of (1.1) on $[0,+\infty)$ with $0 \le u^-(x,t), u^+(x,t) \le 2$ for $t \in [-\tau,+\infty)$, $x \in \mathbb{R}$, and $u^+(x,s) \ge u^-(x,s)$ for $s \in [-\tau, 0]$, $x \in \mathbb{R}$, we have $u^+(x,t) \ge u^-(x,t)$ for all $x \in \mathbb{R}$, $t \ge 0$.*

4.2. Existence of entire solutions. We now study the following ordinary differential equation:

$$\frac{d}{dt}p(t) = c + Ne^{\lambda_{02}p(t)}, \quad t \le 0, \tag{4.6}$$

where λ_{02} is defined in Lemma 3.1; N satisfies

$$N \ge \max_{w\in[0,1]} |b''(w)| \max\left\{ \frac{2K_0}{\eta}, \frac{2K_0}{\eta\gamma}, \frac{K_0^2 e^{2\lambda_{02}(X_0 - c\tau)}}{\eta k} \right\},$$

in which X_0 is given in the proof of Lemma 3.3. Let

$$\rho = p(0), \qquad \omega = \rho - \frac{1}{\lambda_{02}} \ln\left\{1 + \frac{N}{c}e^{\lambda_{02}\rho}\right\}. \tag{4.7}$$

Set

$$\chi = -\frac{1}{\lambda_{02}} \ln\left(1 + \frac{N}{c}\right) < 0.$$

It is easy to see that the function $\omega = \omega(\rho)$ is strictly increasing on $(-\infty, 0]$ with $\omega(0) = \chi$. Hence, $\omega = \omega(\rho)$ is invertible. Thus, for any $\omega \in (-\infty, \chi]$, there exists a unique $\rho = \rho(\omega) \in (-\infty, 0]$ such that $\rho = \rho(\omega)$ is increasing and (4.7) holds.

Now, let $p(0) = \rho(\omega) \le 0$. Then, solving equation (4.6) explicitly, we obtain

$$p(t;\omega) = \rho(\omega) + ct - \frac{1}{\lambda_{02}} \ln\left\{1 + \frac{N}{c}e^{\lambda_{02}\rho(\omega)}(1 - e^{c\lambda_{02}t})\right\}.$$

For any $(\omega, \widetilde{\omega}) \in (-\infty, \chi]^2$, let

$$\widetilde{\rho}(\omega, \widetilde{\omega}) = \widetilde{\omega} + \frac{1}{\lambda_{02}} \ln \left\{ 1 + \frac{N}{c} e^{\lambda_{02}\rho(\omega)} \right\}$$

and

$$\widetilde{p}(t; \omega, \widetilde{\omega}) = \widetilde{\rho}(\omega, \widetilde{\omega}) + ct - \frac{1}{\lambda_{02}} \ln \left\{ 1 + \frac{N}{c} e^{\lambda_{02}\rho(\omega)} (1 - e^{c\lambda_{02}t}) \right\}.$$

It is easy to see that $p(t; \omega)$ and $\widetilde{p}(t; \omega, \widetilde{\omega})$ are increasing on $t \in (-\infty, 0]$. In addition, when $t \in (-\infty, 0]$, $p(t; \omega)$ and $\widetilde{p}(t; \omega, \widetilde{\omega})$ are increasing on $\omega \in (-\infty, \chi]$ and $\widetilde{\omega} \in (-\infty, \chi]$. Also, we can see that

$$p(t; \omega) - ct - \omega = \widetilde{p}(t; \omega, \widetilde{\omega}) - ct - \widetilde{\omega} = -\frac{1}{\lambda_{02}} \ln \left\{ 1 - \frac{\kappa e^{c\lambda_{02}t}}{1 + \kappa} \right\},$$

where $\kappa = (N/c)e^{\lambda_{02}\rho(\omega)}$. Then, we obtain

$$0 < p(t; \omega) - ct - \omega = \widetilde{p}(t; \omega, \widetilde{\omega}) - ct - \widetilde{\omega} \le \mathcal{R}_0 e^{c\lambda_{02}t}, \quad t \le 0, \tag{4.8}$$

for some positive constant $\mathcal{R}_0$, independent of $\omega \in (-\infty, \chi]$ and $\widetilde{\omega} \in (-\infty, \chi]$. Clearly, if $\widetilde{\omega} \le \omega$, then $\widetilde{\rho}(\omega, \widetilde{\omega}) \le \rho(\omega)$, and hence, $\widetilde{p}(t; \omega, \widetilde{\omega}) \le p(t; \omega)$.

Now, we give $\omega_1, \omega_2 \in (-\infty, \chi)$. If $\omega_1 \le \omega_2$, then we let $p_2(t; \omega_1, \omega_2) = p(t; \omega)$ and $p_1(t; \omega_1, \omega_2) = \widetilde{p}(t; \omega, \widetilde{\omega})$ with $\omega = \omega_2$ and $\widetilde{\omega} = \omega_1$. If $\omega_2 \le \omega_1$, then let $p_1(t; \omega_1, \omega_2) = p(t; \omega)$ and $p_2(t; \omega_1, \omega_2) = \widetilde{p}(t; \omega, \widetilde{\omega})$ with $\omega = \omega_1$ and $\widetilde{\omega} = \omega_2$. For the sake of convenience, we denote $p_i(t; \omega_1, \omega_2)$ by $p_i(t)$ in the following, where $i = 1, 2$.

Lemma 4.5. *Suppose that $\underline{u}(x,t)$ and $\overline{u}(x,t)$ are a sub-solution and a super-solution of (1.1) on $(x,t) \in \mathbb{R} \times (-\infty, -T]$ for some $T \in \mathbb{R}$, respectively, and satisfy $\underline{u}(x,t) \le \overline{u}(x,t)$ on $(x,t) \in \mathbb{R} \times (-\infty, -T]$. Then, there exists an entire solution $u(x,t)$ of (1.1) such that*

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \quad \text{for } (x,t) \in \mathbb{R} \times (-\infty, -T].$$

Remark 4.6. From Lemma 4.5, we can see that the construction of entire solutions is reduced to finding a suitable pair of sub–super solutions.

Lemma 4.7. *There exists a $T<0$, independent of ω_1 and ω_2, such that $\overline{u}(x,t)$, defined by*

$$\overline{u}(x,t) := U(x+p_1(t)) + U(-x+p_2(t)),$$

is a super-solution of (1.1) *on* $(-\infty, T)$.

Proof. Without loss of generality, assume that $\omega_1 \le \omega_2 < \chi$, and hence, $p_1(t) \le p_2(t)$ for all $t \le 0$. For convenience, set

$$\mathcal{L}[u](x,t) = \frac{\partial u}{\partial t} - J*u + u + du - \int_{\mathbb{R}} K(y) b(u(x-y,t-\tau))\,dy.$$

With computation, we obtain

$$\begin{aligned}\mathcal{L}[\overline{u}(x,t)] = {} & p_1'(t)U'(x+p_1(t)) + p_2'(t)U'(-x+p_2(t)) \\ & - \int_{\mathbb{R}} J(y)[U(x-y+p_1(t)) + U(-x+y+p_2(t))]\,dy \\ & + (1+d)[U(x+p_1(t)) + U(-x+p_2(t))] \\ & - \int_{\mathbb{R}} K(y) b(U(x-y+p_1(t-\tau)) \\ & \qquad + U(-x+y+p_2(t-\tau)))\,dy.\end{aligned}$$

By (2.1), we further obtain

$$\begin{aligned}\mathcal{L}[\overline{u}(x,t)] &= (p_1'(t)-c)U'(x+p_1(t)) + (p_2'(t)-c)U'(-x+p_2(t)) \\ -\mathcal{G}(x,t) &= [U'(x+p_1(t)) + U'(-x+p_2(t))] \\ & \quad \cdot \Big\{ N e^{\lambda_{02} p_2(t)} - \mathcal{R}(x,t) \Big\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}(x,t) := {} & \int_{\mathbb{R}} K(y) b(U(x-y+p_1(t-\tau)) + U(-x+y+p_2(t-\tau)))\,dy \\ & - \int_{\mathbb{R}} K(y) b(U(x-y+p_1(t)-c\tau))\,dy \\ & - \int_{\mathbb{R}} K(y) b(U(-x+y+p_2(t)-c\tau))\,dy\end{aligned}$$

and

$$\mathcal{R}(x,t) := \frac{\mathcal{G}(x,t)}{U'(x+p_1(t)) + U'(-x+p_2(t))}.$$

For $\tau > 0$ and $i = 1, 2$, we have

$$\begin{aligned}
p_i(t-\tau) &= p(0) + c(t-\tau) - \frac{1}{\lambda_{02}} \ln\left\{1 + \frac{N}{c} e^{\lambda_{02} p(0)} \left(1 - e^{c\lambda_{02}(t-\tau)}\right)\right\} \\
&= p_i(t) - c\tau + \frac{1}{\lambda_{02}} \\
&\cdot \ln\left\{\frac{1+(N/c)e^{\lambda_{02}p(0)}(1-e^{c\lambda_{02}t})}{1+(N/c)e^{\lambda_{02}p(0)}(1-e^{c\lambda_{02}t})+(N/c)e^{\lambda_{02}p(0)}e^{c\lambda_{02}t}(1-e^{c\lambda_{02}\tau})}\right\} \\
&\le p_i(t) - c\tau.
\end{aligned}$$

This gives

$$\begin{aligned}
\mathcal{G}(x,t) \le \int_{\mathbb{R}} & K(y) b(U(x-y+p_1(t)-c\tau) \\
& \qquad + U(-x+y+p_2(t)-c\tau))\, dy \\
& - \int_{\mathbb{R}} K(y) b(U(x-y+p_1(t)-c\tau))\, dy \\
& - \int_{\mathbb{R}} K(y) b(U(-x+y+p_2(t)-c\tau))\, dy \\
\le & \int_{\mathbb{R}} K(y)\, dy \int_0^1 b'(U(x-y+p_1(t)-c\tau) \\
& \qquad + \theta U(-x+y+p_2(t)-c\tau)) \\
& \times U(-x+y+p_2(t)-c\tau)\, d\theta \\
& - \int_{\mathbb{R}} K(y)\, dy \int_0^1 b'(\theta U(-x+y+p_2(t)-c\tau)) \\
& \times U(-x+y+p_2(t)-c\tau)\, d\theta \\
\le & \max_{w\in[0,1]} |b''(w)| \int_{\mathbb{R}} K(y) U(x-y+p_1(t)-c\tau) \\
& \times U(-x+y+p_2(t)-c\tau)\, dy \\
\le & \max_{w\in[0,1]} |b''(w)| U(x+p_1(t)) \\
& \times \int_0^{+\infty} K(y) U(-x+y+p_2(t)-c\tau)\, dy \\
& + \max_{w\in[0,1]} |b''(w)| U(-x+p_2(t))
\end{aligned}$$

$$\times \int_{-\infty}^{0} K(y)U(x-y+p_1(t)-c\tau)\,dy$$
$$\leq \max_{w\in[0,1]} |b''(w)| \Big[U(x+p_1(t))(K*U)(-x+p_2(t)) + U(-x+p_2(t))(K*U)(x+p_1(t)) \Big],$$

where
$$(K*U)(x) = \int_{\mathbb{R}} K(y)U(x-y-c\tau)\,dy.$$

Note that $p_i(t) < 0$ for all $t \leq 0$. For convenience, let $B_0 = \max_{w\in[0,1]} |b''(w)|$.

