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ON FACTORABLE RINGS

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ABSTRACT. In this short note, we introduce the notions of "factorable ring" and "fully factorable ring" for commutative rings based upon the notion of "factorable domain" advanced by Anderson, Kim and Park [**1**]. Using a novel sufficient condition for an ideal to be a product of nonfactorable ideals, we classify the Artinian rings that are (fully) factorable. We also explore the intersection of the class of factorable rings with the class of Noetherian rings. An analogue for multiplication rings of a characterization result due to Butts [**3**] concerning when such a unique factorization occurs is provided.

1. Introduction. Throughout this note, all rings are commutative with $1 \neq 0$. In an effort to bridge the gap between the definitions of "prime ideal" in the contexts of classical algebraic number theory and commutative ring theory, respectively, in 1964, Butts advanced the concept of a "nonfactorable ideal" [**3**]. Specifically, a *nonfactorable ideal* I of a commutative ring R is a nonzero, proper ideal of R such that, whenever $I = JK$ for some ideals J and K of R, it must be the case that either $J = R$ or $K = R$, see also [**4**]. Butts then demonstrated that R is a Dedekind domain if and only if R is a domain for which every nonzero, proper ideal of R can be factored uniquely (up to the order of the factors) as a product of nonfactorable ideals of R [**3,** Theorem]. Capitalizing upon the value of this notion, in 2002, Anderson, Kim and Park introduced and explored *factorable domains*, that is, domains with the property that every nonzero, proper ideal of the domain is a product of nonfactorable ideals of the domain, cf., [**1,** Definition 3]). In particular, they expanded Butts's characterization to include

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factorable Prüfer domains, as well [**1,** Theorem 5], and extended their explorations to the context of star operations on a domain.

Here, we wish to go beyond the context of integral domains to investigate the phenomenon of factorization of ideals as a product of nonfactorable ideals in arbitrary commutative rings. To this end, we develop some very useful results in Proposition 2.2 and the related Theorem 2.7. Utilizing these results, we establish that all SPIRs are factorable (Corollary 2.4) and create a characterization of which Artinian rings are (fully) factorable (Theorem 2.11). As a corollary to the former, we easily have a natural analogue for multiplication rings of the aforementioned result of Butts (Corollary 2.5). We further establish that every local Noetherian ring is factorable (Theorem 2.13). We conclude with examples to show the lack of a general relationship between the length of a (reduced) primary decomposition and the length of a nonfactorable ideal decomposition for Noetherian domains.

For the sake of the requisite background, we provide the following definitions. A *multiplication ring* R is a ring satisfying the property that, for any pair of ideals $I \subseteq J$ of R, there exists an ideal K of R for which $I = JK$. In fact, a domain is a multiplication ring if and only if it is a Dedekind domain. A ZPI-*ring* is a ring such that every nonzero, proper ideal of the ring is uniquely expressible (up to order) as a product of prime ideals of the ring. It is easy to see that every ZPI-ring is a multiplication ring. By eliminating the requirement of uniqueness and enlarging the relevant set of ideals to include the zero ideal, we have the associated notion of a *general* ZPI-*ring*. A *special principal ideal ring* (SPIR) is a local principal ideal ring with nilpotent maximal ideal. Note that SPIRs can be characterized as those local rings for which every ideal of the ring is a power of the unique maximal ideal of the ring (sometimes referred to as "special primary rings," or SPRs, in the literature).

Any unexplained terminology is standard, as in [**2, 5, 6**].

2. Results. We begin with the analogous definition of "factorable domain" for commutative rings, finding a slight variation is afforded in this more general context.

Definition 2.1. Let R be a ring. We say that R is *factorable* if every nonzero, proper ideal of R can be expressed as a product of nonfactorable ideals of R. We say that R is *fully factorable* if R is factorable and the zero ideal can be expressed as a product of nonfactorable ideals of R.

Before exploring various types of (fully) factorable rings, it is critical to provide some sufficient conditions for an ideal to be a product of nonfactorable ideals. Along these lines, Proposition 2.2 not only provides for a wealth of nonfactorable ideals but will be used to establish our central result in this endeavor, Theorem 2.7.

