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R-DUALITY IN G-FRAMES

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ABSTRACT. Recently, the concept of g-Riesz dual sequences for g-Bessel sequences has been introduced. In this paper, we investigate under what conditions a g-Riesz sequence $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ is the g-Riesz dual sequence of a given g-frame $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$.

1. Introduction and preliminaries. Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [**6**] in 1952 to study some deep questions in non-harmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [**5**], and popularized from then on. Frames are generalizations of bases in Hilbert spaces. A frame such as an orthonormal basis allows each element in the underlying Hilbert space to be written as an unconditionally convergent linear combination of the frame elements; however, in contrast to a basis, the coefficients might not be unique. Frames have been used in signal processing, image processing, data compression, filter bank theory, sigma-delta quantization, and wireless communications.

G-frame, introduced by Sun [**14**], is a generalization of a frame which covers many extensions of frames, e.g., pseudo-frames, outer frames, oblique frames, continuous frames, fusion frames, and a class of time-frequency localization operators.

The concept of Riesz dual sequences (R-dual sequences) for Bessel sequences in a separable Hilbert space was introduced by Casazza, Kutyniok and Lammers [**2**], in order to obtain a generalization of the Ron-Shen duality principle [**12**] and the Wexler-Raz biorthogonality relations [**15**] to abstract frame theory.

Let $(e_i)_{i\in\mathcal{I}}, (h_i)_{i\in\mathcal{I}}$ be orthonormal bases for H, and let $(f_i)_{i\in\mathcal{I}}$ be a Bessel sequence in H. The Riesz dual sequence (the R-dual sequence)

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of $(f_i)_{i\in\mathcal{I}}$ with respect to the orthonormal bases $(e_i)_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ is the sequence $(w_j)_{j\in\mathcal{I}}$, such that, for every $j\in\mathcal{I}$,

$$w_j=\sum_{i\in\mathcal{I}}\langle f_i,e_j\rangle h_i.$$

R-duality has been favored by many authors. R-duality with respect to orthonormal bases has been discussed in [**2, 3, 4**]. In [**13**], the authors introduced various alternative R-duals and showed their relations with Gabor frames. In [**7**], the authors proved that the duality principle extends to any dual pairs of projective unitary representations of countable groups.

In [**11**], the authors introduced the concept of g-Riesz dual sequences (g-R-dual sequences) for g-Bessel sequences. In this paper, for a given g-frame $\Lambda=\{\Lambda_i\in L(H,H_i):i\in\mathcal{I}\}$, a given g-Riesz sequence $\Phi=\{\Phi_i\in L(H,H_i):i\in\mathcal{I}\}$, and a given g-orthonormal basis $\Gamma=\{\Gamma_i\in L(H,H_i):i\in\mathcal{I}\}$, we introduce a new sequence

$$(\Pi_i)_{i\in\mathcal{I}}\in(L(H,H_i))_{i\in\mathcal{I}}$$

that can be used to check whether or not Φ is the g-Riesz dual of Λ. Then we study the relation between $(\Pi_i)_{i\in\mathcal{I}}$ and $(\Lambda_i)_{i\in\mathcal{I}}$. Also, we show how Parseval g-frame sequences can be dilated to g-orthonormal bases for H. Then, we investigate under what conditions Φ is the g-Riesz dual sequence of Λ. Throughout this paper, H denotes a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, $\mathcal{I}$ denotes a countable index set and $\{H_i:i\in\mathcal{I}\}$ is a sequence of separable Hilbert spaces. Also, for every $i\in\mathcal{I}$, $L(H,H_i)$ is the set of all bounded, linear operators from H to H_i.

In the rest of this section we review several well-known definitions and results. The new results are stated in Section 2.

For every sequence $\{H_i\}_{i\in\mathcal{I}}$, the space

$$\left(\sum_{i\in\mathcal{I}}\bigoplus H_i\right)_{\ell^2}=\left\{(f_i)_{i\in\mathcal{I}}:f_i\in H_i,i\in\mathcal{I},\sum_{i\in\mathcal{I}}\|f_i\|^2<\infty\right\}$$

with pointwise operations and the following inner product is a Hilbert space

$$\langle(f_i)_{i\in\mathcal{I}},(g_i)_{i\in\mathcal{I}}\rangle=\sum_{i\in\mathcal{I}}\langle f_i,g_i\rangle.$$

A sequence

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

is called a g-*frame* for H with respect to $\{H_i : i \in \mathcal{I}\}$, if there exist $0 < A \le B < \infty$ such that, for every $f \in H$,

$$A\|f\|^2 \le \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \le B\|f\|^2,$$

A, B are called g-frame bounds. We call Λ a *tight* g-*frame* if $A = B$ and a *Parseval* g-*frame* if $A = B = 1$. If only the right-hand inequality is required, Λ is called a g-*Bessel sequence*. We simply call Λ a g-frame for H whenever the space sequence $\{H_i : i \in \mathcal{I}\}$ is clear.

