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ON NONHOMOGENEOUS ELLIPTIC PROBLEMS INVOLVING THE HARDY POTENTIAL AND CRITICAL SOBOLEV EXPONENT

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ABSTRACT. In this paper, we are concerned with elliptic equations with Hardy potential and critical Sobolev exponents where $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $N \geq 3$, $0 \leq \mu < \overline{\mu} = (N-2)^2/4$, $\mathbf{\Omega} \subset \mathbb{R}^N$ an open bounded set. For $\lambda \in [0, \lambda_1)$ with λ_1 being the first eigenvalue of the operator $-\Delta - \mu/|x|^2$ with zero Dirichlet boundary condition, and for $f \in H_0^1(\mathbf{\Omega})^{-1} = H^{-1}$, $f \neq 0$, we show that (1.1) admits at least two distinct nontrivial solutions u_0 and u_1 in $H_0^1(\mathbf{\Omega})$. Furthermore, $u_0 \geq 0$ and $u_1 \geq 0$ whenever $f \geq 0$.

1. Introduction and main result. In this paper, we shall study the existence and multiplicity of nontrivial solutions of the critical elliptic problem

$$(1.1)\qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u + f & \text{in } \mathbf{\Omega}, \\ u = 0 & \text{on } \partial\mathbf{\Omega}, \end{cases}$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $N \geq 3$, $0 \leq \mu < \overline{\mu} = (N-2)^2/4$, $\mathbf{\Omega} \subset \mathbb{R}^N$ an open bounded set. For $\lambda \in [0, \lambda_1)$ with λ_1 being the first eigenvalue of the operator $-\Delta - \mu/|x|^2$ with zero Dirichlet boundary condition, and for $f \in H_0^1(\mathbf{\Omega})^{-1} = H^{-1}$, $f \neq 0$, satisfying

$$\|f\|_{H^{-1}} < C_N \left(1 - \frac{\lambda}{\lambda_1}\right)^{(N+2)/4} S_\mu^{N/4},$$

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where

$$C_N = (2^* - 2)\left(\frac{1}{2^* - 1}\right)^{(2^*-1)/(2^*-2)},$$

and

$$S_\mu = \inf\left\{ \int_{\mathbf{\Omega}} \left(|\nabla u|^2 - \frac{\mu u^2}{|x|^2}\right) : u \in H_0^1(\mathbf{\Omega}),\ |u|_{2^*} = 1\right\},$$

and where $\mathbf{\Omega}$ is a bounded domain in $\mathbb{R}^N$, $2^* = 2N/(N-2)$, $N \geq 3$, $0 \leq \mu < \overline{\mu} = (N-2)^2/4$, and $0 \leq \lambda < \lambda_1$ is a positive constant, where λ_1 is the first eigenvalue of the operator $-\Delta - \mu/|x|^2$ with zero Dirichlet boundary condition, $f \in H^{-1}$ satisfies a suitable condition and $f \neq 0$, and we denote the dual space of $H_1^0(\mathbf{\Omega})$ by H^{-1}.

The existence of solutions of the problems related to (1.1) has been studied extensively. The Hardy potential is critical in nonrelativistic quantum mechanics, as it represents an intermediate threshold between regular and singular potentials, for more details see [**14**]. Problem (1.1) was studied in [**5, 8, 16**] where $f = 0$, $\lambda \neq 0$, and many interesting results have been obtained. If $f \not\equiv 0$, $\mu = \lambda = 0$, Tarantello [**17**] established a possibly sharp estimate for the upper bound of the norm of f, under which problem (1.1) was proved to have at least two distinct solutions. For problem (1.1) on $\mathbb{R}^{\mathbb{N}}$ with $f \not\equiv 0$ and $\mu = \lambda = 0$, some similar results can be found in [**4, 10**] and the references therein. If $f \not\equiv 0$, $\mu \neq 0$ and $\lambda = 0$, (1.1) is a special case of the problem considered in [**18**]. Chen and Zhao [**9**] considered problem (1.1) with $\lambda = 0$ and f replaced by σf, and they proved the existence of two solutions for all $\sigma \in (0, \sigma^*)$ with $0 < \sigma^* < +\infty$, but they could not give an explicit estimate of σ^*.

Let λ_i, $i = 1, 2, \ldots$, be eigenvalues of operator $-\Delta - \mu/|x|^2$ with zero Dirichlet boundary conditions. In view of [**12, 13**], each eigenvalue λ_i is positive, isolated and has finite multiplicity, the smallest eigenvalue λ_1 is simple and $\lambda_i \to \infty$ as $i \to \infty$. Moreover, each L^2 normalized eigenfunction e_i corresponding to λ_i belongs to $H_1^0(\mathbf{\Omega})$, and e_1 is positive.

Consider the classic elliptic problems involving Hardy potential

$$\begin{cases} -\Delta u = \mu u/|x|^2 + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

For $0<\mu<\overline{\mu}$, setting $\beta=\sqrt{\overline{\mu}-\mu}$, Catrina and Wang **[7]** proved that all positive solutions of (1.2) are of the form $u_\varepsilon(x)=\varepsilon^{2-N/2}u(x/\varepsilon)$, $\varepsilon>0$, where

$$u(x)=\frac{C}{|x|^{(N-2)/2-\beta}(1+|x|^{4\beta/(N-2)})^{(N-2)/2}}$$

for an appropriate constant $C>0$. These solutions achieve S_μ, where

$$S_\mu=\inf\left\{\int_{\mathbb{R}^N}\left(|\nabla u|^2-\frac{\mu u^2}{|x|^2}\right):u\in H_0^1(\mathbb{R}^N),|u|_{2^*}=1\right\}.$$

It is well known that solutions of problem (1.1) are the critical points of the functional $I_\mu:H_0^1(\mathbf{\Omega})\to\mathbb{R}$ given by

$$I_\mu(u)=\frac{1}{2}\int_{\mathbf{\Omega}}\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}-\lambda u^2\right)dx-\frac{1}{2^*}\int_{\mathbf{\Omega}}|u|^{2^*}dx-\int_{\mathbf{\Omega}}fu.$$

We observe that $I_\mu(u)$ is bounded from below in the manifold:

$$\Lambda=\{u\in H_0^1(\mathbf{\Omega}):I'_\mu(u)u=0\}.$$

Thus, a natural question to ask is whether or not $I_\mu(u)$ achieves a minimum in Λ.

We assume that:

$$(*)\ \|f\|_{H^{-1}}<C_N\left(1-\frac{\lambda}{\lambda_1}\right)^{(N+2)/4}S_\mu^{N/4},\ C_N=(2^*-2)\left(\frac{1}{2^*-1}\right)^{(2^*-1)/(2^*-2)}.$$

In this paper, we take advantage of the method applied in **[6, 17]** and obtain at least two weak solutions in $H_0^1(\mathbf{\Omega})$.