Now we estimate $\mathcal{R}(x,t)$. The discussion is divided into two cases: $\lambda_{02} \geq -\lambda_{11}$ and $\lambda_{02} < -\lambda_{11}$.

Case I. $\lambda_{02} \geq -\lambda_{11}$. $\mathbb{R}$ is divided into three parts.

(i) $p_2(t) \leq x \leq -p_1(t)$. When $0 \leq x \leq -p_1(t)$, from
$$U'(x+p_1(t)) + U'(-x+p_2(t)) \geq U'(x+p_1(t))$$
and (3.13)–(3.16), we obtain

(4.9)
$$\begin{aligned}
&\mathcal{R}(x,t) \\
&\quad \leq B_0 \frac{U(x+p_1(t))(K*U)(-x+p_2(t)) + U(-x+p_2(t))(K*U)(x+p_1(t))}{U'(x+p_1(t)) + U'(-x+p_2(t))} \\
&\quad \leq B_0 \left\{ \frac{U(x+p_1(t))(K*U)(-x+p_2(t))}{U'(x+p_1(t))} + \frac{U(-x+p_2(t))(K*U)(x+p_1(t))}{U'(x+p_1(t))} \right\} \\
&\quad \leq B_0 \left\{ \frac{U(x+p_1(t))}{\eta U(x+p_1(t))}(K*U)(-x+p_2(t)) + \frac{(K*U)(x+p_1(t))}{U'(x+p_1(t))} U(-x+p_2(t)) \right\} \\
&\quad \leq \frac{B_0}{\eta}[(K*U)(-x+p_2(t)) + U(-x+p_2(t))]
\end{aligned}$$

$$\leq \frac{2B_0K_0}{\eta} e^{\lambda_{02}(-x+p_2(t))} \leq \frac{2B_0K_0}{\eta} e^{\lambda_{02}p_2(t)}.$$

When $p_2(t) \leq x \leq 0$, from

$$U'(x+p_1(t)) + U'(-x+p_2(t)) \geq U'(-x+p_2(t))$$

and (3.13)–(3.16), we obtain

(4.10)

$$\begin{aligned}
&\mathcal{R}(x,t) \\
&\quad \leq B_0 \frac{U(x+p_1(t))(K*U)(-x+p_2(t)) + U(-x+p_2(t))(K*U)(x+p_1(t))}{U'(x+p_1(t)) + U'(-x+p_2(t))} \\
&\quad \leq B_0 \left\{ \frac{U(x+p_1(t))(K*U)(-x+p_2(t))}{U'(-x+p_2(t))} \right. \\
&\qquad\qquad \left. + \frac{U(-x+p_2(t))(K*U)(x+p_1(t))}{U'(-x+p_2(t))} \right\} \\
&\quad \leq B_0 \left\{ \frac{(K*U)(-x+p_2(t))}{\eta U(-x+p_2(t))} U(x+p_1(t)) \right. \\
&\qquad\qquad \left. + \frac{U(-x+p_2(t))}{U'(-x+p_2(t))} (K*U)(x+p_1(t)) \right\} \\
&\quad \leq \frac{B_0}{\eta} [U(x+p_1(t)) + (K*U)(x+p_1(t))] \\
&\quad \leq \frac{2B_0K_0}{\eta} e^{\lambda_{02}(x+p_1(t))} \leq \frac{2B_0K_0}{\eta} e^{\lambda_{02}p_2(t)}.
\end{aligned}$$

(ii) $x \leq p_2(t)$. Following (3.20), (3.14) and $-x+p_2(t) \geq 0$, we have

(4.11)

$$\begin{aligned}
\mathcal{R}(x,t) &\leq \frac{B_0(U(x+p_1(t)) + K*U(x+p_1(t)))}{U'(-x+p_2(t))} \leq \frac{2B_0K_0 e^{\lambda_{02}(x+p_1(t))}}{\eta\gamma e^{\lambda_{11}(-x+p_2(t))}} \\
&\leq \frac{2B_0K_0}{\eta\gamma} \frac{e^{\lambda_{02}p_2(t)}}{e^{(-\lambda_{02}-\lambda_{11})x} e^{\lambda_{11}p_2(t)}} \leq \frac{2B_0K_0}{\eta\gamma} e^{\lambda_{02}p_2(t)}.
\end{aligned}$$

(iii) $x \geq -p_1(t)$. By a similar argument as in (4.11), we obtain

$$\mathcal{R}(x,t) \leq \frac{2B_0K_0}{\eta\gamma} e^{\lambda_{02}p_2(t)}. \tag{4.12}$$

Combining (4.9), (4.10), (4.11) and (4.12), we obtain

$$\mathcal{L}[\overline{u}(x,t)] = [U'(x+p(t)) + U'(-x+p(t))]\left\{Ne^{\lambda_{02}p(t)} - \mathcal{R}(x,t)\right\} \geq 0.$$

Case II. $0 < \lambda_{02} < -\lambda_{11}$. In this case, we have $b'(0) > b'(1)$. Indeed, let

$$\mathcal{J}(s) = \int_{\mathbb{R}} J(y)e^{sy}dy = \int_0^{+\infty} J(y)(e^{sy} + e^{-sy})\,dy.$$

Then, we have

$$\mathcal{J}'(s) = \int_0^{+\infty} J(y)y(e^{sy} - e^{-sy})\,dy > 0$$

for all $s > 0$. Hence, $\mathcal{J}(-\lambda_{11}) > \mathcal{J}(\lambda_{02})$, that is,

$$\int_{\mathbb{R}} J(y)e^{-\lambda_{11}y}dy > \int_{\mathbb{R}} J(y)e^{\lambda_{02}y}dy = \int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}dy.$$

Similarly, we can prove

$$e^{-\lambda_{11}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{11}y}dy > e^{-\lambda_{02}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{02}y}dy.$$

Note that λ_{02} and λ_{11} satisfy

$$\int_{\mathbb{R}} J(y)e^{-\lambda_{02}y}dy - 1 - c\lambda_{02} - d + b'(0)e^{-\lambda_{02}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{02}y}dy = 0,$$
$$\int_{\mathbb{R}} J(y)e^{-\lambda_{11}y}dy - 1 - c\lambda_{11} - d + b'(1)e^{-\lambda_{11}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{11}y}dy = 0.$$

Thus, we obtain

$$b'(0)e^{-\lambda_{02}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{02}y}dy > b'(1)e^{-\lambda_{11}c\tau}\int_{\mathbb{R}} K(y)e^{-\lambda_{11}y}dy,$$

which implies $b'(0) > b'(1) \geq 0$.

Since $b'(u)$ is continuous on $[0,2]$, there exists a $\delta_1 \in (0,1)$ such that

$$0 < b'(u) < b'(0) \quad \text{for } u \in [1-\delta_1, 1+\delta_1],$$

We translate $U(\xi)$ along the ξ-axis so that

$$1 - \delta_1 < U(\xi) \leq 1 \quad \text{for } \xi \geq -X_0 - c\tau,$$

where X_0 may be seen as in the proof of Lemma 3.3. Take $T_1 < 0$, which is independent of $p_2(t)$, so that $U(2p_2(t) + X_0 - c\tau) \le \delta_1$ and $p_2(t) + X_0 - c\tau < 0$ for $t \le T_1$. Thus, for $t \le T_1$, $x \ge -p_1(t)$ and $|y| \le X_0$, we have

$$0 \le b'(U(x - y + p_1(t) - c\tau) + \theta U(-x + y + p_2(t) - c\tau)) \le b'(0),$$

where $\theta \in [0, 1]$. Hence, for any $t \le T_1$ and $x \ge -p(t)$, we get

(4.13)

$$\begin{aligned}
&\mathcal{G}(x,t)\\
&\quad \le \int_{\mathbb{R}} K(y) \int_0^1 b'(U(x - y + p_1(t) - c\tau) + \theta U(-x + y + p_2(t) - c\tau))\\
&\qquad \times U(-x + y + p_2(t) - c\tau)\, d\theta\, dy\\
&\qquad - \int_{\mathbb{R}} K(y) \int_0^1 b'(\theta U(-x + y + p_2(t) - c\tau))\\
&\qquad \times U(-x + y + p_2(t) - c\tau)\, d\theta\, dy\\
&\quad \le \int_{\mathbb{R}} K(y) b'(0) U(-x + y + p_2(t) - c\tau)\, d\theta\, dy\\
&\qquad - \int_{\mathbb{R}} K(y) U(-x + y + p_2(t) - c\tau)\\
&\qquad \times \int_0^1 b'(\theta U(-x + y + p_2(t) - c\tau))\, d\theta\, dy\\
&\quad \le B_0 \int_{\mathbb{R}} K(y) U^2(-x + y + p_2(t) - c\tau)\, dy\\
&\quad \le B_0 U(-x + X_0 + p_2(t) - c\tau) \int_{\mathbb{R}} K(y) U(-x + y + p_2(t) - c\tau)\, dy.
\end{aligned}$$