We say that

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

is a g-*frame sequence*, if it is a g-frame for

$$\overline{\operatorname{span}\{\Lambda_i^*(H_i)\}_{i\in\mathcal{I}}}.$$

If Λ is a g-Bessel sequence, then the *synthesis operator* for Λ is the linear operator,

$$T_\Lambda : \left(\sum_{i\in\mathcal{I}} \bigoplus H_i\right)_{\ell^2} \longmapsto H, \qquad T_\Lambda(f_i)_{i\in\mathcal{I}} = \sum_{i\in\mathcal{I}} \Lambda_i^* f_i.$$

We call the adjoint of the synthesis operator the *analysis operator*. The analysis operator is the linear operator,

$$T_\Lambda^* : H \longmapsto \left(\sum_{i\in\mathcal{I}} \bigoplus H_i\right)_{\ell^2}, \qquad T_\Lambda^* f = (\Lambda_i f)_{i\in\mathcal{I}}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the g-*frame operator* of Λ. If $\Lambda = (\Lambda_i)_{i\in\mathcal{I}}$ is a g-frame with lower and upper g-frame bounds A and B, respectively, then the g-frame operator of Λ is a bounded, positive, and invertible operator on H, and

$$S_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Lambda_i f,$$

$$A\langle f, f\rangle \le \langle S_\Lambda f, f\rangle \le B\langle f, f\rangle, \quad f \in H,$$

so

$$AI \le S_\Lambda \le BI.$$

The canonical dual g-frame for $(\Lambda_i)_{i\in\mathcal{I}}$ is defined by $(\widetilde{\Lambda_i})_{i\in\mathcal{I}} = (\Lambda_i S_\Lambda^{-1})_{i\in\mathcal{I}}$, which is also a g-frame for H with $1/B$ and $1/A$ as its lower and upper frame bounds, respectively. Also, for every $f\in H$, we have

$$f = \sum_{i\in\mathcal{I}} \Lambda_i^* \widetilde{\Lambda_i} f = \sum_{i\in\mathcal{I}} \widetilde{\Lambda_i}^* \Lambda_i f.$$

All of the g-frames

$$\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\},$$

which satisfy

$$\sum_{i\in\mathcal{I}} \Lambda_i^* \Gamma_i f = f, \quad \text{for all } f \in H,$$

are called dual g-frames of Λ.

A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is g-*complete*, if $\{f : \Lambda_i f = 0, \text{ for all } i \in \mathcal{I}\} = \{0\}$ and we call it a g-*orthonormal basis* for H, if

$$\langle \Lambda_i^* f_i, \Lambda_j^* f_j \rangle = \delta_{i,j} \langle f_i, f_j \rangle,$$

for all $f_i \in H_i$, $f_j \in H_j$, $i, j \in \mathcal{I}$ and

$$\sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 = \|f\|^2 \quad \text{for all } f \in H.$$

A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is a g-*Riesz sequence* if there exist $0 < A \le B < \infty$ such that, for every finite subset $F \subset \mathcal{I}$, $f_i \in H_i$, and $i \in F$,

$$A \sum_{i\in F} \|g_i\|^2 \le \left\| \sum_{i\in F} \Lambda_i^* g_i \right\|^2 \le B \sum_{i\in F} \|g_i\|^2.$$

The g-Riesz sequence

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

is called a g-*Riesz basis*, if it is g-complete, too.

Let

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

and

$$\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$$

be g-Bessel sequences with g-Bessel bounds B and C, respectively. The operator $S_{\Lambda\Gamma} : H \mapsto H$ defined by

$$S_{\Lambda\Gamma} f = \sum_{i \in \mathcal{I}} \Lambda_i^* \Gamma_i f, \quad f \in H,$$

is a bounded operator, $\|S_{\Lambda\Gamma}\| \leq \sqrt{BC}$, $S_{\Lambda\Gamma}^* = S_{\Gamma\Lambda}$ and $S_{\Lambda\Lambda} = S_\Lambda$.

For more details about g-frames, see **[8, 14]**.

2. Main results. In **[11]**, the authors introduced the concept of g-Riesz dual sequences for g-Bessel sequences as follows.

Definition 2.1 ([**11**]). Let

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

be a g-Bessel sequence for H, and let

$$\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\},$$
$$\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$$

be g-orthonormal bases for H. For every $j \in \mathcal{I}$, define

$$\Phi_j f = \sum_{i \in \mathcal{I}} \Gamma_j \Lambda_i^* \Upsilon_i f = \Gamma_j S_{\Lambda\Upsilon} f, \quad f \in H,$$

where Λ_i^* is the adjoint operator of Λ_i, for every $i \in \mathcal{I}$. $(\Phi_j)_{j \in \mathcal{I}}$ is called the g-*Riesz dual sequence* of Λ with respect to g-orthonormal bases Γ and Υ.