Theorem 1.1. *Let $f\neq 0$ satisfy $(*)$. Then*

(1) $\inf_\Lambda I_\mu=c_0$ *is achieved at a point $u_0\in\Lambda$ which is a critical point of I_μ and $u_0\geq 0$ whenever $f\geq 0$;*
(2) u_0 *is a local minimum of I_μ and $\|u_0\|_\mu^2-\lambda u_0^2-(2^*-1)|u_0|_{2^*}^{2^*}\geq 0$.*

Similarly to the method used in **[17]**, we split Λ into three parts:

$$\Lambda^+=\{u\in\Lambda:\ \|u\|_\mu^2-\lambda u^2-(2^*-1)|u|_{2^*}^{2^*}>0\},$$
$$\Lambda^0=\{u\in\Lambda:\ \|u\|_\mu^2-\lambda u^2-(2^*-1)|u|_{2^*}^{2^*}=0\},$$
$$\Lambda^-=\{u\in\Lambda:\ \|u\|_\mu^2-\lambda u^2-(2^*-1)|u|_{2^*}^{2^*}<0\}.$$

It turns out that assumption $(*)$ implies $\Lambda^0 = \{0\}$ (see Lemma 2.3 below). Therefore, for $f \neq 0$, we obtain $u_0 \in \Lambda^+$, and consequently,

$$c_0 = \inf_{\Lambda} I_\mu = \inf_{\Lambda^+} I_\mu.$$

So we are led to investigate a second minimization problem, namely,

$$c_1 = \inf_{\Lambda^-} I_\mu.$$

Theorem 1.2. *Let $f \neq 0$ satisfy $(*)$,*

$$\beta > \min\left\{1, \max\left\{\frac{(N-2)^2}{2(N+2)}, \frac{N-2}{4}\right\}\right\}.$$

Then $c_1 > c_0$ and $c_1 = \inf_{\Lambda^-} I_\mu$ is achieved at a point $u_1 \in \Lambda^-$ which defines a critical point for I_μ. Furthermore, $u_1 \geq 0$ whenever $f \geq 0$.

As an immediate consequence of Theorem 1.1 and Theorem 1.2, we have the following conclusion.

Theorem 1.3. *Problem* (1.1) *has at least two weak solutions $u_0,\ u_1 \in H_0^1(\mathbf{\Omega})$ for $f \neq 0$ satisfying $(*)$. Moreover, $u_0 \geq 0,\ u_1 \geq 0$ for $f \geq 0$.*

The remainder of this paper is organized as follows. In Section 2, we obtain the first solution of (1.1) which is a local minimum of I_μ. In Section 3, we verify the PS condition and get the second solution of (1.1).

2. The first solution. Throughout this paper, we denote the norm of $L^p(\mathbf{\Omega})$ by $|u|_p = (\int_{\mathbf{\Omega}} |u|^p)^{1/p}$. Denote the scalar product in $H_0^1(\mathbf{\Omega})$ by

$$\langle u,\ v\rangle_\mu = \int_{\mathbf{\Omega}} \left(\nabla u \nabla v - \frac{\mu}{|x|^2} uv\right) dx,$$

and the corresponding norm by $\|u\|_\mu = \langle u,\ u\rangle_\mu^{1/2}$. Note that $0 \leq \mu < \overline{\mu}$, and by the Hardy inequality,

$$\int_{\mathbf{\Omega}} \frac{u^2}{|x|^2}\, dx \leq \frac{1}{\overline{\mu}} \int_{\mathbf{\Omega}} |\nabla u|^2 dx \quad \text{for all } u \in H_0^1(\mathbf{\Omega}),$$

it is easy to see that $\|u\|_\mu$ is equivalent to the usual norm

$$\|u\| = \left(\int_{\mathbf{\Omega}} |\nabla u|^2 dx \right)^{1/2} \quad \text{on } H_0^1(\mathbf{\Omega}),$$

see **[8]**. Denote by $B_l(x)$ an open ball in $\mathbb{R}^{\mathbb{N}}$, which is concentrated at x with radius l.

In the following discussion, we denote various positive constants as C or C_i, $i = 0, 1, 2, 3, \ldots$, for convenience.

Since $0 \le \lambda < \lambda_1$ and $\lambda_1 = \inf_{u \neq 0} \|u\|_\mu^2 / |u|_2^2$, we can obtain

$$\int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2} - \lambda u^2 \ge \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_\mu^2,$$

so that $\|u\|_\mu^2$ is equivalent to $\int_\Omega |\nabla u|^2 - \mu(u^2/|x|^2) - \lambda u^2$.

To obtain the main results, several preliminary lemmas are in order.

Lemma 2.1. *Let $f \neq 0$ satisfy $(*)$. For every $u \in H_0^1(\mathbf{\Omega})$, $u \neq 0$, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+ u \in \Lambda^-$. In particular,*

$$t^+ > t_{\max} = \left(\frac{\|u\|_\mu^2 - \lambda |u|_2^2}{(2^* - 1)|u|_{2^*}^{2^*}} \right)^{1/(2^*-2)}$$

and

$$I_\mu(t^+ u) = \max_{t \ge t_{\max}} I_\mu(tu).$$

Moreover, if $\int_{\mathbf{\Omega}} fu > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^- u \in \Lambda^+$. In particular,

$$t^- < \left(\frac{\|u\|_\mu^2 - \lambda |u|_2^2}{(2^* - 1)|u|_{2^*}^{2^*}} \right)^{1/(2^*-2)}$$

and $I_\mu(t^- u) \le I_\mu(tu)$, for all $t \in [0, t^+]$.

Proof. Set $\varphi(t) = t(\|u\|_\mu^2 - \lambda |u|_2^2) - t^{2^*-1} |u|_{2^*}^{2^*}$. Easy computations show that φ is concave and achieves its maximum at

$$t_{\max} = \left(\frac{\|u\|_\mu^2 - \lambda |u|_2^2}{(2^* - 1)|u|_{2^*}^{2^*}} \right)^{1/(2^*-2)}.$$

And

$$\varphi(t_{\max}) = \left(\frac{1}{2^*-1}\right)^{(2^*-1)/(2^*-2)}(2^*-2)\left(\frac{(\|u\|_\mu^2 - \lambda|u|_2^2)^{2^*-1}}{|u|_{2^*}^{2^*}}\right)^{1/(2^*-2)},$$

so that

$$\varphi(t_{\max}) = C_N \frac{(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4}}{|u|_{2^*}^{N/2}}.$$

Therefore, if $\int_{\Omega} fu \leq 0$, then there exists a unique $t^+ > t_{\max}$ such that $\varphi(t^+) = \int_{\Omega} fu$ and $\varphi'(t^+) < 0$. Equivalently, $t^+u \in \Lambda^-$ and $I(t^+u) \geq I(tu)$, for all $t \geq t_{\max}$.