Similarly, for any $t \le T_1$ and $x \le p_2(t)$, we obtain

$$\begin{aligned}
\mathcal{G}(x,t) &\le B_0 U(x + X_0 + p_1(t) - c\tau)\\
&\quad \times \int_{\mathbb{R}} K(y) U(x - y + p_1(t) - c\tau)\, dy.
\end{aligned} \tag{4.14}$$

As in the proof of Case I, we divide $\mathbb{R}$ into three intervals $[p_2(t), -p_1(t)]$, $(-\infty, p_2(t)]$ and $[-p_1(t), +\infty)$. In the interval $[p_2(t), -p_1(t)]$, since we do not need the fact that $\lambda_{02} < -\lambda_{11}$, we can obtain the same estimate as (4.9) for $U(x,t)$. For $x \geq -p_1(t) > 0$, by (4.13), we get

$$\begin{aligned}\mathcal{R}(x,t) &\leq \frac{B_0 U(-x+X_0+p_2(t)-c\tau)}{U'(-x+p_2(t))}\\ &\quad\times \int_{\mathbb{R}} K(y)U(-x+y+p_2(t)-c\tau)\,dy\\ &\leq \frac{B_0K_0^2 e^{2\lambda_{02}(-x+X_0+p_2(t)-c\tau)}}{\eta k e^{\lambda_{02}(-x+p_2(t))}}\\ &\leq \frac{B_0K_0^2 e^{2\lambda_{02}(X_0-c\tau)}}{\eta k}\,e^{\lambda_{02}p_2(t)}.\end{aligned}$$

For $x < p_2(t) < 0$, by (4.14), we have

$$\mathcal{R}(x,t) \leq \frac{B_0K_0^2 e^{2\lambda_{02}(X_0-c\tau)}}{\eta k}\,e^{\lambda_{02}p_2(t)}.$$

Thus, for any $t < T_1$,

$$\mathcal{L}[\overline{u}(x,t)] = [U'(x+p_1(t)) + U'(-x+p_2(t))]\{Ne^{\lambda_{02}p_2(t)} - \mathcal{R}(x,t)\} \geq 0,$$

Now, let $T = 0$ when $\lambda_{02} \geq -\lambda_{11}$ and $T = T_1$ when $\lambda_{02} < -\lambda_{11}$. For any $t < T$, we always have $\mathcal{L}[\overline{u}(x,t)] \geq 0$. By Remark 4.3, we show that, for every $T' < T$, $\overline{v}(x,t) = \overline{u}(x,t+T')$, where $(x,t) \in \mathbb{R}\times[-\tau, T-T')$ is a super-solution of (1.1) on $\mathbb{R}\times[0, T-T')$. This completes the proof. □

Lemma 4.8. *$\underline{u}(x,t)$, defined by*

$$\underline{u}(x,t) := \max\{U(x+ct+\omega_1), U(-x+ct+\omega_2)\},$$

is a sub-solution of (1.1) *on* $(-\infty, 0)$.

Proof. When $x \geq (\omega_2-\omega_1)/2$, that is, $x+\omega_1 \geq -x+\omega_2$, then $\underline{u}(x,t) := U(x+ct+\omega_1)$; otherwise, $\underline{u}(x,t) := U(-x+ct+\omega_2)$. It is easy to see that, for $x \geq (\omega_2-\omega_1)/2$,

$$\begin{aligned}\mathcal{L}[\underline{u}(x,t)] &= cU'(x+ct+\omega_1) - \int_{\mathbb{R}} J(y)\underline{u}(x-y,t)\,dy\\ &\quad + (1+d)U(x+ct+\omega_1) - \int_{\mathbb{R}} K(y)b(\underline{u}(x-y,t-t\tau))\,dy\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} J(y)U(x-y+ct+\omega_1)\,dy - \int_{\mathbb{R}} J(y)\underline{u}(x-y,t)\,dy \\
&\quad + \int_{\mathbb{R}} K(y)b(U(x-y+c(t-\tau)+\omega_1))\,dy \\
&\quad - \int_{\mathbb{R}} K(y)b(\underline{u}(x-y,t-\tau))\,dy \\
&\le 0.
\end{aligned}
$$

Similarly, it can be proved that, for $x \le (\omega_2-\omega_1)/2$, $\mathcal{L}[\underline{u}(x,t)] \le 0$.

Obviously, for every $T^*<0$, $w(x,t)$, defined by $w(x,t) = \underline{u}(x,t+T^*)$, is a subsolution of (1.1) on $\mathbb{R} \times [-\tau, -T^*)$. The proof is complete. $\square$

Now, we consider the following initial value problem:

(4.15)
$$
\begin{cases}
\frac{\partial}{\partial t} u_n(x,t) = J * u_n - u_n - du_n \\
\qquad + \int_{\mathbb{R}} K(y)b(u_n(x-y,t-\tau))\,dy, & x\in\mathbb{R},\ t>-n, \\
u_n(x,-n+s) = u_{n,0}(x,s) := \underline{u}(x,-n+s), & x\in\mathbb{R},\ s\in[-\tau,0].
\end{cases}
$$

From **[35,** Lemma 4.1], we can see that $|U'| \le (1+b(1))/c$. Therefore, the initial functions $u_n(x,-n+s)$ are globally Lipschitz in x, and there exists a constant L_0 independent of n and s such that

$$
|u_n(x_1,-n+s) - u_n(x_2,-n+s)| \le L_0|x_1-x_2|
$$

for any $n\in\mathbb{N}$, $s\in[-\tau,0]$ and $x_1,x_2\in\mathbb{R}$.

Lemma 4.9. *Assume that* (H) *and* (B) *hold. Then, there exists a constant $C>0$, which is independent of x, t, n such that, for any $n\in\mathbb{N}$, $t\ge -n+1$ and $x\in\mathbb{R}$, the solutions $u_n(x,t)$ of* (4.15) *satisfy*

$$
|(u_n)_t(x,t)| \le C, \qquad |(u_n)_{tt}(x,t)| \le C.
$$

In addition, there exist positive constants M_1 and M_2, which are independent of n and t, such that

$$
|u_n(x+h,t) - u_n(x,t)| \le M_1 h \tag{4.16}
$$

and

$$\left|\frac{\partial u_n}{\partial t}(x+h,t) - \frac{\partial u_n}{\partial t}(x,t)\right| \le M_2 h \tag{4.17}$$

for any $x \in \mathbb{R}$, $t > -n$ *and* $h > 0$.

Proof. By the comparison principle and $0 < u_n(x, -n+s) < 1$ for $\xi \in \mathbb{R}$ and $s \in [-\tau, 0]$, we have $0 \le u_n(x,t) \le 1$. By (4.15), we have, for $x \in \mathbb{R}$, $t \ge -n$,

$$\begin{aligned} |(u_n)_t| &\le |J * u_n| + (1+d)|u_n| + \max_{u\in[0,1]} b(u) \\ &= 2 + d + \max_{u\in[0,1]} b(u) \\ &=: C_1. \end{aligned}$$

Using the estimate for $(u_n)_t$ and applying a similar argument, we obtain

$$\begin{aligned} |(u_n)_{tt}| &= \Big| J * (u_n)_t - (u_n)_t - d(u_n)_t \\ &\qquad + \int_{\mathbb{R}} K(y) b'(u_n(x-y, t-\tau))(u_n)_t(x-y, t-\tau)\, dy \Big| \\ &\le \Big(2 + d + \max_{u\in[0,1]} b'(u)\Big) C_1 =: C_2. \end{aligned}$$

Take $C = \max\{C_1, C_2\}$. Then, the first statement of Lemma 4.9 follows.

Now, we prove (4.16) and (4.17). For any $h > 0$, let

$$\delta u_n(x,t) = u_n(x+h,t) - u_n(x,t).$$

Then, by (4.15), $\delta u_n(x,t)$ satisfies

$$\begin{cases} (\delta u_n)_t = \int_{\mathbb{R}} (J(x-y+h) - J(x-y)) u_n(y,t)\, dy - (1+d)\, \delta u_n \\ \qquad\qquad + \int_{\mathbb{R}} (K(x-y+h) - K(x-y)) b(u_n(y, t-\tau))\, dy, \\ (\delta u_n)(x, -n+s) = \delta u_{n,0}(x,s), \quad x \in \mathbb{R},\ s \in [-\tau, 0]. \end{cases}$$

Note that

$$\begin{aligned} \delta u_n(x,t) &= \delta u_{n,0}(x,0) e^{-(1+d)t} \\ &\qquad + \int_0^t e^{-(1+d)(t-s)} \mathcal{N}(u_n(x,s))\, ds, \end{aligned} \tag{4.18}$$

where

$$\mathcal{N}(u(x,t)) = \int_{\mathbb{R}} (J(x-y+h) - J(x-y)) u_n(y,t)\, dy \\ + \int_{\mathbb{R}} (K(x-y+h) - K(x-y)) b(u_n(y,t-\tau))\, dy.$$

Since $J' \in L^1$ by (H), there exists an $L_1 > 0$ such that

$$\int_{\mathbb{R}} |J(x+h) - J(x)|\, dx = h \int_{\mathbb{R}} \left| \int_0^1 J'(x+\theta h)\, d\theta \right| dx \\ \leq h \int_{\mathbb{R}} \int_0^1 |J'(x+\theta h)|\, d\theta\, dx \leq L_1 h$$

for any $h > 0$. Similarly, for $L_2 > 0$, the following holds:

$$\int_{\mathbb{R}} |K(x+h) - K(x)|\, dx \leq L_2 h.$$

Then, from (4.18), we get

$$|\delta u_n(x,t)| \leq |\delta u_{n,0}(x,0)| + \int_0^t e^{-(1+d)(t-s)} |\mathcal{N}(u_n(x,s))|\, ds \\ \leq L_0 h + \int_0^t e^{-(1+d)(t-s)} (L_1 h + \widetilde{M} L_2 h)\, ds \\ \leq L_0 h + (L_1 h + \widetilde{M} L_2 h) \int_0^t e^{-(1+d)(t-s)} ds \\ \leq L_0 h + \frac{1}{1+d} (L_1 h + \widetilde{M} L_2 h) =: M_1 h,$$

where $\widetilde{M} := \max_{u \in [0,2]} b(u)$. Hence, (4.16) holds.