Lemma 2.2 ([**11**]). *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a* g-*Bessel sequence, and let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be the* g-*Riesz dual sequence of Λ with respect to* g-*orthonormal bases*

$$\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\},$$
$$\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}.$$

Then, for every $i \in \mathcal{I}$,

$$\Lambda_i f = \sum_{j \in \mathcal{I}} \Upsilon_i {\Phi_j}^* \Gamma_j f = \Upsilon_i S_{\Phi\Gamma} f, \quad f \in H, \tag{2.1}$$

that is, Λ *is the* g*-Riesz dual sequence of* Φ *with respect to* Υ *and* Γ*.*

Note that, with the assumptions of Lemma 2.2, we can easily conclude that Φ is the g-Riesz dual of Λ with respect to Γ and Υ if and only if Λ is the g-Riesz dual of Φ with respect to Υ and Γ.

Our first aim is to characterize the g-Riesz duals of a given g-Bessel sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$. By Lemma 2.2, the g-Riesz duals are precisely the sequences $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ for which we can find two g-orthonormal bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ bases for H such that (2.1) holds. On the other hand, by **[11,** Proposition 3.7], Λ is a g-frame for H with bounds A, B if and only if Φ is a g-Riesz sequence for H with bounds A, B. Thus we arrive at the following question:

> *Let* $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ *be a* g*-frame for* H *and* $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ *be a* g*-Riesz sequence for* H*. Under what conditions can we find* g*-orthonormal bases* $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ *for* H *such that* (2.1) *holds?*

We first show that, for a given g-Riesz sequence $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$, a given sequence $\Lambda = \{\Lambda_j \in L(H, H_j) : j \in \mathcal{I}\}$, and a given g-orthonormal basis $\Gamma = \{\Gamma_j \in L(H, H_j) : j \in \mathcal{I}\}$, we can characterize the sequences $\Upsilon = \{\Upsilon_j \in L(H, H_j) : j \in \mathcal{I}\}$ such that (2.1) holds. Then we investigate under what conditions at least one of these sequences forms a g-orthonormal basis for H.

Let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a g-Riesz sequence in H. Since Φ is a g-Riesz sequence, then it is a g-Riesz basis for $W = \overline{\mathrm{span}_{j \in \mathcal{I}} \Phi_j^*(H_j)}$. Let $\widetilde{\Phi} = \{\widetilde{\Phi}_j \in L(W, H_j) : j \in \mathcal{I}\}$ be the canonical dual of Φ. It is well known that $\widetilde{\Phi}$ is the unique dual g-frame of Φ, and $\widetilde{\Phi}$ is a g-Riesz basis for W.

Since H is a Hilbert space and W is a closed subspace of H, by **[9,** Corollary 1.0.4], for every $j \in \mathcal{I}$, there exists a $\Psi_j \in L(H, H_j)$ such that $\Psi_j(f) = \widetilde{\Phi_j}(f)$ for every $f \in W$ and $||\Psi_j|| = ||\widetilde{\Phi_j}||$. Replacing $\widetilde{\Phi_j}$ by Ψ_j, we can suppose that $\widetilde{\Phi_j} \in L(H, H_j)$, for every $j \in \mathcal{I}$.

Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a sequence. For every $i \in \mathcal{I}$, we define:

$$\Pi_i f = \Lambda_i S_{\Gamma\widetilde{\Phi}} f, \quad f \in H. \tag{2.2}$$

It is easy to check that Π_i is a well-defined operator and $\Pi_i \in L(H, H_i)$, for every $i \in \mathcal{I}$.

Theorem 2.3. *Let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a g-Riesz basis for $W = \overline{\operatorname{span}_{i\in\mathcal{I}} \Phi_i^*(H_i)}$ and with the canonical dual $\widetilde{\Phi}_j = \{\widetilde{\Phi}_j \in L(H, H_j) : j \in \mathcal{I}\}$. Let $\Gamma = \{\Gamma_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a g-orthonormal basis for H, and let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a sequence. Then the following statements hold.*

(a) *There exists a sequence $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ such that*

$$\Lambda_i = \Upsilon_i S_{\Phi\Gamma}, \quad \text{for all } i \in \mathcal{I}. \tag{2.3}$$

(b) *The sequences satisfying* (2.3) *are characterized by*

$$\Upsilon_i = \Pi_i + \Theta_i, \tag{2.4}$$

where $\overline{(\Pi_i)_{i\in\mathcal{I}}}$ is given by (2.2)*, $\Theta_i \in L(H, H_i)$, and $W = \overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)} \subseteq \ker(\Theta_i)$, for every $i \in \mathcal{I}$.*

(c) *If Φ is a g-Riesz basis for H, then* (2.3) *has the unique solution*

$$\Upsilon_i = \Pi_i, \quad \text{for all } i \in \mathcal{I}.$$

Proof.