Proposition 2.2. *Let R be a ring, M a maximal ideal of R and I an ideal of R such that $M^2 \subsetneq I \subseteq M$. Then I is nonfactorable.*

Proof. Suppose $M^2 \subsetneq I \subseteq M$, and put $I = JK$ with J and K proper ideals of R. Note that $J \subseteq M$ or $K \subseteq M$. Without loss of generality, suppose that $J \subseteq M$. Let N be a maximal ideal of R such that $K \subseteq N$. Thus, $I \subseteq JK \subseteq MN$. Observe that $M^2 \subsetneq I \subseteq MN$ implies $M = N$. However, this means that $I \subseteq M^2$, a contradiction. The result follows. $\square$

Corollary 2.3. *Every maximal ideal M of a ring R such that $M^2 \neq M$ is nonfactorable.*

We are now able to provide through Corollaries 2.4 and 2.5 one of the promised goals of this paper, the natural analogue for multiplication rings of Butts's characterization result regarding domains that exhibit a uniqueness of ideal factorability in terms of nonfactorable ideals. Note that, in Corollary 2.4, a nontrivial SPIR refers to an SPIR that is not a field.

Corollary 2.4. *Let R be an* SPIR*, respectively, a nontrivial* SPIR*. Then R is factorable, respectively, fully factorable. Moreover, every nonzero, proper ideal of R is uniquely a product of nonfactorable ideals of R.*

Proof. Let R be an SPIR and M the maximal ideal of R. Since fields are vacuously factorable, we may assume that R is not a field. Suppose that I is a proper ideal of R. Then, $I = M^n$ for some $n \in \mathbb{N}$.

Since $M^2 \neq M$ in R, Corollary 2.3 guarantees that I is a product of nonfactorable ideals, whence R is fully factorable. The "moreover" statement follows from the fact that M is clearly the only nonfactorable ideal of R and, if $M^i \neq 0$, then $M^i = M^j$ implies $i = j$. □

Corollary 2.5. *Let R be a multiplication ring. Then R is a* ZPI*-ring if and only if every nonzero, proper ideal of R is uniquely* (*up to the order of the factors*) *a product of nonfactorable ideals of R.*

Proof. Observe that ZPI-rings are characterized as either Dedekind domains or SPIRs [**5,** Corollary 39.3]. As such, the forward direction readily follows from Corollary 2.4 and the fact that the nonfactorable ideals in a Dedekind domain, that is not a field, are precisely the maximal ideals.

Conversely, suppose that R is a multiplication ring such that every nonzero, proper ideal of R is uniquely (up to the order of the factors) a product of nonfactorable ideals of R. Let I be a nonfactorable ideal of R, and let M be a maximal ideal of R containing I. Then, there exists an ideal J of R for which $I = JM$. However, this would mean that $J = R$, and so $I = M$. Therefore, every nonfactorable ideal of R must be maximal, whence the hypothesis guarantees that R is a ZPI-ring. □

In light of Corollary 2.5, it is curious to note that a general ZPI-ring need not even be factorable, the simplest example being the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$; however, Corollary 2.10 will give a characterization of which general ZPI-rings are factorable. In addition, Example 2.6 reveals that Corollary 2.5 is best possible, in the sense that the "multiplication ring" assumption may not be dispensed with.

Example 2.6. There exists a ring R for which every nonzero, proper ideal of R is uniquely (up to the order of the factors) a product of nonfactorable ideals of R, but R is not a ZPI-ring. Moreover, it may be arranged that R is quasilocal and zero-dimensional. Let k be a field and $\{X_i\}_{i\in I}$ a collection of (commuting, algebraically independent) indeterminates over k, where $|I| > 1$. Put $R = k[\{X_i\}_{i\in I}]/M^2$, where $M = (\{X_i\}_{i\in I})$. Then every nonzero, proper ideal of R is a nonfactorable ideal of R, see Proposition 2.2, and, moreover, is (trivially) uniquely a product of nonfactorable ideals of R. However, since $|I| > 1$, the ring R is clearly not a ZPI-ring.