Finally, we show that (4.17) holds. For any $h > 0$, $x \in \mathbb{R}$ and $t > 0$,

$$
\begin{aligned}
&|(\delta u)_t(x,t)| \\
&\quad \le \left| \int_{\mathbb{R}} (J(x+h-y) - J(x-y)) u_n(y,t)\, dy - (1+d)\, \delta u(x,t) \right. \\
&\qquad \left. + \int_{\mathbb{R}} (K(x+h-y) - K(x-y)) b(u_n(y,t-\tau))\, dy \right| \\
&\quad \le \int_{\mathbb{R}} |J(x+h-y) - J(x-y)| |u_n(y,t)|\, dy + (1+d) |\, \delta u(x,t)| \\
&\qquad + \int_{\mathbb{R}} |K(x+h-y) - K(x-y)| |b(u_n(y,t-\tau))|\, dy \\
&\quad \le L_1 h + (1+d) M_1 h + \widetilde{M} L_2 h =: M_2 h.
\end{aligned}
$$

The proof is complete. □

Combining the above lemmas, we obtain the following result.

Theorem 4.10. *Assume that* (H) *and* (B) *hold. Then, there exists an entire solution* $\Phi(x,t) := \Phi(x,t;\omega_1,\omega_2)$ *of* (1.1) *such that*

$$
\underline{u}(x,t) \le \Phi(x,t) \le \overline{u}(x,t), \quad (x,t) \in \mathbb{R} \times (-\infty, 0], \tag{4.19}
$$

where $\overline{u}(x,t)$ *and* $\underline{u}(x,t)$ *are given as in Lemmas* 4.7 *and* 4.8*. Moreover, positive constants* C_1 *and* C_2 *exist such that*

$$
|\Phi(x+h,t) - \Phi(x,t)| \le C_1 h
$$

and

$$
\left| \frac{\partial \Phi}{\partial t}(x+h,t) - \frac{\partial \Phi}{\partial t}(x,t) \right| \le C_2 h
$$

for any $(x,t) \in \mathbb{R}^2$ *and* $h > 0$.

Proof. Recall that $u_n(x,t)$ is the unique solution of the initial value problem (4.15). By the a priori estimate (Lemma 4.9) and the Arzela-Ascoli theorem, there exists a subsequence $\{u_{n_k}(x,t)\}_{k\in\mathbb{N}}$ of $u_n(x,t)$ such that $u_{n_k}(x,t)$ converges to a function $\Phi(x,t)$ in the sense of the topology $\mathcal{T}$, that is, for any compact set $S \subset \mathbb{R}^2$, $u_{n_k}(x,t)$ and $(\partial/\partial t) u_{n_k}(x,t)$ converge uniformly to functions $\Phi(x,t)$ and $(\partial/\partial t)\Phi(x,t)$ as $n \to +\infty$, respectively. Since $u_{n_k}(x,t)$ satisfies equation (4.15), the limit function $\Phi(x,t)$ is an entire solution of (1.1) such that

$$\underline{u}(x,t) \le \Phi(x,t) \le \overline{u}(x,t) \quad \text{for } (x,t) \in \mathbb{R} \times (-\infty, 0].$$

Set $C_1 = 2 + M_1$, where M_1 is defined as in Lemma 4.9. Fix $(x,t) \in \mathbb{R}^2$ and $h > 0$. Let $S \subset \mathbb{R}^2$ with $(x,t) \in S$ and $(x+h,t) \in S$ be a compact subset. Then, there exists an $N_0 \in \mathbb{N}$ such that, for any $k > N_0$,

$$|\Phi(x,t) - u_{n_k}(x,t)| \le h \quad \text{for any } (x,t) \in S.$$

Thus, we have

$$\begin{aligned} |\Phi(x+h,t) - \Phi(x,t)| &\le |\Phi(x+h,t) - u_{n_k}(x+h,t)| \\ &\quad + |u_{n_k}(x+h,t) - u_{n_k}(x,t)| \\ &\quad + |u_{n_k}(x,t) - \Phi(x,t)| \\ &\le C_1 h. \end{aligned}$$

Similarly, we can prove

$$\left| \frac{\partial \Phi}{\partial t}(x+h,t) - \frac{\partial \Phi}{\partial t}(x,t) \right| \le C_2 h$$

for any $(x,t) \in \mathbb{R}^2$ and $h > 0$. The proof is complete. □

4.3. Qualitative properties of entire solutions. In this subsection, we continue to investigate the qualitative properties of the entire solutions of (1.1).

Theorem 4.11. *Let $\Phi(x,t)$ be the entire solution of* (1.1) *as stated in Theorem* 4.10. *Then the following properties hold*:

(i)

$$\lim_{t \to -\infty} \Big\{ \sup_{x \ge 0} |\Phi(x,t) - U(x + c_0 t + \omega_1)| + \sup_{x \le 0} |\Phi(x,t) - U(-x + c_0 t + \omega_2)| \Big\} = 0. \tag{4.20}$$

(ii) $0 < \Phi(x,t) < 1$ *and* $(\partial/\partial t)\Phi(x,t) > 0$ *for* $(x,t) \in \mathbb{R}^2$.

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|\Phi(x,t)-1|=0$ *and* $\lim_{t\to-\infty}\sup_{|x|\le M_1}\Phi(x,t)=0$ *for any* $M_1\in\mathbb{R}^+$.

(iv) $\lim_{|x|\to+\infty}\sup_{t\ge t_1}|\Phi(x,t)-1|=0$ *for any* $t_1\in\mathbb{R}$.

(v) *For any* $(x,t)\in\mathbb{R}^2$, $\Phi(x,t;\omega_1,\omega_2)$ *is increasing with respect to* $(\omega_1,\omega_2)\in\mathbb{R}^2$.

(vi) $\Phi(x,t;\omega_1,\omega_2)$ *converges to*

$$\begin{cases} U(-x+c_0t+\omega_2) & \text{as } \omega_1\to-\infty \text{ in the sense of topology } \mathcal{T};\\ U(x+c_0t+\omega_1) & \text{as } \omega_2\to-\infty \text{ in the sense of topology } \mathcal{T}.\end{cases}$$

Proof. The proofs of parts (iii)–(iv) are trivial and omitted.

(i) Without loss of generality, assume that $\omega_1\le\omega_2$. When $x\ge 0$, by estimates (4.8) and (4.19), we obtain

$$\begin{aligned} 0&\le \Phi(x,t)-U(x+c_0t+\omega_1)\le \overline{u}(x,t)-U(x+c_0t+\omega_1)\\ &=U(x+p_1(t))-U(x+c_0t+\omega_1)+U(-x+p_2(t))\\ &\le \sup_{x\in\mathbb{R}}|U'(x)|(p_1(t)-c_0t-\omega_1)+K_0e^{\lambda_{02}(-x+p_2(t))}\\ &\le N_0e^{\lambda_{02}c_0t}+K_0e^{\lambda_{02}p_2(t)},\end{aligned}$$

where N_0 is some positive constant. This implies that

$$\lim_{t\to-\infty}\sup_{x\ge0}|\Phi(x,t)-U(x+c_0t+\omega_1)|=0.$$

Similarly, when $x\le 0$, we have

$$0\le\Phi(x,t)-U(-x+c_0t+\omega_2)\le N_0e^{\lambda_{02}c_0t}+K_0e^{\lambda_{02}p_2(t)}.$$

Thus, (4.20) follows.

(ii) We denote a solution of (1.1) with initial data $\varphi\in C_{[0,1]}$ by $u(x,t;\varphi)$. Define

$$u_n(x,t)=u(x,t;\varphi_n),\qquad \varphi_n(x,s)=\underline{u}(x,T-n+s),$$
$$(x,s)\in\mathbb{R}\times[-\tau,0],\quad n\in\mathbb{N}.$$

Since $0<u_n(x,t)<1$, by the comparison principle we obtain that $0<\Phi(x,t)<1$ for $(x,t)\in\mathbb{R}^2$.

Now, we show $(\partial/\partial t)\Phi(x,t)>0$ on $\mathbb{R}^2$. We first prove that $u_n(x,t)$ is increasing in $t\in[-n,+\infty)$ for any $x\in\mathbb{R}$. Since $\underline{u}(x,t)$ is a subsolution of (1.1), then

$$u_n(x,t) = u(x,t;\varphi_n) \geq \underline{u}(x,t+T-n) \quad \text{for all } (x,t) \in \mathbb{R}\times[-\tau,-T+n].$$

For any $\epsilon > 0$, $\underline{u}(x,\cdot+\epsilon) \geq \underline{u}(x,\cdot)$ on $\mathbb{R}$, it follows that $u_n(x,T-n+s+\epsilon) = u(x,s+\epsilon;\varphi_n) \geq \varphi_n(x,s)$ for all $(x,s) \in \mathbb{R}\times[-\tau,0]$. By comparison and the uniqueness of solutions, we have

$$u_n(x,t+\epsilon) = u(x,t;u(\cdot,\epsilon+\cdot;\varphi_n)) \geq u_n(x,t)$$
$$\text{for any } (x,t) \in \mathbb{R}\times(0,+\infty).$$

Due to the arbitrariness of ϵ, we see that $u_n(x,t)$ is increasing on t. Therefore, $\Phi_t(x,t) \geq 0$ for $(x,t) \in \mathbb{R}^2$. Since $\Phi_t(x,t)$ satisfies

$$\begin{aligned}\Phi_{tt}(x,t) &= J*\Phi_t - \Phi_t - d\Phi_t \\ &\quad + \int_{\mathbb{R}} J(y)b'(\Phi(x-y,t-\tau))\Phi_t(x-y,t-\tau)\,dy,\end{aligned} \tag{4.21}$$

we have

$$\begin{aligned}\Phi_t(x,t) &= \Phi_t(x,s)e^{-(1+d)(t-s)} \\ &\quad + \int_s^t e^{-(1+d)(t-r)}\mathcal{G}(\Phi)(x,r)\,dr \\ &\geq \Phi_t(x,s)e^{-(1+d)(t-s)}\end{aligned} \tag{4.22}$$

for any $s < t$, where

$$\mathcal{G}(\Phi)(x,t) = J*\Phi_t + \int_{\mathbb{R}} J(y)b'(\Phi(x-y,t-\tau))\Phi_t(x-y,t-\tau)\,dy \geq 0.$$