(a) Since Φ is a g-Riesz basis, then $\langle \widetilde{\Phi_j}^* g_j, \Phi_k^* g_k\rangle = \delta_{jk}\langle g_j, g_k\rangle$, for every $g_j \in H_j$, $g_k \in H_k$ and $j, k \in \mathcal{I}$. For every $f \in H$, $g_i \in H_i$, and $i \in \mathcal{I}$, we have

$$\begin{aligned}\langle \Pi_i S_{\Phi\Gamma} f, g_i\rangle &= \left\langle \sum_{j\in\mathcal{I}} \Phi_j^*\Gamma_j f, \Pi_i^* g_i \right\rangle = \left\langle \sum_{j\in\mathcal{I}} \Phi_j^*\Gamma_j f, \sum_{k\in\mathcal{I}} \widetilde{\Phi_k}^* \Gamma_k \Lambda_i^* g_i \right\rangle \\ &= \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \langle \Phi_j^*\Gamma_j f, \widetilde{\Phi_k}^* \Gamma_k\Lambda_i^* g_i\rangle = \sum_{j\in\mathcal{I}} \langle \Gamma_j f, \Gamma_j \Lambda_i^* g_i\rangle\end{aligned}$$

$$= \left\langle f, \sum_{j\in\mathcal{I}} \Gamma_j^*\Gamma_j\Lambda_i^* g_i \right\rangle = \langle f, \Lambda_i^* g_i\rangle = \langle \Lambda_i f, g_i\rangle.$$

Therefore, $\Pi_i S_{\Phi\Gamma} = \Lambda_i$, for every $i \in \mathcal{I}$. Hence, $\Upsilon_i = \Pi_i$ satisfies (2.3), for every $i \in \mathcal{I}$.

(b) Suppose that the sequence $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ satisfies (2.3). We can write, $\Upsilon_i = \Pi_i + \Theta_i$ with $\Theta_i = \Upsilon_i - \Pi_i$, for every $i \in \mathcal{I}$. Therefore, $\Theta_i \in L(H, H_i)$, for every $i \in \mathcal{I}$. By Lemma 2.2, for every $i \in \mathcal{I}$,

$$\Lambda_i = \Upsilon_i S_{\Phi\Gamma} = \Pi_i S_{\Phi\Gamma}.$$

This implies that, for every $i \in \mathcal{I}$,

$$S_{\Gamma\Phi}(\Upsilon_i^* - \Pi_i^*) = 0.$$

Since Γ is a g-orthonormal basis, the above relation implies that $\Phi_j(\Upsilon_i^* - \Pi_i^*) = 0$, which is equivalent to $(\Upsilon_i - \Pi_i)\Phi_j^* g_j = 0$, for every $g_j \in H_j$, and $i, j \in \mathcal{I}$.

Suppose that $x \in \operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)$. By definition of $\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)$, there exist a finite subset $F \subset \mathcal{I}$ and $\{g_j \in H_j : j \in F\}$ such that $x = \sum_{j\in F} \Phi_j^* g_j$. For every $i \in \mathcal{I}$, we have

$$\begin{aligned}\Theta_i(x) = (\Upsilon_i - \Pi_i)x &= (\Upsilon_i - \Pi_i)\sum_{j\in F} \Phi_j^* g_j \\ &= \sum_{j\in F} (\Upsilon_i - \Pi_i)\Phi_j^* g_j = 0.\end{aligned}$$

Since $\overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)} = W$ and Θ_i is continuous, then $\Theta_i(x) = 0$, for every $x \in W$. Thus, $W \subseteq \ker \Theta_i$, for every $i \in \mathcal{I}$.

(c) If $(\Phi_j)_{j\in\mathcal{I}}$ is a g-Riesz basis for H, then $W = H$. Therefore, $\Theta_i = 0$ and $\Upsilon_i = \Pi_i$ for every $i \in \mathcal{I}$. □

In the next proposition, we study the relation between $(\Pi_i)_{i\in\mathcal{I}}$ and $(\Lambda_i)_{i\in\mathcal{I}}$.

Proposition 2.4. *Let $\Gamma = \{\Gamma_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a* g-*orthonormal basis for H, and let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a* g-*Riesz basis for $W = \overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)}$ with* g-*Riesz bounds A_1, B_1, and with the canonical dual $\{\widetilde{\Phi}_j \in L(H, H_j) : j \in \mathcal{I}\}$. Let $\Lambda = \{\Lambda_i \in L(H, H_i) :$*

$i \in \mathcal{I}\}$ be a sequence and $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$. Then the following statements hold.

(a) *If Λ is a* g-*Bessel sequence for H with* g-*Bessel bound B, then $(\Pi_i)_{i\in\mathcal{I}}$ is a* g-*Bessel sequence for W with* g-*Bessel bound B/A_1.*

(b) *If Λ is a* g-*frame for H with* g-*frame bounds A and B, then $(\Pi_i)_{i\in\mathcal{I}}$ is a* g-*frame for W with* g-*frame bounds $A/B_1, B/A_1$.*

(c) *If Λ is a* g-*Bessel sequence for H, then for every $(g_i)_{i\in\mathcal{I}} \in (\sum_{i\in\mathcal{I}} \oplus H_i)_{\ell^2}$, we have*

$$\left\|\sum_{j\in\mathcal{I}} \Pi_j^* g\right\|^2 \le \frac{1}{A_1}\left\|\sum_{j\in\mathcal{I}} \Lambda_j^* g_j\right\|^2,$$

$$\left\|\sum_{j\in\mathcal{I}} \Lambda_j^* g_j\right\|^2 \le B_1\left\|\sum_{j\in\mathcal{I}} \Pi_j^* g_j\right\|^2.$$

(d) *If Λ is a* g-*Riesz basis for H with* g-*Riesz bounds A and B, then $(\Pi_i)_{i\in\mathcal{I}}$ is a* g-*Riesz for W with* g-*Riesz bounds $A/B_1, B/A_1$.*

Proof.