If $\int_{\Omega} fu > 0$, by assumption $(*)$, we have that

$$\begin{aligned}\int_{\Omega} fu < C_N \frac{((1-(\lambda/\lambda_1))\|u\|_\mu^2)^{(N+2)/4}}{|u|_{2^*}^{N/2}} &\leq \varphi(t_{\max}) \\ &= C_N \frac{(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4}}{|u|_{2^*}^{N/2}}.\end{aligned}$$

Consequently, we have a unique $0 < t^- < t_{\max} < t^+$ such that

$$\varphi(t^-) = \int_{\Omega} fu = \varphi(t^+)$$

and

$$\varphi'(t^-) > 0 > \varphi'(t^+).$$

Equivalently, $t^+u \in \Lambda^-$ and $t^-u \in \Lambda^+$. Also, we have $I(t^+u) \geq I(tu)$ for all $t \geq t^-$ and $I(t^-u) \leq I(tu)$ for all $t \in [0, t^+]$. □

Lemma 2.2. *If f satisfies $(*)$, then*

$$C_2 := \inf_{|u|_{2^*}=1}\left(C_N(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4} - \int_{\Omega} fu\right) > 0.$$

Proof. For $u \in H_0^1(\mathbf{\Omega})$ with $|u|_{2^*} = 1$, we have that

$$\begin{aligned}
&C_N(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4} - \int_{\mathbf{\Omega}} fu \\
&\quad\geq C_N(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4} - \|f\|_{H^{-1}}\|u\|_\mu \\
&\quad> \left(C_N\Big(1 - \frac{\lambda}{\lambda_1}\Big)^{(N+2)/4}\|u\|_\mu^{N/2} - C_N\Big(1 - \frac{\lambda}{\lambda_1}\Big)^{(N+2)/4} S_\mu^{N/4} + \xi_0\right)\|u\|_\mu \\
&\quad> \frac{1}{2}\|u\|_\mu \xi_0 > 0,
\end{aligned}$$

where ξ_0 is some positive constant. This completes the proof. $\square$

For $u \neq 0$, set

$$\psi(u) = C_N \frac{(\|u\|_\mu^2 - \lambda|u|_2^2)^{(N+2)/4}}{|u|_{2^*}^{N/2}} - \int_{\mathbf{\Omega}} fu.$$

Fixing $\nu > 0$, it follows from Lemma 2.2 that

$$\inf_{|u|_{2^*} \geq \nu} \psi(u) \geq C_2 \nu.$$

Lemma 2.3. *Let f satisfy $(*)$. For every $u \in \Lambda$, $u \neq 0$, we have*

$$\|u\|_\mu^2 - \lambda|u|_2^2 - (2^* - 1)|u|_{2^*}^{2^*} \neq 0,$$

i.e., $\Lambda_0 = \{0\}$.

Proof. Arguing by contradiction, we assume that, for some $u \in \Lambda$, $u \neq 0$,

$$\|u\|_\mu^2 - \lambda|u|_2^2 - (2^* - 1)|u|_{2^*}^{2^*} = 0,$$

which implies

$$|u|_{2^*} \geq \left(\Big(1 - \frac{\lambda}{\lambda_1}\Big)\frac{S_\mu}{2^* - 1}\right)^{1/(2^*-2)} = \nu_0.$$

For $u \in \Lambda$, we have

$$0 = \|u\|_\mu^2 - \lambda|u|_2^2 - |u|_{2^*}^{2^*} - \int_{\mathbf{\Omega}} fu = (2^* - 2)|u|_{2^*}^{2^*} - \int_{\mathbf{\Omega}} fu.$$

By Lemma 2.2,

$$
\begin{aligned}
0 &< C_2\nu_0 \le \psi(u)\\
&= \left(\frac{1}{2^*-1}\right)^{(2^*-1)/(2^*-2)}(2^*-2)\left(\frac{(\|u\|_\mu^2-\lambda|u|_2^2)^{2^*-1}}{|u|_{2^*}^{2^*}}\right)^{1/(2^*-2)}-\int_{\mathbf{\Omega}} fu\\
&= (2^*-2)\left(\left(\frac{1}{2^*-1}\right)^{(2^*-1)/(2^*-2)}\left(\frac{(\|u\|_\mu^2-\lambda|u|_2^2)^{2^*-1}}{|u|_{2^*}^{2^*}}\right)^{1/(2^*-2)}-|u|_{2^*}^{2^*}\right)\\
&= (2^*-2)|u|_{2^*}^{2^*}\left(\left(\frac{\|u\|_\mu^2-\lambda|u|_2^2}{(2^*-1)|u|_{2^*}^{2^*}}\right)^{2^*-1/2^*-2}-1\right)=0,
\end{aligned}
$$

which yields a contradiction. □

As a consequence of Lemma 2.3, we obtain the next lemma.

Lemma 2.4. *Let $f \neq 0$ satisfy $(*)$. Given $u \in \Lambda$, $u \neq 0$, there are a $\delta > 0$ and a differentiable function $t = t(v) > 0$, $v \in H$, $\|v\|_\mu < \delta$, satisfying*

$$t(0) = 1, \qquad t(v)(u-v) \in \Lambda, \quad \textit{for } \|v\|_\mu < \delta,$$

and

$$\langle t'(0),\ v\rangle = \frac{2\int_{\mathbf{\Omega}}(\nabla u\nabla v-\mu(uv/|x|^2)-\lambda uv) - 2^*\int_{\mathbf{\Omega}}|u|^{2^*-2}uv-\int_{\mathbf{\Omega}} fv}{\|u\|_\mu^2-\lambda|u|_2^2-(2^*-1)|u|_{2^*}^{2^*}}. \tag{2.1}$$

Proof. Define $F : \mathbb{R}\times H_0^1(\mathbf{\Omega}) \to \mathbb{R}$ as follows:

$$F(t,\, v) = t(\|u-v\|_\mu^2 - \lambda|u-v|_2^2) - t^{2^*-1}|u-v|_{2^*}^{2^*} - \int_{\mathbf{\Omega}} f(u-v).$$

Since $F(1,0) = 0$, and by Lemma 2.3, we have $F_t(1,0) = \|u\|_\mu^2 - \lambda|u|_2^2 - (2^*-1)|u|_{2^*}^{2^*} \neq 0$, we can apply the implicit function theorem at the point $(1,0)$ to obtain the result. □

Next, we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. We now show that I_μ is bounded from below in Λ. Indeed, for $u \in \Lambda$, we have

$$\int_{\mathbf{\Omega}} |\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2 - \int_{\mathbf{\Omega}}|u|^{2^*} - \int_{\mathbf{\Omega}} fu = 0,$$

so that

$$\begin{aligned}
I_\mu(u) &= \frac{1}{2}\int_{\mathbf{\Omega}} (|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2)\,dx - \frac{1}{2^*}\int_{\mathbf{\Omega}} |u|^{2^*}\,dx - \int_{\mathbf{\Omega}} fu \\
&= \frac{1}{N}\int_{\mathbf{\Omega}} \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2\right) - \left(1 - \frac{1}{2^*}\right)\int_{\mathbf{\Omega}} fu \\
&\geq \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|_\mu^2 - \frac{N+2}{2N}\|f\|_{H^{-1}}\|u\|_\mu \\
&\geq -\frac{\lambda_1}{16N(\lambda_1 - \lambda)}\left((N+2)\|f\|_{H^{-1}}\right)^2.
\end{aligned}$$

In particular,

$$c_0 \geq -\frac{\lambda_1}{16N(\lambda_1 - \lambda)}\left((N+2)\|f\|_{H^{-1}}\right)^2.$$

In order to get an upper bound for c_0, let $w \in H_0^1(\mathbf{\Omega})$ be the unique solution for

$$-\Delta u - \mu\frac{u}{|x|^2} = f.$$

Therefore, for $f \neq 0$,

$$\int_{\mathbf{\Omega}} fw = \|w\|_\mu^2 > 0.$$

Set $t_0 = t^-(w) > 0$ as defined by Lemma 2.1. Then $t_0 w \in \Lambda^+$, and consequently,

$$\begin{aligned}
I_\mu(t_0 w) &= \frac{t_0^2}{2}(\|w\|_\mu^2 - \lambda|w|_2^2) - \frac{t_0^{2^*}}{2^*}|w|_{2^*}^{2^*} - t_0\int_{\mathbf{\Omega}} fw \\
&= -\frac{t_0^2}{2}(\|w\|_\mu^2 - \lambda|w|_2^2) + \frac{2^* - 1}{2^*}t_0^{2^*}|w|_{2^*}^{2^*} \\
&< -\frac{t_0^2}{N}(\|w\|_\mu^2 - \lambda|w|_2^2) \leq -\frac{t_0^2}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|w\|_\mu^2 \\
&= -\frac{t_0^2}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|f\|_{H^{-1}}^2.
\end{aligned}$$

This yields

$$c_0 < -\frac{t_0^2}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|f\|_{H^{-1}}^2 < 0. \tag{2.2}$$

By Ekeland's variational principle, see **[1]**, a minimizing sequence $\{u_n\} \subset \Lambda$ of the minimization problem $\inf_\Lambda I_\mu = c_0$ exists such that

(i) $I_\mu(u_n) < c_0 + 1/n$;
(ii) $I_\mu(v) \geq I_\mu(u_n) - (1/n)\|v - u_n\|_\mu$, for all $v \in \Lambda$.