If $(x_0,t_0) \in \mathbb{R}^2$ exists such that $\Phi_t(x_0,t_0) = 0$, then, by (4.22), we have $\Phi_t(x_0,t) = 0$ for all $t \leq t_0$. In view of $\Phi_t(x,t) \geq 0$, for any $(x,t) \in \mathbb{R}^2$, we obtain $\Phi_{tt}(x_0,t) = 0$ for any $t \leq t_0$. Then, by (4.21), we obtain

$$(J*\Phi_t)(x_0,t) + \int_{\mathbb{R}} J(y)b'(\Phi(x_0-y,t-\tau))\Phi_t(x_0-y,t-\tau)\,dy = 0$$

for any $t < t_1$. Since $b'(u) \geq 0$ for $u \in [0,1]$ and $\Phi_t(x,t) \geq 0$ for any $(x,t) \in \mathbb{R}^2$, we have

$$(J*\Phi_t)(x_0,t) = 0 \quad \text{for all } t \leq t_0.$$

In view of $J \in C^1$ and $\int_{\mathbb{R}} J(x)\,dx = 1$, by induction, we further have $\Phi_t(x,t) = 0$ for any $x \in \mathbb{R}$ and $t < t_1$, which contradicts (4.19).

(v) From the proof of Theorem 4.10, we see that $\underline{u}_n(x,t) \le \Phi(x,t) \le \overline{u}_n(x,t)$ for any $(x,t) \in \mathbb{R} \times (-\infty, T)$. Then, by the comparison principle, we further obtain

$$\Phi(x,t) \ge \max\{U(x+ct+\omega_1), U(-x+ct+\omega_2)\}, \quad (x,t) \in \mathbb{R}^2.$$

Since

$$\max\{U(x+ct+\widetilde{\omega}_1), U(-x+ct+\widetilde{\omega}_2)\} \\ \ge \max\{U(x+ct+\omega_1), U(-x+ct+\omega_2)\},$$

for any $(\omega_1,\omega_2) \in (-\infty,\vartheta)^2$ and $(\widetilde{\omega}_1,\widetilde{\omega}_2) \in (-\infty,\vartheta)^2$ with $\widetilde{\omega}_1 \ge \omega_1$ and $\widetilde{\omega}_2 \ge \omega_2$, it is not difficult to show that property (v) holds.

(vi) In order to prove $\Phi(x,t;\omega_1,\omega_2) \to U(-x+c_0t+\omega_2)$ as $\omega_1 \to -\infty$ in the sense of the topology of $\mathcal{T}$, let $\{\omega_1^k\}$ satisfy $\omega_1^{k+1} \le \omega_1^k \le \omega_2$ for any $k \in \mathbb{N}$ and $\omega_1^k \to -\infty$ as $k \to \infty$. Then (1.1) admits entire solutions $\Phi^k(x,t;\omega_1^k,\omega_2)$ such that, for any $t \le T$,

$$\begin{aligned} U(-x+ct+\omega_2) &\le \max\{U(x+ct+\omega_1^k),\ U(-x+ct+\omega_2)\} \\ &\le \Phi^k(x,t;\omega_1^k,\omega_2) \\ &\le U(x+p(t;\omega_1^k,\omega_2)) + U(-x+p(t;\omega_1^k,\omega_2)) \end{aligned} \tag{4.23}$$

for $x \in \mathbb{R}$ and $k \in \mathbb{N}$. By Lemma 4.9 and by a diagonal extraction process, $\widetilde{\Phi}(x,t) := \widetilde{\Phi}(x,t;-\infty,\omega_2)$ exists such that $\Phi^k(x,t;\omega_1^k,\omega_2)$ converges to $\widetilde{\Phi}(x,t)$ as $k \to \infty$ (up to extraction of some subsequence) in the sense of the topology of $\mathcal{T}$. Thus, $\widetilde{\Phi}(x,t,-\infty,\omega_2)$ is an entire solution of (1.1). By the monotonicity of $\Phi^k(x,t;\omega_1^k,\omega_2)$ with respect to ω_1^k, $\Phi^k(x,t;\omega_1^k,\omega_2)$ converges to $\widetilde{\Phi}(x,t,-\infty,\omega_2)$ in the sense of the topology of $\mathcal{T}$. Obviously, $\widetilde{\Phi}(x,t,-\infty,\omega_2)$ is independent of $k \in \mathbb{N}$. Thus, by (4.23), we obtain

$$U(-x+ct+\omega_2) \le \widetilde{\Phi}(x,t;-\infty,\omega_2) \le U(-x+p(t;\omega_1^k,\omega_2))$$

for all $x \in \mathbb{R}$ and $t \le T$. It follows from [**32,** Lemma 4.2] that three positive numbers β_0 (independent of U), σ_0 and $\overline{\delta}$ exist such that, for any $\delta \in (0,\overline{\delta}]$ and every $\xi_0 \in \mathbb{R}$, the function w^+ defined by

$$w^+(x,t) := U(x+ct+\xi_0+\sigma_0\delta(e^{\beta_0\tau} - e^{-\beta_0 t})) + \delta e^{-\beta_0 t}$$

is a super-solution of (1.1) on $[0,+\infty)$. Fix any $t_1<0$. Set

$$\eta := \sup_{x\in\mathbb{R}} ||\widetilde{\Phi}(x,t_1+\cdot;-\infty,\omega_2) - U(-x+c(t_1+\cdot)+\omega_2)||_{L^\infty[-\tau,0]}.$$

According to $p(t)-ct-\omega_2\to 0$ as $t\to-\infty$, for any $\delta>0$, $t_2<t_1-\tau$ exists such that

$$\begin{aligned} U(-x+c(t_2+s)+\omega_2) &\le \widetilde{\Phi}(x,t+t_2;-\infty,\omega_2) \\ &\le U(x+c(t_2+s)+\omega_2 \\ &\qquad + \sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0 s})) + \delta e^{-\beta_0 s} \end{aligned}$$

for $x\in\mathbb{R}$ and $s\in[-\tau,0]$. Take $t_0=t_1-t_2>0$. Then, we have

$$\begin{aligned} U(-x+c(t_1+s)+\omega_2) &\le \widetilde{\Phi}(x,t+t_1;-\infty,\omega_2) \\ &\le U(x+c(t_1+s) \\ &\qquad +\omega_2+\sigma_0\delta(e^{\beta_0\tau}-e^{-\beta_0(t_0+s)})) + \delta e^{-\beta_0(t_0+s)} \end{aligned}$$

for $x\in\mathbb{R}$ and $s\in[-\tau,0]$. Thus,

$$\begin{aligned} &\sup_{x\in\mathbb{R}} ||\widetilde{\Phi}(x,t_1+\cdot;-\infty,\omega_2) - U(x+c(t_1+\cdot)+\omega_2)||_{L^\infty[-\tau,0]} \\ &\quad \le \delta\Big(1+\sigma_0 e^{-\beta_0\tau}\max_{x\in\mathbb{R}} U'(x)\Big). \end{aligned}$$

Since δ is arbitrary, we get $\eta=0$. Hence, we obtain that $\widetilde{\Phi}(x,t;-\infty,\omega_2) = U(-x+ct+\omega_2)$ for any $t<T$ and $x\in\mathbb{R}$. We can similarly prove the remainder results of (vi). The proof is complete. □

4.4. Uniqueness and stability of entire solutions.

Lemma 4.12. *There exist constants $\delta_0>0$, $\nu_0>0$ and $\sigma_0>0$ such that, for any $\eta\in\mathbb{R}$, $\delta\in(0,\delta_0]$ and $\sigma\ge\sigma_0$, the functions*

$$W^{\pm}(x,t) = \Phi(x,t+\eta\pm\sigma\delta[1-e^{-\nu_0 t}]) \pm \delta e^{-\nu_0 t}$$

are a pair of super- and sub-solutions of (1.1) *on* $[0,+\infty)$.

Proof. We only need show that $W^+(x,t)$ is a super-solution. The other case can be proved similarly. Since

$$\lim_{(\nu,\vartheta)\to(0,b'(0))} [-\nu+d-\vartheta e^{\nu\tau}] = d-b'(0) > 0$$

and

$$\lim_{(\nu,\vartheta)\to(0,b'(1))}[-\nu+d-\vartheta e^{\nu\tau}]=d-b'(1)>0,$$

we can fix $\nu_0>0$ and $\delta_0>0$ such that

$$-\nu_0+d-\vartheta e^{\nu_0\tau}>0 \quad \text{for any } \vartheta\in[b'(0)-\delta_0,b'(0)+\delta_0] \tag{4.24}$$

and

$$-\nu_0+d-\vartheta e^{\nu_0\tau}>0 \quad \text{for any } \vartheta\in[b'(1)-\delta_0,b'(1)+\delta_0]. \tag{4.25}$$

Let $\delta_1\in(0,\delta_0)$ satisfy

$$\delta_1 e^{\nu_0\tau}\left[1+\max_{u\in[0,1]}|b'(u)|+\max_{u\in[0,1]}|b''(u)|\right]\le\frac{\delta_0}{4}. \tag{4.26}$$

Since

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}K(y)b'(\Phi(x-y,t-\tau))\,dy-b'(1)\right|=0,$$

there exists a $T_1>0$ such that, for any $t\in(T_1,+\infty)$ and $x\in\mathbb{R}$,

$$\int_{\mathbb{R}}K(y)b'(\Phi(x-y,t-\tau))\,dy\in\left[b'(1)-\frac{\delta_0}{2},\ b'(1)+\frac{\delta_0}{2}\right]. \tag{4.27}$$