(a) Let Λ be a g-Bessel sequence for H with g-Bessel bound B. Since Φ is a g-Riesz basis for W with g-Riesz bounds A_1 and B_1, then $\widetilde{\Phi}$ is a g-Riesz basis W with g-Riesz bounds $1/B_1$ and $1/A_1$. Consequently, $\widetilde{\Phi}$ is a g-frame for W with bounds $1/B_1$ and $1/A_1$. For every $f \in W$, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 &= \sum_{i\in\mathcal{I}} \|\Lambda_i S_{\Gamma\widetilde{\Phi}} f\|^2 \le B\|S_{\Gamma\widetilde{\Phi}} f\|^2\\ &= B\left\|\sum_{j\in\mathcal{I}} \Gamma_j^* \widetilde{\Phi}_j f\right\|^2 = B\sum_{j\in\mathcal{I}} \|\widetilde{\Phi}_j f\|^2 \le \frac{B}{A_1}\|f\|^2.\end{aligned}$$

Therefore, $(\Pi_i)_{i\in\mathcal{I}}$ is a g-Bessel sequence for W with g-Bessel bound B/A_1.

(b) Let Λ be a g-frame for H with g-frame bounds A and B. Using (a) implies that $(\Pi_i)_{i\in\mathcal{I}}$ is a g-Bessel sequence for W with g-Bessel bound B/A_1. In order to complete the proof of (b) it is enough to prove that $(\Pi_i)_{i\in\mathcal{I}}$ satisfies the lower bound condition. For every $f \in W$, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}}\|\Pi_i f\|^2 &= \sum_{i\in\mathcal{I}}\|\Lambda_i S_{\Gamma\widetilde{\Phi}} f\|^2 \geq A\|S_{\Gamma\widetilde{\Phi}} f\|^2\\ &= A\Big\|\sum_{j\in\mathcal{I}}\Gamma_j^*\widetilde{\Phi}_j f\Big\|^2\\ &= A\sum_{j\in\mathcal{I}}\|\widetilde{\Phi}_j f\|^2 \geq \frac{A}{B_1}\|f\|^2.\end{aligned}$$

Therefore, $(\Pi_i)_{i\in\mathcal{I}}$ is a g-frame for W with bounds A/B_1 and B/A_1.

(c) Let $(g_i)_{i\in\mathcal{I}}\in(\sum_{i\in\mathcal{I}}\oplus H_i)_{\ell^2}$. We have

$$\begin{aligned}\Big\|\sum_{j\in\mathcal{I}}\Pi_j^* g_j\Big\|^2 &= \Big\|\sum_{j\in\mathcal{I}} S_{\widetilde{\Phi}\Gamma}\Lambda_j^* g_j\|^2 = \Big\|S_{\widetilde{\Phi}\Gamma}\sum_{j\in\mathcal{I}}\Lambda_j^* g_j\Big\|^2\\ &= \Big\|\sum_{k\in\mathcal{I}}\widetilde{\Phi_k}^*\Gamma_k\sum_{j\in\mathcal{I}}\Lambda_j^* g_j\Big\|^2\\ &\leq \frac{1}{A_1}\sum_{k\in\mathcal{I}}\Big\|\Gamma_k\sum_{j\in\mathcal{I}}\Lambda_j^* g_j\Big\|^2 \leq \frac{1}{A_1}\Big\|\sum_{j\in\mathcal{I}}\Lambda_j^* g_j\Big\|^2.\end{aligned}$$

By the proof of Theorem 2.3 (a), $\Lambda_i = \Pi_i S_{\Phi\Gamma}$, for every $i\in\mathcal{I}$. For every $(g_i)_{i\in\mathcal{I}}\in(\sum_{i\in\mathcal{I}}\oplus H_i)_{\ell^2}$, we have

$$\begin{aligned}\Big\|\sum_{j\in\mathcal{I}}\Lambda_j^* g_j\Big\|^2 &= \Big\|\sum_{j\in\mathcal{I}} S_{\Gamma\Phi}\Pi_j^* g_j\Big\|^2 = \Big\|S_{\Gamma\Phi}\sum_{j\in\mathcal{I}}\Pi_j^* g_j\Big\|^2\\ &= \Big\|\sum_{k\in\mathcal{I}}\Gamma_k^*\Phi_k\sum_{j\in\mathcal{I}}\Pi_j^* g_j\Big\|^2 = \sum_{k\in\mathcal{I}}\Big\|\Phi_k\sum_{j\in\mathcal{I}}\Pi_j^* g_j\Big\|^2\\ &\leq B_1\Big\|\sum_{j\in\mathcal{I}}\Pi_j^* g_j\Big\|^2.\end{aligned}$$