Taking n large enough, from (2.2), we obtain

$$\begin{aligned} I_\mu(u_n) &= \frac{1}{N}\int_{\mathbf{\Omega}}\left(|\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2} - \lambda|u|_2^2\right) - \frac{N+2}{2N}\int_{\mathbf{\Omega}} f u_n \\ &< c_0 + \frac{1}{n} < -\frac{t_0^2}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|f\|_{H^{-1}}^2. \end{aligned} \tag{2.3}$$

This implies

$$\int_{\mathbf{\Omega}} f u_n \geq \frac{2}{N+2} t_0^2\left(1 - \frac{\lambda}{\lambda_1}\right)\|f\|_{H^{-1}}^2 > 0. \tag{2.4}$$

Consequently, $u_n \neq 0$, and combining (2.3) and (2.4), we derive for n large,

$$\frac{2t_0^2}{N+2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|f\|_{H^{-1}} \leq \|u_n\|_\mu \leq \frac{N+2}{2}\left(\frac{\lambda_1}{\lambda_1 - \lambda}\right)\|f\|_{H^{-1}}. \tag{2.5}$$

Proposition 2.5. *$\|I'_\mu(u_n)\| \to 0$ as $n \to +\infty$.*

Proof. Since $u_n \in \Lambda$, by Lemma 2.4, we can find $\varepsilon_n > 0$ and a differentiable functional $t_n = t_n(v) > 0$, $v \in H_0^1(\mathbf{\Omega})$, $\|v\|_\mu < \varepsilon_n$ such that

$$w_n = t_n(v)(u_n - v) \in \Lambda \quad \text{for } \|v\|_\mu < \varepsilon_n.$$

By the continuity of $t_n(v)$ and $t_n(0) = 1$, without loss of generality, we can assume that ε_n satisfies $1/2 \leq t_n(v) \leq 3/2$ for $\|v\|_\mu < \varepsilon_n$.

It follows from condition (ii) that

$$I_\mu(t_n(v)(u_n - v)) - I_\mu(u_n) \geq -\frac{1}{n}\|t_n(v)(u_n - v) - u_n\|_\mu,$$

that is,

$$\begin{aligned} &\langle I'_\mu(u_n),\, t_n(v)(u_n - v) - u_n\rangle + o(\|t_n(v)(u_n - v) - u_n\|_\mu) \\ &\qquad\qquad \geq -\frac{1}{n}\|t_n(v)(u_n - v) - u_n\|_\mu. \end{aligned}$$

Consequently,

$$\begin{aligned} &t_n(v)\langle I'_\mu(u_n), v\rangle + (1-t_n(v))\langle I'_\mu(u_n), u_n\rangle \\ &\qquad \le \frac{1}{n}\|(t_n(v)-1)u_n - t_n(v)v\|_\mu + o(\|t_n(v)(u_n-v)-u_n\|_\mu). \end{aligned}$$

By the choice of ε_n, we obtain

$$\begin{aligned} \langle I'_\mu(u_n), v\rangle \le \frac{C}{n}|\langle t'_n(0), v\rangle| + o(\|v\|_\mu) \\ + \frac{1}{n}\|v\|_\mu + o(|\langle t'_n(0), v\rangle|\|u_n\|_\mu + \|v\|_\mu). \end{aligned} \tag{2.6}$$

If we can prove that

$$|\langle t'_n(0), v\rangle| \le \|v\|_\mu, \tag{2.7}$$

then, from (2.6), we get

$$\langle I'_\mu(u_n), v\rangle \le \frac{C}{n}\|v\|_\mu + \frac{1}{n}\|v\|_\mu + o(\|v\|_\mu) \quad \text{for } \|v\|_\mu \le \varepsilon_n.$$

Hence, for any $0 < \varepsilon < \varepsilon_n$, we have

$$\|I'_\mu(u_n)\| = \frac{1}{\varepsilon}\sup_{\|v\|_\mu=\varepsilon}\langle I'_\mu(u_n), v\rangle \le \frac{C}{n} + \frac{1}{\varepsilon}o(\varepsilon), \tag{2.8}$$

for some $C > 0$ independent of ε and n. Taking $\varepsilon \to 0$, we obtain $\|I'_\mu(u_n)\| \to 0$ as $n \to +\infty$.

We now turn to proving (2.7). Indeed, by (2.1), we have

$$\begin{aligned} \langle t'_n(0), v\rangle &< \frac{2\int_{\mathbf{\Omega}}|\nabla u\nabla v - \mu(uv/|x|^2)| + 2^*\int_{\mathbf{\Omega}}|u|^{2^*-1}|v| + |\int_{\mathbf{\Omega}} fv|}{|\|u\|_\mu^2 - \lambda|u|_2^2 - (2^*-1)|u|_{2^*}^{2^*}|} \\ &\le \frac{(2\|u_n\|_\mu + 2^*\|u_n\|_\mu^{2^*-1} + \|f\|_{H^{-1}})\|v\|_\mu}{|\|u\|_\mu^2 - \lambda|u|_2^2 - (2^*-1)|u|_{2^*}^{2^*}|}. \end{aligned}$$

Noting (2.5), in order to prove (2.7), we only need to show that

$$|\|u\|_\mu^2 - \lambda|u|_2^2 - (2^*-1)|u|_{2^*}^{2^*}| > \rho, \tag{2.9}$$

for some $\rho > 0$ and n large. We argue by way of contradiction. Assume that, for a subsequence, still called $\{u_n\}$, we have

$$|\|u\|_\mu^2 - \lambda|u|_2^2 - (2^*-1)|u|_{2^*}^{2^*}| = o(1). \tag{2.10}$$

From estimates (2.5) and (2.10) we derive

$$|u_n|_{2^*} \geq \nu > 0$$

and

$$\left(\frac{\|u_n\|_\mu^2 - \lambda |u|_2^2}{2^* - 1}\right)^{(2^*-1)/(2^*-2)} - (|u_n|_{2^*}^{2^*})^{(2^*-1)/(2^*-2)} = o(1).$$