Since

$$\lim_{\xi\to-\infty}U(\xi)=0,\quad \lim_{\xi\to-\infty}\int_{\mathbb{R}}K(y)b(U(\xi-y-c\tau))\,dy=b'(0),$$
$$\lim_{\xi\to+\infty}\int_{\mathbb{R}}K(y)b(U(\xi-y-c\tau))=b'(1),$$

there exists an $X_1>0$ such that

$$\int_{\mathbb{R}}K(y)U(\xi-y)\,dy\max_{x\in[0,1]}|b''(u)|<\frac{\delta_0}{8}\quad\text{for } \xi\le -X_1, \tag{4.28}$$

(4.29)

$$\int_{\mathbb{R}}K(y)b'(U(\xi-y-c\tau))\,dy\in\left[b'(0)-\frac{\delta_0}{2},\ b'(0)+\frac{\delta_0}{2}\right]\quad\text{for } \xi\le -X_1,$$

and

$$\int_{\mathbb{R}} K(y)b'(U(\xi - y - c\tau))\,dy \in \left[b'(1) - \frac{\delta_0}{2},\ b'(1) + \frac{\delta_0}{2}\right] \quad \text{for } \xi \geq X_1, \tag{4.30}$$

Since $p(t) - ct - \omega \to 0$ as $t \to -\infty$, there exists a $T_2 < T$, where T is defined as in Lemma 4.7 and such that, for $t \leq T_2$,

$$2(p(t) - ct - \omega) \max_{u\in[0,1]} |b''(u)| \cdot \max_{\xi\in\mathbb{R}} U'(\xi) \in \left(0, \frac{\delta_0}{8}\right). \tag{4.31}$$

Let $\kappa_1 = \min_{\xi\in[-X_1,X_1]} U'(\xi) > 0$. Then, there exists a large $\sigma_1 > 0$ such that

$$\frac{1}{2}c\sigma_1\nu_0\kappa_1 - \nu_0 + d - \max_{u\in[0,2]} b'(u)e^{\nu_0\tau} \geq 0. \tag{4.32}$$

Let $\Psi(x,t) := U(x + ct + \omega) + U(-x + ct + \omega)$. It is easy to prove that

$$\lim_{t\to-\infty} ||\Phi - \Psi||_{C(\mathbb{R}\times(-\infty,t])} = 0.$$

By (1.1), we further obtain that $\lim_{t\to-\infty} ||\Phi - \Psi||_{\mathbb{R}\times C^1(-\infty,t]} = 0$. Thus, there exists a $T_3 \leq T_2$ such that, for any $t \leq T_3$,

$$\sup_{x\in\mathbb{R}} ||\Phi - \Psi||_{C^1(-\infty,t]} \leq \frac{1}{2}c\kappa_1, \tag{4.33}$$

Since

$$\lim_{|x|\to+\infty} \max_{t\in[T_3,T_1]} \left| \int_{\mathbb{R}} K(y)b'(\Phi(x - y, t - \tau)) - b'(1) \right| = 0,$$

there exists a large, positive number X_2 such that, for any $|x| \geq X_2$ and $t \in [T_3, T_1]$,

$$\int_{\mathbb{R}} K(y)b'(\Phi(x - y, t - \tau))\,dy \in \left[b'(1) - \frac{\delta_0}{2}, b'(1) + \frac{\delta_0}{2}\right]. \tag{4.34}$$

Let

$$\kappa_2 = \min_{\substack{|x|\leq X_2 \\ t\in[T_3,T_1]}} \frac{\partial\Phi(x,t)}{\partial t} > 0.$$

Take $\sigma_2 > 0$ such that

$$\sigma_2\nu_0\kappa_2 - \nu_0 + d - \max_{u\in[0,2]} b'(u)e^{-\nu_0\tau} \geq 0. \tag{4.35}$$

Let $\xi(t) = t+\eta+\sigma\delta(1-e^{-\nu_0 t})$. Then, $W^+(x,t) = \Phi(x,\xi(t))+\delta e^{-\nu_0 t}$, and we have

$$\begin{aligned}
\mathcal{L}[W^+](x,t) &= \frac{\partial W^+}{\partial t} - J * W^+ + W^+ + dW^+ \\
&\quad - \int_{\mathbb{R}} K(y)b(W^+(x-y,t-\tau))\,dy \\
&= \Phi_2'(x,\xi(t))(1+\sigma\delta\nu_0 e^{-\nu_0 t}) - \delta\nu_0 e^{-\nu_0 t} \\
&\quad - J * \Phi(x,\xi(t)) + \Phi(x,\xi(t)) + d\Phi(x,\xi(t)) + d\delta e^{-\nu_0 t} \\
&\quad - \int_{\mathbb{R}} K(y)b(\Phi(x-y,\xi(t-\tau)) + \delta e^{-\nu_0(t-\tau)})\,dy \\
&= \Phi_2'(x,\xi(t))\sigma\delta\nu_0 e^{-\nu_0 t} - \delta\nu_0 e^{-\nu_0 t} + d\delta e^{-\nu_0 t} \\
&\quad + \int_{\mathbb{R}} K(y)b(\Phi(x-y,\xi(t)-\tau))\,dy \\
&\quad - \int_{\mathbb{R}} K(y)b(\Phi(x-y,\xi(t-\tau)) + \delta e^{-\nu_0(t-\tau)})\,dy \\
&\geq \Phi_2'(x,\xi(t))\sigma\delta\nu_0 e^{-\nu_0 t} - \delta\nu_0 e^{-\nu_0 t} + d\delta e^{-\nu_0 t} \\
&\quad + \int_{\mathbb{R}} K(y)b(\Phi(x-y,\xi(t)-\tau))\,dy \\
&\quad - \int_{\mathbb{R}} K(y)b(\Phi(x-y,\xi(t)-\tau) + \delta e^{-\nu_0(t-\tau)})\,dy \\
&= \delta e^{-\nu_0 t}\Big[\Phi_2'(x,\xi(t))\sigma\nu_0 - \nu_0 + d - e^{\nu_0\tau} \\
&\qquad \times \int_{\mathbb{R}} J(y)b'(\Phi(x-y,\xi(t)-\tau) + \theta\delta e^{-\nu_0(t-\tau)})\,dy\Big],
\end{aligned}$$

where $\theta \in [0,1]$ and $\Phi_2'(x,t) = (\partial\Phi(x,t))/\partial t$. Let $\sigma_0 = \max\{\sigma_1,\sigma_2\}$.

Now we consider six cases.

Case (i). $x \in \mathbb{R}$ and $\xi(t) > T_1$. By (4.25) and (4.27), we have $\mathcal{L}[W^+](x,t) > 0$.

Case (ii). $\xi(t) \leq T_3$ and $|x| + c\xi(t) + \omega \geq X_1$. Since

$$\Phi(x,\xi(t)) \geq U(x + c\xi(t) + \omega),$$

$$\begin{aligned}\Phi(x-y,\xi(t)-\tau) &\geq U(x-y+c\xi(t)-c\tau+\omega),\\ \Phi(x-y,\xi(t)-\tau) &\leq U(x-y+p(\xi(t)-\tau))\\ &\quad + U(-x+y+p(\xi(t)-\tau)),\end{aligned}$$

we have

$$\begin{aligned}b'(\Phi(x-y,\xi(t)-\tau)) \leq b'(U(x-y+c\xi(t)-c\tau+\omega))&\\ + U(-x+y+c\xi(t)-c\tau+\omega)\max_{u\in[0,1]}|b''(u)|&\\ + 2[p(\xi(t))-c\xi(t)-\omega]\max_{x\in[0,1]}|b''(u)|\cdot\max_{\xi\in\mathbb{R}}U'(\xi).&\end{aligned}$$

By (4.25), (4.26), (4.28), (4.30) and (4.31), for $\xi(t)\leq T_3$, $x>0$ with $x+c\xi(t)+\omega\geq X_1$, we get $\mathcal{L}[W^+](x,t)\geq 0$. By symmetry, we also obtain $\mathcal{L}[W^+](x,t)\geq 0$ for $\xi(t)\leq T_3$, $x<0$ with $-x+c\xi(t)+\omega\geq X_1$.

Case (iii). $\xi(t)\leq T_3$, $|x|+c\xi(t)+\omega\leq -X_1$. By (4.24), (4.26), (4.28), (4.29) and (4.31), we have $\mathcal{L}[W^+](x,t)\geq 0$.

Case (iv). $\xi(t)\leq T_3$, $-X_1\leq |x|+c\xi(t)+\omega\leq X_1$. From (4.32) and (4.33), it follows that $\mathcal{L}[W^+](x,t)\geq 0$.

Case (v). $T_3\leq \xi(t)\leq T_1$, $|x|>X_2$. By (4.25), (4.26) and (4.34), we obtain $\mathcal{L}[W^+](x,t)\geq 0$.