(d) Using (b) and (c), the claim is obvious. □

Proposition 2.5. *Let* $\Phi = \{\Phi_j\in L(H,H_j) : j\in\mathcal{I}\}$ *be a* g-*Riesz basis for* $W = \overline{\mathrm{span}_{j\in\mathcal{I}}\Phi_j^*(H_j)}$ *with* g-*Riesz bounds* A_1 *and* B_1*, and with the canonical dual* $\{\widetilde{\Phi}_j\in L(H,H_j) : j\in\mathcal{I}\}$*. Let*

$\Gamma = \{\Gamma_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a g*-orthonormal basis for H, and let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a sequence and $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$. Then the following statements hold.*

(a) *If $(\Pi_i)_{i\in\mathcal{I}}$ is a* g*-Bessel sequence for W with* g*-Bessel bound B, then Λ is a* g*-Bessel sequence for H with* g*-Bessel bound BB_1.*
(b) *If $(\Pi_i)_{i\in\mathcal{I}}$ is a* g*-frame for W with* g*-frame bounds A and B, then Λ is a* g*-frame for H with* g*-frame bounds AA_1, BB_1.*
(c) *If $(\Pi_i)_{i\in\mathcal{I}}$ is a* g*-Riesz basis for W with* g*-Riesz bounds A and B, then Λ is a* g*-Riesz for H with* g*-Riesz bounds AA_1, BB_1.*

Proof. By the proof of Theorem 2.3 (a), for every $f \in H$ and $i \in \mathcal{I}$, we have

$$\Lambda_i f = \Pi_i S_{\Phi\Gamma} f.$$

Now, the proof is similar to that of Proposition 2.4. Therefore, we omit it. □

Proposition 2.6. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a* g*-frame for H, and let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a* g*-Riesz basis for $W = \overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)}$, with the canonical dual $\{\widetilde{\Phi}_j \in L(H, H_j) : j \in \mathcal{I}\}$. Then there exists a* g*-orthonormal basis $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$ is a Parseval* g*-frame for W if and only if there exists a unitary operator $M : H \to W$ such that $S_\Phi = M S_\Lambda M^*$, where S_Λ and S_Φ are the* g*-frame operators for Λ and Φ, respectively.*

Proof. Suppose that there exists a g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$ is a Parseval g-frame for W. Then, for every $f \in W$, we have

$$\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 = \|f\|^2.$$

Since $(\widetilde{\Phi_j})_{j\in\mathcal{I}}$ is a g-Riesz basis for W, then $(\Upsilon_j)_{j\in\mathcal{I}} = (\widetilde{\Phi_j} S_{\widetilde{\Phi}}^{-1/2})_{j\in\mathcal{I}}$ is a g-orthonormal basis for W. Consider

$$M : H \longrightarrow W$$

by

$$Mf = S_{\Upsilon\Gamma} f = \sum_{i\in\mathcal{I}} \Upsilon_i^* \Gamma_i f, \quad f \in H.$$

Then $M^* = S_{\Gamma\Upsilon}$ and M is a unitary operator. By the definition of M^* we have

$$M^* S_{\widetilde{\Phi}}^{1/2} = S_{\Gamma\widetilde{\Phi}}.$$

For every $f \in W$, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 &= \sum_{i\in\mathcal{I}} \|\Lambda_i S_{\Gamma\widetilde{\Phi}} f\|^2 = \sum_{i\in\mathcal{I}} \langle \Lambda_i S_{\Gamma\widetilde{\Phi}} f, \Lambda_i S_{\Gamma\widetilde{\Phi}} f\rangle \\ &= \sum_{i\in\mathcal{I}} \langle \Lambda_i^* \Lambda_i S_{\Gamma\widetilde{\Phi}} f, S_{\Gamma\widetilde{\Phi}} f\rangle = \langle S_\Lambda S_{\Gamma\widetilde{\Phi}} f, S_{\Gamma\widetilde{\Phi}} f\rangle \\ &= \langle S_\Lambda M^* S_{\widetilde{\Phi}}^{1/2} f, M^* S_{\widetilde{\Phi}}^{1/2} f\rangle = \langle S_{\widetilde{\Phi}}^{1/2} M S_\Lambda M^* S_{\widetilde{\Phi}}^{1/2} f, f\rangle.\end{aligned}$$

Since $(\Pi_i)_{i\in\mathcal{I}}$ ia a Parseval g-frame, then, for every $f \in W$,

$$\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 = \|f\|^2.$$

Thus, for every $f \in W$,

$$\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 = \langle S_{\widetilde{\Phi}}^{1/2} M S_\Lambda M^* S_{\widetilde{\Phi}}^{1/2} f, f\rangle = \langle f, f\rangle.$$

So $S_{\widetilde{\Phi}}^{1/2} M S_\Lambda M^* S_{\widetilde{\Phi}}^{1/2} = I$ implies that $M S_\Lambda M^* = S_{\widetilde{\Phi}}^{-1}$. On the other hand, $S_{\widetilde{\Phi}}^{-1} = S_\Phi$; therefore, $S_\Phi = M S_\Lambda M^*$.