By (2.10) and the fact that $u_n \in \Lambda$, we obtain

$$\int_{\mathbf{\Omega}} f u_n = (2^* - 2)|u_n|_{2^*}^{2^*} + o(1).$$

The above equality, together with Lemma 2.2, implies

$$\begin{aligned}
0 &< C_2 \nu^{(N+2)/2} \\
&\leq |u_n|_{2^*}^{2^*/(2^*-2)} \psi(u) \\
&= |u_n|_{2^*}^{2^*/(2^*-2)} \left(C_N \frac{(\|u_n\|_\mu^2 - \lambda |u|_2^2)^{(N+2)/4}}{|u_n|_{2^*}^{N/2}} - \int_{\mathbf{\Omega}} f u_n \right) \\
&= (2^* - 2)\left(\left(\frac{\|u_n\|_\mu^2 - \lambda |u|_2^2}{2^* - 1}\right)^{(2^*-1)/(2^*-2)} - (|u_n|_{2^*}^{2^*})^{(2^*-1)/(2^*-2)} \right) \\
&= o(1),
\end{aligned}$$

which is impossible. So we conclude that

$$\|I'_\mu(u_n)\| \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{2.11}$$

□

Let $u_0 \in H_0^1(\mathbf{\Omega})$ be the weak limit of u_n. By equation (2.4), the following holds:

$$\int_{\mathbf{\Omega}} f u_0 > 0,$$

and, from (2.11), we have

$$\langle I'_\mu(u_0), v \rangle = 0, \quad \text{for all } v \in H_0^1(\mathbf{\Omega}),$$

i.e., u_0 is a weak solution for (1.1). Therefore, $u_0 \in \Lambda$, and hence,

$$c_0 \leq I_\mu(u_0) = \frac{1}{N}(\|u_0\|_\mu^2 - \lambda |u_0|_2^2) - \frac{N+2}{2N} \int_{\mathbf{\Omega}} f u_0 \leq \lim_{n \to +\infty} I_\mu(u_n) = c_0.$$

Consequently, by the above equation, $u_n \to u_0$ strongly in $H_0^1(\mathbf{\Omega})$ and $I_\mu(u_0) = c_0 = \inf_\Lambda I_\mu$. Also, from Lemma 2.1 and (2.11) it is necessarily that $u_0 \in \Lambda^+$, see **[6]**.

Next, we claim that u_0 is a local minimum of I_μ. For every $u \in H_0^1(\mathbf{\Omega})$ with $\int_{\mathbf{\Omega}} fu > 0$, from Lemma 2.1, we have

$$I_\mu(su) \geq I_\mu(t^- u)$$

for every

$$0 < s < \left(\frac{\|u\|_\mu^2 - \lambda|u|_2^2}{(2^* - 1)|u|_{2^*}^{2^*}}\right)^{1/(2^*-2)}.$$

In particular, for $u = u_0 \in \Lambda^+$, we have

$$t^- = 1 < \left(\frac{\|u_0\|_\mu^2 - \lambda|u|_2^2}{(2^* - 1)|u_0|_{2^*}^{2^*}}\right)^{1/(2^*-2)}. \tag{2.12}$$

Let $\delta > 0$ be sufficiently small so that

$$1 < \frac{\|u_0 - v\|_\mu^2 - \lambda|u - v|_2^2}{(2^* - 1)|u_0 - v|_{2^*}^{2^*}}$$

for $\|v\|_\mu < \delta$. From Lemma 2.4, let $t(v) > 0$ be such that $t(v)(u_0 - v) \in \Lambda$ for every $\|v\|_\mu < \delta$. Since $t(v) \to 1$ as $\|v\|_\mu \to 0$, we can always assume that

$$t(v) < \left(\frac{\|u_0 - v\|_\mu^2 - \lambda|u - v|_2^2}{(2^* - 1)|u_0 - v|_{2^*}^{2^*}}\right)^{1/(2^*-2)}$$

for every $\|v\|_\mu < \delta$. By the above inequality, $t(v)(u_0 - v) \in \Lambda^+$, and for

$$0 < s < \left(\frac{\|u_0 - v\|_\mu^2 - \lambda|u - v|_2^2}{(2^* - 1)|u_0 - v|_{2^*}^{2^*}}\right)^{1/(2^*-2)},$$

we can obtain

$$I_\mu(u_0) \leq I_\mu(t(v)(u_0 - v)) \leq I_\mu(s(u_0 - v)).$$

By equation (2.12), we can take $s = 1$, and obtain

$$I_\mu(u_0) \leq I_\mu(u_0 - v), \quad \text{for all } v \in H, \ \|v\|_\mu < \delta,$$

so that u_0 is a local minimum for I_μ.

Furthermore, if $f \geq 0$, take $t_0 = t^-(|u_0|) > 0$ with $t_0|u_0| \in \Lambda^+$, we also easily see from (2.12) that $t_0 \geq 1$, and Lemma 2.1 gives that

$$I_\mu(u_0) \leq I_\mu(t_0|u_0|) \leq I_\mu(|u_0|) \leq I_\mu(u_0),$$

so we can always take $u_0 \geq 0$. By the maximum principle for a weak solution, see [**15,** Theorem 8.19], we can show that, if $f \geq 0$, $f \not\equiv 0$, then $u_0 > 0$ in $\mathbb{R}^{\mathbb{N}}$. $\square$

3. The second solution. Now, we will illustrate that I_μ satisfies the (P.S) condition at the levels below some constant.

Proposition 3.1. *Every sequence* $\{u_n\} \subset H_0^1(\mathbf{\Omega})$ *satisfying*

(a) $I_\mu(u_n) \to c$ *with* $c < c_0 + (1/N)S_\mu^{N/2}$, *where* c_0 *is defined as in Theorem* 1.1 (1);
(b) $\|I'_\mu(u_n)\| \to 0$

has a convergent subsequence.

Proof. By the standard method, it is easy to get that $\|u_n\|_\mu$ is uniformly bounded. Going, if necessary, to a subsequence, called u_n, we can assume that

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\mathbf{\Omega}).$$

And, according to condition (b), we have

$$\langle I'_\mu(u), v\rangle = 0, \quad \text{for all } v \in H_0^1(\mathbf{\Omega}).$$

That means that u is a weak solution for (1.1). In particular, $u \neq 0$, $u \in \Lambda$ and $I_\mu(u) \geq c_0$.

Let $u_n = u + v_n$ with $v_n \rightharpoonup 0$ weakly in $H_0^1(\mathbf{\Omega})$. According to [**2,** Lemma], we have

$$|u_n|_{2^*}^{2^*} = |u + v_n|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + |v_n|_{2^*}^{2^*} + o(1).$$

Hence, taking n large enough that

$$\begin{aligned} c_0 + \frac{1}{N} S_\mu^{N/2} &> I_\mu(u+v_n) \\ &= I_\mu(u) + \frac{1}{2}(\|v_n\|_\mu^2 - \lambda |v_n|_2^2) - \frac{1}{2^*}|v_n|_{2^*}^{2^*} + o(1) \\ &\geq c_0 + \frac{1}{2}(\|v_n\|_\mu^2 - \lambda |v_n|_2^2) - \frac{1}{2^*}|v_n|_{2^*}^{2^*} + o(1), \end{aligned}$$

gives

$$\frac{1}{2}\|v_n\|_\mu^2 - \frac{1}{2^*}|v_n|_{2^*}^{2^*} < \frac{1}{N} S_\mu^{N/2} + o(1). \tag{3.1}$$

And, from (b), the following holds

$$\begin{aligned} o(1) &= \langle I'_\mu(u_n), u_n \rangle \\ &= \|u\|_\mu^2 - \lambda |u|_2^2 - |u|_{2^*}^{2^*} - \int_{\boldsymbol{\Omega}} fu + \|v_n\|_\mu^2 \\ &\quad - \lambda |v_n|_2^2 - |v_n|_{2^*}^{2^*} + o(1) \\ &= \langle I'_\mu(u), u \rangle + \|v_n\|_\mu^2 - \lambda |v_n|_2^2 - |v_n|_{2^*}^{2^*} + o(1), \end{aligned}$$

and by the fact $\langle I'_\mu(u), u \rangle = 0$ and $|v_n|_2^2 = o(1)$, we obtain

$$\|v_n\|_\mu^2 - |v_n|_{2^*}^{2^*} = o(1). \tag{3.2}$$

Now, we claim that conditions (3.1) and (3.2) hold simultaneously if and only if a subsequence $\{v_{n_k}\}$ of $\{v_n\}$, converges strongly to zero, i.e., $\|v_{n_k}\|_\mu^2 \to 0$, as $k \to +\infty$.