Case (vi). $T_3\leq \xi(t)\leq T_1$, $|x|\leq X_2$. It is easy to see that (4.35) implies $\mathcal{L}[W^+](x,t)\geq 0$. The proof is complete. □

Theorem 4.13. *Assume that* $\Phi(x,t)$ *is the entire solution of* (1.1) *given in Theorem* 4.10. *If* $\widetilde{\Phi}(x,t)$ *is an entire solution of* (1.1) *satisfying* (4.20), *then, for some* $(x_0,t_0)\in\mathbb{R}^2$,

$$\Phi(x,t)=\widetilde{\Phi}(x+x_0,t+t_0)\quad \textit{for any } (x,t)\in\mathbb{R}^2.$$

Proof. Set $\Pi(x,t)=\widetilde{\Phi}(x+x_0,t+t_0)$ for any $(x,t)\in\mathbb{R}^2$. Then, we only need to prove that $\Phi(x,t)=\Pi(x,t)$ holds for any $(x,t)\in\mathbb{R}^2$. Fix an arbitrary $t_1<0$. Define

$$\eta := \sup_{x\in\mathbb{R}}||\Phi(x,\cdot+t_1)-\Pi(x,\cdot+t_1)||_{L^\infty([-\tau,0])}.$$

In order to prove $\Phi(x,t)=\Pi(x,t)$, it suffices to show that $\eta=0$. For any small $\delta\in(0,\delta_0]$, where δ_0 is determined by Lemma 4.12, there exists a $t_2 < t_1 - \tau < -\tau$ such that

$$\sup_{x\in\mathbb{R}} ||\Phi(x,\cdot+t_1+t_2) - \Pi(x,\cdot+t_1+t_2)||_{L^\infty([-\tau,0])} \le \delta,$$

that is,

$$\Phi(x,s+t_1+t_2) - \delta \le \Pi(x,s+t_1+t_2) \le \Phi(x,s+t_1+t_2) + \delta$$

for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. Due to the monotonicity of Φ with respect to t, we further have

$$\begin{aligned}
&\Phi(x, s+t_1+t_2+\sigma_0\delta(1-e^{\nu_0\tau}) - \sigma_0\delta(1-e^{-\nu_0 s})) - \delta e^{-\nu_0 s}\\
&\quad \le \Pi(x, s+t_1+t_2)\\
&\quad \le \Phi(x, s+t_1+t_2-\sigma_0\delta(1-e^{\nu_0\tau}) + \sigma_0\delta(1-e^{-\nu_0 s})) + \delta e^{-\nu_0 s},
\end{aligned}$$

where σ_0 and ν_0 are given in Lemma 4.12. By the comparison principle, for all $x \in \mathbb{R}$ and $t \ge 0$, we obtain

$$\begin{aligned}
&\Phi(x, t+t_1+t_2+\sigma_0\delta(1-e^{\nu_0\tau}) - \sigma_0\delta(1-e^{-\nu_0 t})) - \delta e^{-\nu_0 t}\\
&\quad \le \Pi(x, t+t_1+t_2)\\
&\quad \le \Phi(x, t+t_1+t_2-\sigma_0\delta(1-e^{\nu_0\tau}) + \sigma_0\delta(1-e^{-\nu_0 t})) + \delta e^{-\nu_0 t}.
\end{aligned}$$

Set $t \in [-t_2-\tau, -t_2]$, $\widehat{r} = t+t_1+t_2+\sigma_0\delta(1-e^{\nu_0\tau}) - \sigma_0\delta(1-e^{-\nu_0 t})$, and

$$\mathcal{M} = \sup_{x\in\mathbb{R}} \left\| \frac{\partial\Phi(x,t)}{\partial t} \right\|_{L^\infty(\mathbb{R})}.$$

Then, by the mean-value theorem, it follows that

$$\begin{aligned}
&|\Pi(x,t+t_1+t_2) - \Phi(x,t+t_1+t_2)|\\
&\quad \le 2\delta + |\Phi(x,\widehat{r}+2\sigma_0\delta(e^{\nu_0\tau} - e^{-\nu_0 t})) - \Phi(x,\widehat{r})| \le 2(1+e^{\nu_0\tau}\sigma_0\mathcal{M})\delta,
\end{aligned}$$

which implies

$$\sup_{x\in\mathbb{R}} ||\Phi(x,\cdot+t_1) - \Pi(x,\cdot+t_1)||_{L^\infty([-\tau,0])} \le 2(1+e^{\nu_0\tau}\sigma_0\mathcal{M})\delta.$$

This, in turn, implies that

$$\eta \le 2\left(1 + e^{\nu_0\tau}\sigma_0 \sup_{x\in\mathbb{R}} \left\| \frac{\partial\Phi(x,t)}{\partial t} \right\|_{L^\infty}\right)\delta \quad \text{for all } \delta \in (0,\delta_0].$$

By the arbitrariness of δ, we have $\eta = 0$. Therefore, $\widetilde{\Phi}(x+x_0, t+t_0) = \Phi(x,t)$ for $(x,t) \in \mathbb{R}^2$. The proof is complete. $\square$

Theorem 4.14. *The entire solution* $\Phi(x,t) := \Phi_{\omega_1,\omega_2}(x,t)$ *of* (1.1) *given in Theorem* 4.10 *is continuously dependent upon* $(\omega_1, \omega_2) \in (-\infty, 0)$ *in the sense of* $\mathcal{T}$.

Proof. Given (ω_1^0, ω_2^0), take two sequences $\{(\omega_{+,1}^k, \omega_{+,2}^k)\}_{k\in\mathbb{N}}$ and $\{(\omega_{-,1}^k, \omega_{-,2}^k)\}_{k\in\mathbb{N}}$ satisfying $\{(\omega_{\pm,1}^k, \omega_{\pm,2}^k)\}_{k\in\mathbb{N}} \subset (-\infty, \theta)^2$, $\lim_{k\to+\infty} (\omega_{\pm,1}^k, \omega_{\pm,2}^k) \to (\omega_1^0, \omega_2^0)$ and

$$(\omega_{-,1}^k, \omega_{-,2}^k) \le (\omega_{-,1}^{k+1}, \omega_{-,2}^{k+1}) < (\omega_1^0, \omega_2^0) < (\omega_{+,1}^{k+1}, \omega_{+,2}^{k+1}) \le (\omega_{+,1}^k, \omega_{+,2}^k)$$

for any $k \in \mathbb{N}$. By Theorem 4.10, (1.1) admits entire solutions

$$\Phi^0(x,t) := \Phi^0_{\omega_1^0,\omega_2^0}(x,t), \qquad \Phi_\pm^k(x,t) := \Phi^k_{\pm,\omega_{\pm,1}^k,\omega_{\pm,2}^k}(x,t).$$

It also follows that there exist $\Phi_\pm(x,t)$ such that $\Phi_\pm^k$ converge to $\Phi_\pm(x,t)$, respectively, in the sense of $\mathcal{T}$ as $k \to +\infty$. In particular, $\Phi_+(x,t)$ and $\Phi_-(x,t)$ are entire solutions of (1.1).

We now prove that $\Phi_+(x,t) \equiv \Phi^0(x,t)$ for any $(x,t) \in \mathbb{R}^2$. By Theorem 4.11 (v), we obtain

$$\begin{aligned} \Phi_-^k(x,t) &\le \Phi_-^{k+1}(x,t) \le \Phi_-(x,t) \\ &\le \Phi^0(x,t) \le \Phi_+(x,t) \\ &\le \Phi_+^{k+1}(x,t) \le \Phi_+^k(x,t) \end{aligned}$$

for all $(x,t) \in \mathbb{R}^2$ and $k \in \mathbb{N}$. For any $t \le T$, we have

$$\begin{aligned} &\max\{U(x+ct+\omega_{-,1}^k), U(-x+ct+\omega_{-,2}^k)\} \\ &\quad \le \max\{U(x+ct+\omega_{-,1}^{k+1}), U(-x+ct+\omega_{-,2}^{k+1})\} \\ &\quad \le \max\{U(x+ct+\omega_1^0), U(-x+ct+\omega_2^0)\} \\ &\quad \le \max\{U(x+p_1(t;\omega_1^0,\omega_2^0)), U(-x+p_2(t;\omega_1^0,\omega_2^0)\} \\ &\quad \le \max\{U(x+p_1(t;\omega_{+,1}^{k+1},\omega_{+,2}^{k+1})), U(-x+p_2(t;\omega_{+,1}^{k+1},\omega_{+,2}^{k+1})\} \\ &\quad \le \max\{U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k)), U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k)\} \end{aligned}$$

and

$$\begin{aligned} &\max\{U(x+ct+\omega_{-,1}^k), U(-x+ct+\omega_{-,2}^k)\} \le \Phi_+(x,t) \\ &\quad \le \max\{U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k)), U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k)\} \end{aligned}$$

for any $k \in \mathbb{N}$. Recall that T is independent of k. Following this, we have

$$\begin{aligned}
&\sup_{x\geq 0}|\Phi_+(x,t) - U(x+ct+\omega_1^0)| \\
&\qquad + \sup_{x<0}|\Phi_+(x,t) - U(-x+ct+\omega_2^0)| \\
&\quad \leq \sup_{x\geq 0}|U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k) - U(x+ct+\omega_{-,1}^k)| \\
&\qquad + \sup_{x<0}|U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k) - U(-x+ct+\omega_{-,2}^k)| \\
&\qquad + \sup_{x\geq 0}U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k) \\
&\qquad + \sup_{x<0}U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k) \\
&\quad \leq \sup_{x\geq 0}|U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k) - U(x+ct+\omega_{+,1}^k)| \\
&\qquad + \sup_{x<0}|U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k) - U(-x+ct+\omega_{+,2}^k)| \\
&\qquad + \sup_{x\geq 0}U(-x+p_2(t;\omega_{+,1}^k,\omega_{+,2}^k) \\
&\qquad + \sup_{x<0}U(x+p_1(t;\omega_{+,1}^k,\omega_{+,2}^k) \\
&\qquad + \sup_{x\geq 0}|U(x+ct+\omega_{+,1}^k) - U(x+ct+\omega_{-,1}^k)| \\
&\qquad + \sup_{x<0}|U(-x+ct+\omega_{+,2}^k) - U(-x+ct+\omega_{-,2}^k)|.
\end{aligned}$$

By the arbitrariness of $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\sup_{x\geq 0}|\Phi_+(x,t) - U(x+ct+\omega_1^0)| \\
&\qquad + \sup_{x<0}|\Phi_+(x,t) - U(-x+ct+\omega_2^0)| \\
&\quad \leq 2R_0 e^{c\lambda_{02}t}\max_{x\in\mathbb{R}} U'(x) + 2U(ct) \to 0, \quad \text{as } t \to +\infty.
\end{aligned}$$

By Theorem 4.13, we have $\Phi_+(x,t) = \Phi^0(x,t)$ for $(x,t) \in \mathbb{R}^2$. Similarly, we obtain $\Phi_-(x,t) = \Phi^0(x,t)$ for $(x,t) \in \mathbb{R}^2$. In view of the monotonicity of $\Phi_+^k(x,t)$ and $\Phi_-^k(x,t)$ about $k \in \mathbb{N}$, $\Phi_+^k(x,t)$ and $\Phi_-^k(x,t)$ converge to $\Phi^0(x,t)$ in the sense of $\mathcal{T}$ as $k \to \infty$. Now, consider $(\omega_1,\omega_2) \to (\omega_1^0,\omega_2^0)$. It is easy to prove that $\Phi(x,t;\omega_1,\omega_2)$ converges to $\Phi^0(x,t;\omega_1^0,\omega_2^0)$ in the sense of $\mathcal{T}$ as $k \to \infty$. The proof is complete. $\square$

Theorem 4.15. *The entire solution* Φ *of* (1.1) *given in Theorem* 4.10 *is Lyapunov stable in the following sense*: *for any given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that, for any* $\varphi \in C(\mathbb{R} \times [-\tau, 0], [0, 1])$ *and* $||\varphi(\cdot,\cdot) - \Phi(\cdot + x_0, \cdot + t_0)||_{L^\infty(\mathbb{R}\times[-\tau,0])} < \delta$ *for some* $(x_0, t_0) \in \mathbb{R}^2$, *we have* $|u(x, t) - \Phi(\cdot + x_0, \cdot + t_0)| < \epsilon$ *for any* $x \in \mathbb{R}$ *and* $t \geq 0$.

