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ON THE COEFFICIENTS OF TRIPLE PRODUCT L-FUNCTIONS

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ABSTRACT. In this paper, we investigate the average behavior of coefficients of the triple product L-function $L(f\otimes f\otimes f, s)$ attached to a primitive holomorphic cusp form $f(z)$ of weight k for the full modular group $SL(2,\mathbb{Z})$. Here we call $f(z)$ a primitive cusp form if it is an eigenfunction of all Hecke operators simultaneously.

1. Introduction. Let $k\geq 2$ be an even integer. Denote by H_k^* the set of all normalized Hecke primitive cusp forms $f(z)$ of weight k for the full modular group $SL(2,\mathbb{Z})$. Here, and throughout this paper, we call $f(z)$ a primitive cusp form if it is an eigenfunction of all Hecke operators simultaneously. It is known that $f(z)$ has the following Fourier expansion at the cusp ∞,

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{(k-1)/2}e^{2\pi inz}, \quad \Im m z>0, \tag{1.1}$$

where we use $\lambda_f(n)$ to denote the normalized Fourier coefficients, i.e., coefficients which have been divided by $n^{(k-1)/2}$. According to Deligne [**4**], for any prime number p, there are two (complex) numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\alpha_f(p)\beta_f(p)=|\alpha_f(p)|=|\beta_f(p)|=1 \tag{1.2}$$

and

$$\lambda_f(p)=\alpha_f(p)+\beta_f(p). \tag{1.3}$$

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The coefficient $\lambda_f(n)$ is a real multiplicative function of n and satisfies the Deligne inequality,

$$|\lambda_f(n)| \leq d(n), \tag{1.4}$$

for all integers $n \geq 1$, where $d(n)$ is the divisor function.

Let $L(f,s)$ be the Hecke L-function attached to f, which is defined as:

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \big(1-\alpha_f(p)p^{-s}\big)^{-1}\big(1-\beta_f(p)p^{-s}\big)^{-1}, \tag{1.5}$$

for $\Re e\, s > 1$. In the literature, many researchers have investigated the average behavior of various sums concerning $\lambda_f(n)$, for instance, there is a long history on the investigation of the upper estimate for

$$S_f(x) := \sum_{n \leq x} \lambda_f(n). \tag{1.6}$$

In 1927, Hecke [**10**] proved that

$$S_f(x) \ll_f x^{1/2}. \tag{1.7}$$

Subsequent improvement was first given by Wilton [**40**] in which only the case of Ramanujan's τ-function was stated and later generalized by Walfisz [**39**] to other forms. Let θ be a constant satisfying

$$|\lambda_f(n)| \leq n^{\theta}. \tag{1.8}$$

Walfisz proved that

$$S_f(x) \ll_f x^{(1+\theta)/3}. \tag{1.9}$$

Then the works of Kloosterman [**19**], Davenport [**3**], Salié [**32**], Weil [**41**] and Deligne [**4**] on the exponent θ in (1.8) imply better corresponding results in (1.9). In 1989, Hafner and Ivić [**9**] were able to remove the factor x^{ε} of Deligne's result, i.e.,

$$S_f(x) \ll_f x^{1/3}. \tag{1.10}$$

Rankin [**31**] further proved that

$$S_f(x) \ll_f x^{1/3}(\log x)^{-0.0652}. \tag{1.11}$$

In this direction, the best known result is due to Wu [**42**], which states that

$$S_f(x) \ll_f x^{1/3}(\log x)^{-0.1185}. \tag{1.12}$$

For the second moment of $\lambda_f(n)$, Rankin and Selberg independently (see [**30, 33**]) proved that

$$\sum_{n\leq x}\lambda_f(n)^2 = Cx + O(x^{3/5}).$$

For work on the ℓth power sum of $\lambda_f(n)$

$$S_\ell(f;x) := \sum_{n\leq x}\lambda_f(n)^\ell,$$

see Moreno and Shahidi [**27**], Fomenko [**5**], Lü ([**23, 24, 25**]), Lau and Lü [**20**] and Lau, Lü and Wu [**21**].

The triple product L-function $L(f\otimes f\otimes f, s)$ satisfies analogous analytic properties such as those of the Hecke L-functions, and its coefficients also change signs. In this paper, we consider the average behavior of the coefficients $\lambda_{f\otimes f\otimes f}(n)$ of the triple product L-function $L(f\otimes f\otimes f, s)$. We prove:

Theorem 1.1. *For any* $\varepsilon > 0$*, we have*

$$\sum_{n\leq x}\lambda_{f\otimes f\otimes f}(n) \ll_{f,\varepsilon} x^{7/10+\varepsilon}. \tag{1.13}$$

Theorem 1.2. *For any* $\varepsilon > 0$*, we have*

$$\sum_{n\leq x}\lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x) + O_{f,\varepsilon}(x^{175/181+\varepsilon}), \tag{1.14}$$

where $P(t)$ *is a polynomial of degree* 4.