Arguing by contradiction, assume that $\|v_{n_k}\|_\mu^2$ is bounded away from zero, that is, for some constant $C_3 > 0$, $\|v_{n_k}\|_\mu^2 \geq C_3$ holds for all $n \in \mathbb{N}$.

From (3.2), it follows that

$$\|v_n\|_{2^*}^{2^*-2} \geq S_\mu + o(1);$$

therefore,

$$\|v_n\|_{2^*}^{2^*} \geq S_\mu^{N/2} + o(1).$$

This and (3.1) and (3.2) yield, for n large,

$$\frac{1}{N} S_\mu^{N/2} \leq \frac{1}{N}\|v_n\|_{2^*}^{2^*} + o(1) = \frac{1}{2}\|v_n\|_\mu^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + o(1) < \frac{1}{N} S_\mu^{N/2},$$

which is a contradiction. In conclusion, $u_n \to u$ strongly. $\square$

At this point, it would not be difficult to derive Theorem 1.2, if we had the inequality

$$\inf_{\Lambda^-} I_\mu = c_1 < c_0 + \frac{1}{N} S_\mu^{N/2}.$$

We shall obtain it by comparison with a mountain-pass value. In order to get this result recall $u_0 \neq 0$. Following **[3]**, we let $\Sigma \subset \Omega$ be a set of positive measures such that $u_0 > 0$ on Σ (replace u_0 with $-u_0$ and f with $-f$, if necessary).

Let $\eta \in C_0^\infty(\Omega)$, with $\eta(x) \geq 0$ and $\eta(x) = 1$ in a neighborhood of $x = 0$. Set

$$U_\varepsilon(x) = \eta(x) u_\varepsilon(x), \qquad V_\varepsilon(x) = \frac{U_\varepsilon(x)}{|U_\varepsilon(x)|_{2^*}}, \quad x \in \mathbb{R}^{\mathbb{N}},$$

where $u_\varepsilon(x)$ and $\eta(x)$ are defined as before. Then, we have the following estimate, see **[5]** and **[11]**,

$$\int_{\mathbb{R}^{\mathbb{N}}} \left(|\nabla V_\varepsilon|^2 - \mu \frac{V_\varepsilon^2}{|x|^2} \right) dx = S_\mu + O(\varepsilon^{2\beta});$$

$$\int_{\mathbb{R}^{\mathbb{N}}} |V_\varepsilon|^2 = \begin{cases} O(\varepsilon^2) & \beta > 1, \\ O(\varepsilon^{2\beta} |\ln \varepsilon|) & \beta = 1, \\ O(\varepsilon^{2\beta}) & \beta < 1, \end{cases}$$

$$\frac{t^2}{2} \int_{\mathbf{\Omega}} \left(|\nabla V_\varepsilon|^2 - \frac{\mu}{|x|^2} V_\varepsilon^2 \right) - \frac{t^{2^*}}{2^*} \int_{\mathbf{\Omega}} |V_\varepsilon|^{2^*} \leq \frac{1}{N} S_\mu^{N/2} + O(\varepsilon^{2\beta}).$$

Lemma 3.2. *Assume that* $\beta > \min\{1, \max\{(N-2)^2/(2(N+2)), (N-2)/4\}\}$*, for every* $t > 0$*, and almost every* $a \in \Sigma$*,* $\varepsilon_0 = \varepsilon_0(t, a) > 0$ *exists such that*

$$I_\mu(u_0 + tV_\varepsilon) < c_0 + \frac{1}{N} S_\mu^{N/2},$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. By direct calculation, we obtain

$$
\begin{aligned}
I_\mu(u_0+tV_\varepsilon) &= \frac{1}{2}\int_{\mathbf{\Omega}}\left(|\nabla(u_0+tV_\varepsilon)|^2-\frac{\mu}{|x|^2}(u_0+tV_\varepsilon)^2-\lambda(u_0+tV_\varepsilon)^2\right)\\
&\quad -\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0+tV_\varepsilon|^{2^*}-\int_{\mathbf{\Omega}}f(u_0+tV_\varepsilon)\\
&=\frac{1}{2}\int_{\mathbf{\Omega}}\left(|\nabla u_0|^2-\frac{\mu}{|x|^2}u_0^2-\lambda u_0^2\right)-\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0|^{2^*}-\int_{\mathbf{\Omega}}fu_0\\
&\quad +\frac{t^2}{2}\int_{\mathbf{\Omega}}\left(|\nabla V_\varepsilon|^2-\frac{\mu}{|x|^2}V_\varepsilon^2\right)\\
&\quad -\frac{t^{2^*}}{2^*}\int_{\mathbf{\Omega}}|V_\varepsilon|^{2^*}-\frac{t^2}{2}\int_{\mathbf{\Omega}}\lambda V_\varepsilon^2-\int_{\mathbf{\Omega}}ftV_\varepsilon\\
&\quad +\int_{\mathbf{\Omega}}\left(\nabla u_0\nabla tV_\varepsilon-\frac{\mu}{|x|^2}u_0tV_\varepsilon-\lambda u_0tV_\varepsilon\right)\\
&\quad -\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0+tV_\varepsilon|^{2^*}\\
&\quad +\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0|^{2^*}+\frac{t^{2^*}}{2^*}\int_{\mathbf{\Omega}}|V_\varepsilon|^{2^*}.
\end{aligned}
$$

We know that, if $t\to\infty$, then $I_\mu(u_0+tV_\varepsilon)\to-\infty$, so we assume that t is in a bounded set. Because u_0 is a solution of (1.1), the following holds:

$$
\int_{\mathbf{\Omega}}\nabla u_0\nabla(tV_\varepsilon)-\frac{\mu}{|x|^2}u_0tV_\varepsilon-\lambda u_0tV_\varepsilon=\int_{\mathbf{\Omega}}|u_0|^{2^*-1}tV_\varepsilon+\int_{\mathbf{\Omega}}ftV_\varepsilon.
$$

So,

$$
\begin{aligned}
I_\mu(u_0+tV_\varepsilon) &= I_\mu(u_0)+\frac{t^2}{2}\int_{\mathbf{\Omega}}\left(|\nabla V_\varepsilon|^2-\frac{\mu}{|x|^2}V_\varepsilon^2\right)\\
&\quad -\frac{t^{2^*}}{2^*}\int_{\mathbf{\Omega}}|V_\varepsilon|^{2^*}-\frac{t^2}{2}\int_{\mathbf{\Omega}}\lambda V_\varepsilon^2\\
&\quad +\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0|^{2^*}+\frac{t^{2^*}}{2^*}\int_{\mathbf{\Omega}}|V_\varepsilon|^{2^*}+\int_{\mathbf{\Omega}}|u_0|^{2^*-1}tV_\varepsilon\\
&\quad -\frac{1}{2^*}\int_{\mathbf{\Omega}}|u_0+tV_\varepsilon|^{2^*}.
\end{aligned}
$$