Proof. Given any $\epsilon > 0$, choose $\delta_1 = \epsilon/(2M)$. Then, for any $|z| \leq \delta_1$, we have

$$
\begin{aligned}
(4.36)\qquad ||\Phi(\cdot, t) - \Phi(\cdot, t + z)||_{L^\infty(\mathbb{R})} &\leq \sup_{t\in\mathbb{R}} \left\| \frac{\partial}{\partial t}\Phi(x, t) \right\|_{L^\infty(\mathbb{R})} |z| \\
&\leq M\delta_1 \leq \frac{\epsilon}{2}.
\end{aligned}
$$

For any $\varphi \in C([-\tau, 0], [0, 1])$ and $||\varphi - \Phi(x_0 + \cdot, t_0 + \cdot)||_{L^\infty(\mathbb{R}\times[-\tau,0])} < \delta \leq \delta_0$, where $x_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$ are arbitrary constants, the following holds

$$\Phi(x + x_0, s + t_0) - \delta \leq \varphi(x, s) \leq \Phi(x + x_0, s + t_0) + \delta$$

for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. Since $\Phi(x, t)$ is increasing in t for all $x \in \mathbb{R}$, we further have

$$
\begin{aligned}
&\Phi(x + x_0, s + t_0 + \sigma_0\delta(1 - e^{\nu_0\tau}) - \sigma_0\delta(1 - e^{-\nu_0 s})) - \delta e^{-\nu_0 s} \\
&\quad\leq \varphi(x, s) \leq \Phi(x + x_0, s + t_0 + \sigma_0\delta(1 - e^{\nu_0\tau}) \\
&\qquad\qquad + \sigma_0\delta(1 - e^{-\nu_0 s})) + \delta e^{-\nu_0 s}
\end{aligned}
$$

for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, where δ_0, σ_0 and ν_0 are as in Lemma 4.12. Then, by the comparison principle and Lemma 4.12, we obtain

$$
\begin{aligned}
(4.37)\qquad &\Phi(x + x_0, t + t_0 + \sigma_0\delta(1 - e^{\nu_0\tau}) - \sigma_0\delta(1 - e^{-\nu_0 t})) - \delta e^{-\nu_0 t} \\
&\quad\leq u(x, t; \varphi) \\
&\quad\leq \Phi(x + x_0, t + t_0 + \sigma_0\delta(1 - e^{\nu_0\tau}) \\
&\qquad\qquad + \sigma_0\delta(1 - e^{-\nu_0 t})) + \delta e^{-\nu_0 t}
\end{aligned}
$$

for all $x \in \mathbb{R}$ and $t > 0$. Furthermore, let $\delta^* = \{\epsilon/2, (\delta_1 e^{-\nu_0\tau}/\sigma_0), \delta_0\}$. Then, for any $\delta < \delta^*$,

$$
\begin{aligned}
|\sigma_0\delta(1 - e^{\nu_0\tau}) - \sigma_0\delta(1 - e^{-\nu_0 t})| &\leq |\sigma_0\delta(e^{\nu_0\tau} - e^{-\nu_0 t})| \\
&\leq \sigma_0\delta e^{\nu_0\tau} \leq \delta_1.
\end{aligned}
$$

It then follows from (4.36) and (4.37) that

$$|u(x,t;\varphi) - \Phi(x+x_0,t+t_0)| \le M\sigma_0\delta + \delta \le \epsilon$$

for all $x \in \mathbb{R}$ and $t \ge 0$. The proof is complete. □

REFERENCES

1. P.W. Bates, P.C. Fife, X. Ren and X. Wang, *Traveling waves in a convolution model for phase transitions*, Arch. Rat. Mech. Anal. **138** (1997), 105–136.

2. J. Carr and A. Chmaj, *Uniqueness of travelling waves for nonlocal monostable equations*, Proc. Amer. Math. Soc. **132** (2004), 2433–2439.

3. E. Chasseigne, M. Chaves and J.D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pure Appl. **86** (2006), 271–291.

4. X. Chen, *Existence, uniqueness and asymptotic stability of traveling waves in non-local evolution equations*, Adv. Differ. Eq. **2** (1997), 125–160.

5. J. Coville, J. Dávila and S. Martínez, *Nonlocal anisotropic dispersal with monostable nonlinearity*, J. Differ. Eq. **244** (2008), 3080–3118.

6. J. Coville and L. Dupaigne, *On a nonlocal reaction diffusion equation arising in population dynamics*, Proc. Roy. Soc. Edinburgh **137** (2007), 1–29.

7. P.C. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions*, in *Trends in nonlinear analysis*, Springer, Berlin, 2003.

8. Y.-J.L. Guo, *Entire solutions for a discrete diffusive equation*, J. Math. Anal. Appl. **347** (2008), 450–458.

9. J.S. Guo and C.H. Wu, *Entire solutions for a two-component competition system in a lattice*, Tohoku Math. J. **62** (2010), 17–28.

10. F. Hamel and N. Nadirashvili, *Entire solution of the* KPP *eqution*, Comm. Pure Appl. Math. **52** (1999), 1255–1276.

11. ______, *Travelling fronts and entire solutions of the Fisher*-KPP *equation in* R^N, Arch. Rat. Mech. Anal. **157** (2001), 91–163.

12. R. Huang, M. Mei and Y. Wang, *Planar traveling waves for nonlocal dispersion equation with monostable nonlinearity*, Discr. Cont. Dynam. Syst. **32** (2012), 3621–3649.

13. L.I. Ignat and J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal. **251** (2007), 399–437.

14. W.-T. Li, Y.-J. Sun and Z.-C. Wang, *Entire solutions in the Fisher*-KPP *equation with nonlocal dispersal*, Nonlin. Anal. **11** (2010), 2302–2313.

15. W.-T. Li, L. Zhang and G.-B. Zhang, *Invasion entire solutions in a competition system with nonlocal dispersal*, Discr. Cont. Dynam. Syst. **35** (2015), 1531–1560.

16. J. Medlock and M. Kot, *Spreading disease*: *Integro-differential equations old and new*, Math. Biosci. **184** (2003), 201–222.

17. Y. Morita and H. Ninomiya, *Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Differ. Eq. **18** (2006), 841–861.

18. Y. Morita and K. Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal. **40** (2009), 2217–2240.

19. S. Pan, W.-T. Li and G. Lin, *Travelling wave fronts in nonlocal reaction-diffusion systems and applications*, Z. Angew. Math. Phys. **60** (2009), 377–392.

20. ______, *Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay*, Nonlin. Anal. **72** (2010), 3150–3158.

21. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

22. Y.J. Sun, W.T. Li and Z.C. Wang, *Entire solutions in nonlocal dispersal equations with bistable nonlinearity*, J. Differ. Eq. **251** (2011), 551–581.

23. M.X. Wang and G.Y. Lv, *Entire solutions of a diffusive and competitive Lotka-Volterra type system with nonlocal delay*, Nonlinearity **23** (2010), 1609–1630.

24. Z.-C. Wang, W.-T. Li and S. Ruan, *Entire solutions in lattice delayed differential equations with nonlocal interaction*: *Bistable cases*, Math. Mod. Nat. Phenom. **8** (2013), 78–103.

25. Z.-C. Wang, W.-T. Li and J. Wu, *Entire solutions in delayed lattice differential equations with monostable nonlinearity*, SIAM J. Math. Anal. **40** (2009), 2392–2420.

26. D.V. Widder, *The Laplace transform*, Princeton University Press, Princeton, 1941.

27. S.L. Wu, *Entire solutions in a bistable reaction-diffusion system modeling man-environment-man epidemics*, Nonlin. Anal. **13** (2012), 1991–2005.

28. S.L. Wu and P.X. Weng, *Entire solutions for a multi-type* SIS *nonlocal epidemic model in R or Z*, J. Math. Anal. Appl. **394** (2012), 603–615.

29. Shi-Liang Wu and S. Ruan, *Entire solutions for nonlocal dispersal equations with spatio-temporal delay*: *Monostable case*, J. Differ. Eq. **258** (2015), 2435–2470.

30. H. Yagisita, *Back and global solutions characterizing annihilation dynamics of traveling fronts*, Publ. Res. Inst. Math. Sci. **39** (2003), 117–164.

31. G.-B. Zhang, *Traveling waves in a nonlocal dispersal population model with age-structure*, Nonlin. Anal. **74** (2011), 5030–5047.

32. ______, *Nonmonotone traveling waves and entire solutions for a delayed nonlocal dispersal equation with convolution type crossing-monostable nonlinearity*, Appl. Anal. **96** (2017), 1830–1866.

33. G.-B. Zhang, W.-T. Li and Z.-C. Wang, *Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity*, J. Differ. Eq. **252** (2012), 5096–5124.

34. G.-B. Zhang and R. Ma, *Spreading speeds and traveling waves for a nonlocal dispersal equation with convolution type crossing-monostable nonlinearity*, Z. Angew. Math. Phys. **65** (2014), 819–844.

35. G.-B. Zhang and R. Ma, *Existence, uniqueness and stability of traveling wavefronts for a nonlocal dispersal equation with convolution type bistable nonlinearity*, Electr. J. Differ. Eq. **144** (2015), 1–27.

36. L. Zhang, *Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks*, J. Differ. Eq. **197** (2004), 162–196.

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