We now provide a valuable new tool for demonstrating that certain ideals are products of nonfactorable ideals.

Theorem 2.7. *Let R be a ring and I a proper ideal of R such that $M^n \subsetneq I$ for some natural number $n \geq 2$ and maximal ideal M of R. Then, I is a product of nonfactorable ideals of R.*

Proof. We proceed by induction on n. The case of $n = 2$ is settled by Proposition 2.2. Now, assume that a proper ideal J can be expressed as a product of nonfactorable ideals of R whenever $M^{n-1} \subsetneq J$ for some natural number n and some maximal ideal M. Let I be a proper ideal of R and M a maximal ideal of R such that $M^n \subsetneq I$. If I is a nonfactorable ideal, the result follows. Now suppose that I is factorable. Then $I = JK$ for some proper ideals J and K of R. Note that, necessarily, $M^n \subsetneq J \subseteq M$ and $M^n \subsetneq K \subseteq M$. Thus, $I = (J + M^{n-1})(K + M^{n-1})$, where $M^{n-1} \subsetneq J + M^{n-1} \subseteq M$ and $M^{n-1} \subsetneq K + M^{n-1} \subseteq M$. By the induction hypothesis, $J + M^{n-1}$ and $K + M^{n-1}$ can be written as a product of nonfactorable ideals. Hence, I can be written as a product of nonfactorable ideals. □

Corollary 2.8. *Let R be a quasilocal ring with nonzero nilpotent maximal ideal. Then R is fully factorable.*

In order to make full use of Theorem 2.7 in this note, we also require an understanding of when a finite direct product is factorable. Proposition 2.9 does just that.

Proposition 2.9. *Let $R_1, R_2, \dots, R_n$ be rings with $n \geq 2$. Put $R = R_1 \times R_2 \times \cdots \times R_n$. Then the following are equivalent*:

(i) *R is fully factorable*;
(ii) *R is factorable*;
(iii) *each R_i is fully factorable.*

Proof.

(i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Suppose that R is factorable. Let I be a proper ideal of R_i. Then

$$A = R_1 \times R_2 \times \cdots \times R_{i-1} \times I \times R_{i+1} \times \cdots \times R_n$$

is a nonzero, proper ideal of R. Since R is factorable, A is a product of nonfactorable ideals $J_1, J_2, \ldots, J_m$ of R. Then, for each $k = 1, 2, \ldots, m$, it is necessarily the case that

$$J_k = R_1 \times R_2 \times \cdots \times R_{i-1} \times H_k \times R_{i+1} \times \cdots \times R_n$$

for some ideal H_k of R_i. Moreover, $I = H_1 H_2 \cdots H_m$ and each H_k is nonfactorable in R_i since each J_k is nonfactorable in R. Thus, R_i is fully factorable.

(iii) $\Rightarrow$ (i). Suppose that each R_i is fully factorable. Let $I_1 \times I_2 \times \cdots \times I_n$ be a proper ideal of R. By supposition, each I_t that is a proper ideal of R_t is a product of nonfactorable ideals of R_t. Thus, for each such I_t, there exist nonfactorable ideals $J_1, J_2, \ldots, J_m$ of R_t such that $I_t = J_1 J_2 \cdots J_m$. As such, $I_1 \times I_2 \times \cdots \times I_n$ is a product of ideals of R of the form

$$R_1 \times R_2 \times \cdots \times R_{i-1} \times J \times R_{i+1} \times \cdots \times R_n,$$

where J is a nonfactorable ideal of R_i . However, each

$$R_1 \times R_2 \times \cdots \times R_{i-1} \times J \times R_{i+1} \times \cdots \times R_n$$

is nonfactorable since each J is nonfactorable. Therefore, R is fully factorable. □

Corollary 2.10. *Let R be a general* ZPI-*ring. Then R is factorable if and only if either*

(i) *R is a Dedekind domain, or*
(ii) *R is* (*isomorphic to*) *a finite direct product of nontrivial* SPIR*s.*

Moreover, R is fully factorable if and only if condition (ii) *holds.*

Proof. By [**5,** Theorem 39.2], R is a general ZPI-ring if and only if R is (isomorphic to) a finite direct product of Dedekind domains and SPIRs. However, amongst Dedekind domains and SPIRs, only nontrivial SPIRs are fully factorable, cf., Corollary 2.4. The result then follows from Proposition 2.9. □

We now give the promised characterization of when an Artinian ring is (fully) factorable.