By this estimate and the result ([**11,** Lemma 4.1]), we have

$$\frac{t^2}{2}\int_{\mathbf{\Omega}}(|\nabla V_\varepsilon|^2-\frac{\mu}{|x|^2}V_\varepsilon^2)-\frac{t^{2^*}}{2^*}\int_{\mathbf{\Omega}}|V_\varepsilon|^{2^*}\le\frac{1}{N}S_\mu^{N/2}+O(\varepsilon^{2\beta}).$$

And, for $u_0, t, V_\varepsilon > 0$, we have the inequality

$$|u_0+tV_\varepsilon|^{2^*}>u_0^{2^*}+2^*u_0^{2^*-1}tV_\varepsilon+2^*u_0(tV_\varepsilon)^{2^*-1}+(tV_\varepsilon)^{2^*},$$

so that we obtain

$$\begin{aligned}I_\mu(u_0+tV_\varepsilon)<c_0&+\frac{1}{N}S_\mu^{N/2}+O(\varepsilon^{2\beta})-\int_{\mathbf{\Omega}}u_0(tV_\varepsilon)^{2^*-1}\\&-\begin{cases}O(\varepsilon^2) & \beta>1,\\ O(\varepsilon^{2\beta}|\ln\varepsilon|) & \beta=1,\\ O(\varepsilon^{2\beta}) & \beta<1.\end{cases}\end{aligned}$$

Next, let us estimate $\int_{\mathbf{\Omega}}u_0(tV_\varepsilon)^{2^*-1}$. Since t belongs to a bounded set and $V_\varepsilon=U_\varepsilon/|U_\varepsilon|_{2^*}$, so we directly estimate $\int_{\mathbf{\Omega}}u_0U_\varepsilon^{2^*-1}$. Set $u_0=0$ outside Ω and $\eta(x)=1$ in Ω; by the form of u_ε, it follows that

$$\begin{aligned}\int_{\mathbf{\Omega}}u_0U_\varepsilon^{2^*-1}&=\int_{\mathbb{R}^{\mathbb{N}}}u_0\eta(x)u_\varepsilon^{2^*-1}\\&=C\int_{\mathbb{R}^{\mathbb{N}}}u_0\eta(x)\frac{\varepsilon^{(N+2)/4}}{|x|^{((N-2)/2-\beta)(2^*-1)}(\varepsilon+|x|^{4\beta/(N-2)})^{(N+2)/2}}\,dx\\&=C\int_{\mathbb{R}^{\mathbb{N}}}u_0\eta(x)\frac{\varepsilon^{-(N+2)/4}}{\varepsilon^{((N-2)/(4\beta))((N-2)/2-\beta)(2^*-1)}}\\&\qquad\cdot\frac{1}{|x/\varepsilon^{(N-2)/(4\beta)}|^{((N-2)/2-\beta)(2^*-1)}}\\&\qquad\cdot\frac{1}{(1+|x/\varepsilon^{(N-2)/(4\beta)}|^{(4\beta)/N-2)})^{(N+2)/2}}\,dx\\&=C\int_{\mathbb{R}^{\mathbb{N}}}u_0\eta(x)\frac{\varepsilon^{-(N+2)/4}}{\varepsilon^{((N-2)/(4\beta))((N-2)/2-\beta)(2^*-1)}}\psi\left(\frac{x}{\varepsilon^{(N-2)/(4\beta)}}\right)dx\\&=C\varepsilon^{(N-2)^2/(8\beta)}\int_{\mathbb{R}^{\mathbb{N}}}u_0\eta(x)\frac{1}{\varepsilon^{((N-2)/(4\beta))N}}\psi\left(\frac{x}{\varepsilon^{(N-2)/(4\beta)}}\right)dx,\end{aligned}$$

where

$$\psi(x)=\frac{1}{|x|^{((N-2)/2-\beta)(2^*-1)}(1+|x|^{4\beta/(N-2)})^{(N+2)/2}}.$$

Claim. $\psi(x) \in L^1(\mathbb{R}^{\mathbb{N}})$. We know that

$$\int_{\mathbb{R}^{\mathbb{N}}} \psi(x)\,dx = \int_{B_1(0)} \psi(x)dx + \int_{B_1^C(0)} \psi(x)\,dx.$$

Firstly, we consider

$$\begin{aligned}\int_{B_1(0)} \psi(x)\,dx &< \int_{B_1(0)} \frac{1}{|x|^{((N-2)/2-\beta)(2^*-1)}}\,dx \\ &= C\int_0^1 \frac{\rho^{N-1}}{\rho^{((N-2)/2-\beta)(2^*-1)}}\,d\rho \\ &= C\int_0^1 \rho^{N-1-((N-2)/2-\beta)(2^*-1)}\,d\rho \\ &= C\rho^{N-((N-2)/2-\beta)(2^*-1)}|_0^1,\end{aligned}$$

when $N-((N-2)/2-\beta)(2^*-1) > 0$, that is, $\beta > -(N-2)^2/2(N+2)$, so we obtain that

$$\int_{B_1(0)} \psi(x)\,dx < +\infty.$$

Secondly, we consider

$$\begin{aligned}\int_{B_1^C(0)} \psi(x)\,dx &< \int_{B_1^C(0)} \frac{1}{|x|^{((N-2)/2-\beta)(2^*-1)}|x|^{2\beta(N+2)/(N-2)}}\,dx \\ &= C\int_1^{+\infty} \frac{\rho^{N-1}}{\rho^{((N-2)/2-\beta)(2^*-1)+(2\beta(N+2)/(N-2))}}\,d\rho \\ &= C\int_1^{+\infty} \rho^{N-1-((N-2)/2-\beta)(2^*-1)-(2\beta(N+2)/(N-2))}\,d\rho \\ &= C\rho^{N-((N-2)/2-\beta)(2^*-1)-(2\beta(N+2)/(N-2))}|_1^{+\infty},\end{aligned}$$

when

$$N - \left(\frac{N-2}{2} - \beta\right)(2^*-1) - \frac{2\beta(N+2)}{N-2} < 0,$$

that is,

$$\beta > \frac{(N-2)^2}{2(N+2)},$$

so we obtain that

$$\int_{B_1^C(0)} \psi(x)\,dx < +\infty.$$

In conclusion, we obtain that, when $\beta > (N-2)^2/2(N+2)$, $\psi(x)$ is L^1 integrable. Therefore, setting

$$\alpha = \int_{\mathbb{R}^N} \frac{1}{|x|^{((N-2)/2-\beta)(2^*-1)}(1+|x|^{4\beta/(N-2)})^{(N+2)/2}}\, dx,$$

we have

$$\int_{\mathbb{R}^N} u_0\eta(x)\frac{1}{\varepsilon^{((N-2)/(4\beta))N}}\psi\bigg(\frac{x}{\varepsilon^{(N-2)/(4\beta)}}\bigg)\, dx \longrightarrow u_0(a)\alpha$$

for almost every $a \in \Sigma$. In other words,

$$\int_{\mathbf{\Omega}} u_0(U_\varepsilon)^{2^*-1} = C\varepsilon^{(N-2)^2/(8\beta)}u_0(a)\alpha + o(\varepsilon^{(N-2)^2/(8\beta)}).$$