Conversely, suppose that there exists a unitary operator $M : H \to W$ such that $S_\Phi = M S_\Lambda M^*$. Define $\Gamma_i = \widetilde{\Phi_i} S_{\widetilde{\Phi}}^{-1/2} M$, for every $i \in \mathcal{I}$. Since $(\widetilde{\Phi_i})_{i\in\mathcal{I}}$ is a g-Riesz basis for W, then $(\widetilde{\Phi_i} S_{\widetilde{\Phi}}^{-1/2})_{i\in\mathcal{I}}$ is a g-orthonormal basis for W. On the other hand, M is a unitary operator; therefore, $(\Gamma_i)_{i\in\mathcal{I}}$ is a g-orthonormal basis for H. We can easily see that $S_{\Gamma\widetilde{\Phi}} = M^* S_{\widetilde{\Phi}}^{1/2}$. A calculation similar to the above relations implies that, for every $f \in W$,

$$\begin{aligned}\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 &= \sum_{i\in\mathcal{I}} \|\Lambda_i S_{\Gamma\widetilde{\Phi}} f\|^2 = \langle S_\Lambda S_{\Gamma\widetilde{\Phi}} f, S_{\Gamma\widetilde{\Phi}} f\rangle \\ &= \langle S_{\widetilde{\Phi}}^{1/2} M S_\Lambda M^* S_{\widetilde{\Phi}}^{1/2} f, f\rangle = \langle S_{\widetilde{\Phi}}^{1/2} S_\Phi S_{\widetilde{\Phi}}^{1/2} f, f\rangle\end{aligned}$$

$$= \langle S_{\widetilde{\Phi}}^{1/2} S_{\widetilde{\Phi}}^{-1} S_{\widetilde{\Phi}}^{1/2} f, f\rangle = \langle f, f\rangle = \|f\|^2.$$

Therefore, $(\Pi_i)_{i\in\mathcal{I}}$ is a Parseval g-frame for W. $\square$

In the next proposition, we show under what conditions a Parseval g-frame sequence can be dilated to a g-orthonormal basis for H.

Proposition 2.7. *Let* $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ *be a Parseval* g-*frame for* $W = \overline{\operatorname{span}_{i\in\mathcal{I}} \Lambda_i^*(H_i)}$. *Then there exists a* g-*orthonormal basis* $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ *for* H *such that* $\Lambda_i = \Gamma_i P$, *for every* $i \in \mathcal{I}$, *if and only if*

$$\dim \operatorname{Ker} T_\Lambda = \dim(W^\perp),$$

where P *is the orthogonal projection of* H *onto* W *and* T_Λ *is the synthesis operator of* Λ.

Proof. Let $\dim \operatorname{Ker} T_\Lambda = \dim(W^\perp)$. Suppose that $\{e_{jk} : k \in K_j\}$ is an orthonormal basis for H_j, where K_j is a subset of $\mathbb{Z}$, $j \in \mathcal{I}$ and $u_{jk} = \Lambda_j^* e_{jk}$. By [**14,** Theorem 3.1],

$$\Lambda_i f = \sum_{k\in K_i} \langle f, u_{ik}\rangle e_{ik},$$

where $(u_{ik})_{k\in K_i, i\in\mathcal{I}}$ is a Parseval frame for $W = \overline{\operatorname{span}_{i\in\mathcal{I}} \Lambda_i^*(H_i)}$. Let T be a the synthesis operator for $(u_{ik})_{k\in K_i, i\in\mathcal{I}}$. Then

$$\dim \operatorname{Ker} T = \dim \operatorname{Ker} T_\Lambda,$$

see [**1,** Theorem 2.3]. Therefore,

$$\dim \operatorname{Ker} T = \dim \operatorname{Ker} T_\Lambda = \dim(W^\perp).$$

By [**4,** Theorem 2], there exists an orthonormal basis $(\theta_{ik})_{k\in K_i, i\in\mathcal{I}}$ for H such that $u_{ik} = P\theta_{ik}$, where P is the orthogonal projection of H onto W. Let $\Gamma_i f = \sum_{k\in K_i} \langle f, \theta_{ik}\rangle e_{ik}$, for every $i \in \mathcal{I}$. Then $(\Gamma_i)_{i\in\mathcal{I}}$ is a g-orthonormal basis for H and

$$\Gamma_i P f = \sum_{k\in K_i} \langle Pf, \theta_{ik}\rangle e_{ik} = \sum_{k\in K_i} \langle f, u_{ik}\rangle e_{ik} = \Lambda_i f.$$

Conversely, suppose that there exists a g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that $\Lambda_i = \Gamma_i P$. For every $(g_i)_{i\in\mathcal{I}} \in (\sum_{i\in\mathcal{I}} \oplus H_i)_{\ell^2}$ we have

$$\sum_{i\in\mathcal{I}} \Lambda_i^* g_i = \sum_{i\in\mathcal{I}} P\Gamma_i^* g_i = P \sum_{i\in\mathcal{I}} \Gamma_i^* g_i.$$

Then $(g_i)_{i\in\mathcal{I}} \in \operatorname{Ker} T_\Lambda$ if and only if $\sum_{i\in\mathcal{I}} \Gamma_i^* g_i \in W^\perp$. It follows easily that $\dim \operatorname{Ker} T_\Lambda = \dim(W^\perp)$ □

In the next theorem, for a given g-frame $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$, a given g-Riesz sequence $\Phi = \{\Phi_i \in L(H, H_i) : i \in \mathcal{I}\}$ and a given g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, we characterize the existence of a g-orthonormal basis $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ such that Φ is the g-Riesz dual sequence of Λ with respect to Γ and Υ.