Theorem 2.11. *Let R be an Artinian ring with maximal ideals M_1, $M_2, \dots, M_n$. Then, R is fully factorable if and only if $M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} = 0$ implies $\alpha_i > 1$ for each i. Moreover, this condition on the maximal ideals of R characterizes the nontrivial Artinian rings that are factorable.*

Proof. If R is local ($n = 1$), then Corollary 2.8 provides for the desired characterization (in fact, for nontrivial, local Artinian rings, the corresponding condition on the maximal ideal is automatic). Thus, we may assume that $n \geq 2$. By the Chinese remainder theorem, it is the case that

$$R \cong R/M_1^{\alpha_1} \times R/M_2^{\alpha_2} \times \cdots \times R/M_n^{\alpha_n},$$

where $M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} = 0$. If R is (fully) factorable, then Proposition 2.9 asserts that each $R/M_i^{\alpha_i}$ is fully factorable, and thus, necessarily, each $\alpha_i > 1$. Conversely, suppose that each $\alpha_i > 1$. It then follows from Theorem 2.7 that each $R/M_i^{\alpha_i}$ is fully factorable. Another application of Proposition 2.9 yields that R itself is (fully) factorable. □

In light of Theorem 2.11, it is then natural to ask about the more general question of which Noetherian rings are factorable. While clearly not all Noetherian rings are factorable, for example, $\mathbb{Z}_2 \times \mathbb{Z}_2$, Theorem 2.13 provides for some large classes of Noetherian rings which are (although it should be noted that Anderson, Kim, and Park already observed that every Noetherian domain is factorable [**1,** page 4116]). Before presenting Theorem 2.13, we pause to argue in Proposition 2.12 that, for finitely generated ideals in general, testing their nonfactorability can be accomplished even if one restricts to the class of finitely generated ideals.

Proposition 2.12. *Let I be a finitely generated ideal of the ring R. Then I is nonfactorable if and only if, whenever J and K are finitely generated ideals of R for which $I = JK$, then $J = R$ or $K = R$.*

Proof. The "only if" direction is trivial. Conversely, suppose that I is a finitely generated ideal of the ring R and, whenever $I = JK$, where

J and K are finitely generated ideals of R, it must be the case that $J = R$ or $K = R$. Put $I = AB$, where A and B are ideals of R. Since I is finitely generated, $I = (i_1, \ldots, i_n)$ for some $i_1, i_2, \ldots, i_n \in I$. Thus, there must exist a function $\alpha:\{1, 2, \ldots, n\} \to \mathbb{N}$, elements $a_{s,r} \in A$, $1 \leq s \leq n$, $1 \leq r \leq \alpha(s)$, and elements $b_{s,r} \in B$, $1 \leq s \leq n$, $1 \leq r \leq \alpha(s)$, such that

$$i_s = \sum_{r=1}^{\alpha(s)} a_{s,r} b_{s,r} \quad \text{for each } s = 1, 2, \ldots, n.$$

Therefore, $I = (\{a_{s,r}\})(\{b_{s,r}\})$. By assumption, either $(\{a_{s,r}\}) = R$ or $(\{b_{s,r}\}) = R$. As such, either $A = R$ or $B = R$. Hence, I is nonfactorable. $\square$