Consequently,

$$\begin{aligned} I_\mu(u_0+tV_\varepsilon) < c_0 + \frac{1}{N}S_\mu^{N/2} &+ O(\varepsilon^{2\beta}) \\ &- C\varepsilon^{(N-2)^2/(8\beta)}u_0(a)\alpha + o(\varepsilon^{(N-2)^2/(8\beta)}) \\ &- \begin{cases} O(\varepsilon^2) & \beta > 1, \\ O(\varepsilon^{2\beta}|\ln\varepsilon|) & \beta = 1, \\ O(\varepsilon^{2\beta}) & \beta < 1. \end{cases} \end{aligned}$$

Therefore, if $\beta > 1$, so that without consideration of $\int_{\mathbf{\Omega}} u_0(tV_\varepsilon)^{2^*-1}$, we have $I_\mu(u_0+tV_\varepsilon) < c_0 + (1/N)S_\mu^{N/2}$. Otherwise, if $\beta > (N-2)/4$, then there is a $2\beta > (N-2)^2/(8\beta)$.

When we take $\beta > m = \max\{(N-2)^2/(2(N+2)), (N-2)/4\}$, $I_\mu(u_0+tV_\varepsilon) < c_0+(1/N)S_\mu^{N/2}$ holds. In the end, under the assumption of $\beta > \min\{1, m\}$,

$$I_\mu(u_0+tV_\varepsilon) < c_0 + \frac{1}{N}S_\mu^{N/2}, \tag{3.3}$$

holds for all $0 < \varepsilon < \varepsilon_0$. □

Our aim is to state a mountain pass theorem that produces a value which is below the threshold $c_0 + (1/N)S_\mu^{N/2}$ but also compares with the value $c_1 = \inf_{\Lambda^-} I_\mu$. To this end, observe that, under assumption $(*)$, the manifold Λ^- disconnects $H_0^1(\mathbf{\Omega})$ into exactly two connected components U_1 and U_2. To see this, note that, for every $u \in H_0^1(\mathbf{\Omega})$,

$\|u\| = (\|u\|_\mu^2 - \lambda |u|_2^2)^{1/2} = 1$, and by Lemma 2.1, we can find a unique $t^+(u) > 0$ such that

$$t^+(u)u \in \Lambda^-, \qquad I_\mu(t^+(u)u) = \max_{t \geq t_{\max}} I_\mu(tu).$$

The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of u. Set

$$U_1 = \left\{ u = 0 \text{ or } u : \|u\| < t^+\left(\frac{u}{\|u\|}\right)\right\}$$

and

$$U_2 = \left\{ u : \|u\| > t^+\left(\frac{u}{\|u\|}\right)\right\}.$$

Clearly, $H_0^1(\mathbf{\Omega}) \setminus \Lambda^- = U_1 \cup U_2$ and $\Lambda^+ \subset U_1$, in particular, $u_0 \subset U_1$.

Proof of Theorem 1.2. An easy computation shows that, for a suitable constant $C_4 > 0$,

$$0 < t^+(u) < C_4, \quad \text{for all } \|u\| = 1,\ |u|_{2^*} > \delta_1 > 0.$$

Since

$$\frac{|u_0 + t_0 V_\varepsilon|_{2^*}}{\|u_0 + t_0 V_\varepsilon\|} \geq \frac{|V_\varepsilon|_{2^*}}{2\|V_\varepsilon\|} \geq \frac{1}{2(S_\mu + O(\varepsilon^{2\beta}))^{1/2}}$$

for t_0 sufficiently large, we can choose

$$t_0 > \left(\frac{C_4^2 - \|u_0\|^2}{(1 - \lambda/\lambda_1)\, S_\mu}\right)^{1/2} + 1$$

large enough, $\varepsilon_0 > 0$, $\delta_1 > 0$ small enough such that $w_\varepsilon = u_0 + t_0 V_\varepsilon$ satisfies $|w_\varepsilon / \|w_\varepsilon\||_{2^*} > \delta_1$ for all $0 < \varepsilon < \varepsilon_0$. Since

$$\begin{aligned} \|w_\varepsilon\|^2 &= \|u_0 + t_0 V_\varepsilon\|^2 \\ &\geq \|u_0\|^2 + t_0^2\left(1 - \frac{\lambda}{\lambda_1}\right) S_\mu + o(1) \\ &> C_4^2 > \left(t^+\left(\frac{w_\varepsilon}{\|w_\varepsilon\|}\right)\right)^2, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small, we get

$$w_\varepsilon = u_0 + t_0 V_\varepsilon \in U_2. \tag{3.4}$$

For such a choice of t_0, fix $\varepsilon > 0$ such that (3.3) and (3.4) hold for all $0 < \varepsilon < \varepsilon_0$. Set

$$\Gamma = \{\gamma \in C([0,1],\, H_0^1(\mathbf{\Omega})) : \gamma(0) = u_0,\, \gamma(1) = u_0 + t_0 V_\varepsilon(x)\}.$$

Clearly, $\gamma :\ [0,1] \to H_0^1(\mathbf{\Omega})$ given by $\gamma(s) = u_0 + st_0V_\varepsilon$ belongs to Γ. So, by Lemma 3.2, we conclude

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I_\mu(\gamma(s)) < c_0 + \frac{1}{N} S_\mu^{N/2}. \tag{3.5}$$

Also, since the range of $\gamma \in \Gamma$ intersects Λ^-, we have

$$c_1 = \inf_{\Lambda^-} I_\mu \leq c. \tag{3.6}$$

Similar to the proof of Theorem 1.1, we can show that Ekeland's variational principle gives a sequence $\{u_n\} \subset \Lambda^-$ satisfying

$$I_\mu(u_n) \longrightarrow c_1,$$

and

$$\|I'_\mu(u_n)\| \longrightarrow .$$

Furthermore, from (3.5) and (3.6), we have

$$c_1 < c_0 + \frac{1}{N} S_\mu^{N/2}.$$

Therefore, by Lemma 3.2, we obtain a subsequence of $\{u_n\}$, called $\{u_n\}$, and $u_1 \in H_0^1(\mathbf{\Omega})$ such that

$$u_n \longrightarrow u_1 \text{ strongly in } H_0^1(\mathbf{\Omega}).$$

Consequently, u_1 is c critical point for I_μ, and, since Λ^- is closed, we have $u_1 \in \Lambda^-$ and $I_\mu(u_1) = c_1$.

Lastly, we assume that $f \geq 0$ and $f \not\equiv 0$. Let $t^+ > 0$ be such that

$$t^+|u_1| \in \Lambda^-.$$

According to Lemma 2.1, we obtain

$$I_\mu(t^+|u_1|) \leq I_\mu(t^+u_1) \leq \max_{t\geq t_{\max}} I_\mu(tu_1) = I_\mu(u_1).$$

Therefore, we can always take $u_1 \geq 0$. By the maximum principle for weak solutions, see [**15,** Theorem 8.19], we can show that, if $f \geq 0$, $f \not\equiv 0$, then $u_1 > 0$ in $\mathbb{R}^{\mathbb{N}}$. $\square$

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