Theorem 2.8. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-frame for H, and let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-orthonormal basis for H. Let $\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$ be a Riesz basis for $W = \overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)}$, with the canonical dual $\{\widetilde{\Phi}_j \in L(H, H_j) : j \in \mathcal{I}\}$. Then Φ is the g-Riesz dual sequence of Λ with respect to Γ and some g-orthonormal basis $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ if and only if the following statements hold.*

(a) $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$ *is a Parseval g-frame for W.*
(b) $\dim \operatorname{Ker} T_\Lambda = \dim(W^\perp)$, *where T_Λ denotes the synthesis operator of Λ.*

Proof. Suppose that there is a g-orthonormal basis $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that Φ is the g-Riesz dual sequence of Λ with respect to Γ and Υ. Then, by Lemma 2.2,

$$\Lambda_i f = \Upsilon_i S_{\Phi\Gamma} f, \text{ for all } i \in \mathcal{I}, \text{ for all } f \in H.$$

By Theorem 2.3, $(\Upsilon_j)_{j\in\mathcal{I}}$ is characterized by

$$\Upsilon_i = \Pi_i + \Theta_i,$$

where

$$\Pi_i = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}, \qquad \Theta_i \in L(H, H_i) \quad \text{and} \quad W \subseteq ker(\Theta_i),$$

for every $i \in \mathcal{I}$. If P is the orthogonal projection of H onto W, then $\Upsilon_i P = \Pi_i P$, for every $i \in \mathcal{I}$. Since Υ is a g-orthonormal basis for H, for every $f \in W$, we have

$$\sum_{i\in\mathcal{I}} \|\Pi_i f\|^2 = \sum_{i\in\mathcal{I}} \|\Pi_i P f\|^2 = \sum_{i\in\mathcal{I}} \|\Upsilon_i P f\|^2 = \|f\|^2$$

Therefore, Π is a Parseval g-frame for W.

Let

$$(g_i)_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \oplus H_i\right)_{\ell^2}.$$

By [**11,** Lemma 3.6], $(g_i)_{i\in\mathcal{I}} \in \operatorname{Ker} T_\Lambda$ if and only if $\sum_{i\in\mathcal{I}} \Upsilon_i^* g_i \in \operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)^\perp = \overline{\operatorname{span}_{j\in\mathcal{I}} \Phi_j^*(H_j)}^\perp = W^\perp$. From this, it follows easily that

$$\dim \operatorname{Ker} T_\Lambda = \dim(W^\perp).$$

Conversely, suppose that (a) and (b) hold. Since $(\Pi_i)_{i\in\mathcal{I}}$ is a Parseval g-frame for W, by Proposition 2.7, there exists a g-orthonormal basis $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that $\Pi_i = \Upsilon_i P$, where P is the orthogonal projection of H onto W. We can write $\Upsilon_i = \Pi_i + \Theta_i$, with $\Theta_i = \Upsilon_i - \Pi_i$, for every $i \in \mathcal{I}$. For every $x \in W$, we have

$$\Theta_i(x) = \Upsilon_i P(x) - \Pi_i(x) = \Pi_i(x) - \Pi_i(x) = 0.$$

Thus, by Theorem 2.3, we have

$$\Lambda_i f = \Upsilon_i S_{\Phi\Gamma} f, \text{ for all } i \in \mathcal{I}, \text{ for all } f \in H,$$

that is, Λ is the g-Riesz dual sequence of Φ with respect to Υ and Γ. Now, by Lemma 2.2, Φ is the g-Riesz dual sequence of Λ with respect to Γ and Υ. □

Corollary 2.9. *Let*

$$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$$

be a g-*frame for* H, *and let*

$$\Phi = \{\Phi_j \in L(H, H_j) : j \in \mathcal{I}\}$$

be a Riesz basis for $W = \overline{\mathrm{span}_{j\in\mathcal{I}}\,\Phi_j^*(H_j)}$. *Then there exist* g-*orthonormal bases*

$$\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$$

and

$$\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$$

for H *such that* Φ *is the* g-*Riesz dual sequence of* Λ *with respect to* Γ *and* Υ *if and only if the following statements hold.*

(a) *There exists a unitary operator* M *such that* $S_\Phi = MS_\Lambda M^*$.
(b) $\dim\operatorname{Ker} T_\Lambda = \dim(W^\perp)$.

Proof. By Proposition 2.6, there exists a unitary operator M such that $S_\Phi = MS_\Lambda M^*$ if and only if there exists a g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ for H such that $(\Pi_i)_{i\in\mathcal{I}} = (\Lambda_i S_{\Gamma\widetilde{\Phi}})_{i\in\mathcal{I}}$ is a Parseval g-frame for W. Now, by Theorem 2.8, the claim is obvious. □

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