Theorem 2.13. *Any local Noetherian ring or Noetherian domain is factorable.*

Proof. Let R either be a local Noetherian ring or Noetherian domain. Let I be a nonzero, proper ideal of R. Suppose that I cannot be expressed as a product of nonfactorable ideals. In particular, I itself cannot be nonfactorable. Thus, there exist proper ideals J_1 and K_1 of R such that $I = J_1K_1$. Now, at least one of J_1 or K_1 cannot be expressed as a product of nonfactorable ideals. Without loss of generality, suppose it is J_1. Then there exist proper ideals J_2 and K_2 of R such that $J_1 = J_2K_2$. Repeating this process, a chain $I \subseteq J_1 \subseteq J_2 \subseteq \cdots$ of ideals of R can be created. In order to see that each containment is proper suppose that there exists an i such that $J_i = J_{i+1}$. Since $J_i = J_{i+1}K_{i+1}$, it is the case that $J_i = J_iK_{i+1}$. If R is a local Noetherian ring, Nakayama's lemma implies that $J_i = 0$, a contradiction. If R is a Noetherian domain, $K_{i+1} = R$, also a contradiction. However, R cannot have a properly ascending chain of ideals. Hence, I is a product of nonfactorable ideals, and the result is proved. $\square$

With Theorem 2.13 in view, it bears mentioning that quasilocal domains, in general, are not factorable; in particular, any quasilocal domain of the form $D + XL[[X]]$, where D is a quasilocal subring of the field L such that D itself is not a field, is not factorable by **[1,** Corollary 8].

We conclude this paper by comparing the factorization of an ideal in terms of nonfactorable ideals with the classical (reduced) decomposition of the ideal in terms of primary ideals in the context of a Noetherian domain. Specifically, Example 2.14 reveals that there is no relationship, in general, between the number of ideals required for the former type of factorization (dubbed "nonfactorable length" here) and the number of ideals required for the latter type of factorization (dubbed "primary length" here).

Example 2.14. Noetherian domains R and S exist such that, for any natural number n, there exists a primary ideal of R with a nonfactorable length of n, and there exists a nonfactorable ideal of S with a primary length of n.

Let R be a nontrivial DVR with maximal ideal M. Then, the ideal M^n is a primary ideal of R (as every proper ideal of R is primary); however, M^n has a nonfactorable length of n as R is a Dedekind domain. Hence, R has uniqueness of such factorizations.

Put

$$S = k[X_1, X_2, \ldots, X_{n+1}],$$

where k is a field and $X_1, X_2, \ldots, X_{n+1}$ are (commuting, algebraically independent) indeterminates over k. Put

$$P_i = (X_i, X_{n+1})$$

for $i = 1, 2, \ldots, n$, and

$$I = \bigcap_{i=1}^{n} P_i.$$

Since $\{P_i\}_{i=1}^n$ is a set of pairwise incomparable prime ideals of S, it follows that I has a primary length of n. We claim that I is nonfactorable. To this end, suppose that $I = JK$, where J and K are ideals of S. Note that if $J \subseteq I$, then $I = IK$, whence $K = S$, as desired. Similarly, if $K \subseteq I$, then $J = S$, as desired.

Assume then that $J \nsubseteq I$ and $K \nsubseteq I$. Reorder the P_is, if necessary, so that

$$J \subseteq P_i \quad \text{for } i = 1, 2, \ldots, m < n$$

and

$$J \nsubseteq P_i \quad \text{for } i = m+1, m+2, \ldots, n.$$

Then, $K \subseteq P_i$ for $i = m+1, m+2, \ldots, n$. As such,

$$I \subseteq (P_1 \cap P_2 \cap \cdots \cap P_m)(P_{m+1} \cap P_{m+2} \cap \cdots \cap P_n) \subseteq I,$$

and thus,

$$I = (P_1 \cap P_2 \cap \cdots \cap P_m)(P_{m+1} \cap P_{m+2} \cap \cdots \cap P_n).$$

However, this is a contradiction since $X_{n+1} \in I$, and

$$X_{n+1} \notin (P_1 \cap P_2 \cap \cdots \cap P_m)(P_{m+1} \cap P_{m+2} \cap \cdots \cap P_n).$$

The claim is thus proved. $\square$

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