Remark 1.3. The triple product L-function is of degree 8, and the L-function associated with $\lambda_{f\otimes f\otimes f}(n)^2$ has degree 64. Therefore, general summation formulae (see, e.g., [**6,** Proposition 1.1 and Theorem 1.2]) imply

$$\sum_{n\le x}\lambda_{f\otimes f\otimes f}(n)\ll_{f,\varepsilon} x^{1-2/9+\varepsilon},$$
$$\sum_{n\le x}\lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x)+O_{f,\varepsilon}(x^{1-2/65+\varepsilon}).$$

These results largely mean that, for an L-function of degree m, the error term for the sum of its coefficients can be bounded by $x^{1-2/(m+1)+\varepsilon}$.

One can easily find that our results are better than these kinds of general results. The reason is that, in our case, the corresponding L-functions can be decomposed into products of some L-functions of smaller degrees. In principle, such factorizations are definitely helpful (see, e.g., [**6**]).

Theorem 1.4. *Let ℓ denote a positive integer. Then there exists a suitable positive constant c_ℓ such that*

$$\sum_{n\le x}\lambda_{f\otimes f\otimes f}(n)^{2\ell}\sim c_\ell x(\log x)^{\delta_\ell}, \tag{1.15}$$

where

$$\delta_\ell=\frac{1}{3\ell+1}\binom{6\ell}{3\ell}-1.$$

Recently many researchers have been interested in the study of $\mathrm{GL}(3)\times\mathrm{GL}(2)$ L-functions (see, e.g., [**2, 22**]). The $\mathrm{GL}(3)\times\mathrm{GL}(2)$ L-function $L(\mathrm{sym}^2 f\otimes f,s)$ (or $L(\mathrm{Ad}^2\, f\otimes f,s)$) is closely related to the triple product L-function $L(f\otimes f\otimes f,s)$. Similar to Theorems 1.1–1.4, we also have

Theorem 1.5. *Let $\lambda_{\mathrm{sym}^2 f\otimes f}(n)$ denote the nth coefficient of $L(\mathrm{sym}^2 f\otimes f,s)$ in its Dirichlet series expansion in the region of absolute convergence. Then, for any $\varepsilon>0$, we have*

$$\sum_{n\le x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)\ll_{f,\varepsilon} x^{2/3+\varepsilon} \tag{1.16}$$

and

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O_{f,\varepsilon}(x^{17/18+\varepsilon}), \tag{1.17}$$

where $Q(t)$ is a polynomial of degree 1.

Let ℓ denote a positive integer. Then a suitable positive constant d_ℓ exists such that

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^2 f \otimes f}(n)^{2\ell} \sim d_\ell x (\log x)^{\gamma_\ell - 1}, \tag{1.18}$$

where

$$\gamma_\ell = \sum_{k=0}^{2\ell} \binom{2\ell}{k} (-1)^{2\ell-k} \frac{1}{\ell+k+1} \binom{2\ell+2k}{\ell+k}.$$

Remark 1.6. Since, for any positive integer k, the integer $k+1$ divides $\binom{2k}{k}$, i.e.,

$$\frac{((k+1)+k-1)!}{(k+1)!k!} \in \mathbb{Z},$$

the numbers δ_ℓ in Theorem 1.4 and γ_ℓ in Theorem 1.5 are integers. The numbers $\delta_\ell + 1$ and γ_ℓ should agree with the expected order of the pole at $s = 1$ of the L-functions associated with such coefficients. For example, $\delta_1 + 1 = 5$ and $\gamma_1 = 2$ coincide with the order of the pole at $s = 1$ of the corresponding L-functions, respectively, see (2.4) and (6.3). For any $\ell \geq 2$, due to the absence of the corresponding Langlands functoriality results, we proved (1.15) and (1.18) by applying the Sato-Tate conjecture (now a theorem proved by Barnet-Lamb, Geraghty, Harris and Taylor [**1**]) instead.

2. Preliminaries and some lemmas. This section is devoted to recalling and establishing some preliminary results which we shall need in the proof of Theorems 1.1–1.4.

Let $f(z)$ be a normalized Hecke primitive eigencuspform of weight k for the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Recall that the triple product

L-function $L(f\otimes f\otimes f,s)$ is defined by

$$L(f\otimes f\otimes f,s)=\prod_p\left(1-\frac{\alpha_p^3}{p^s}\right)^{-1}\left(1-\frac{\alpha_p}{p^s}\right)^{-3}\left(1-\frac{\beta_p}{p^s}\right)^{-3}\left(1-\frac{\beta_p^3}{p^s}\right)^{-1}$$
$$=\sum_{n=1}^{\infty}\frac{\lambda_{f\otimes f\otimes f}(n)}{n^s},$$

for $\Re e\, s>1$. The jth symmetric power L-function attached to f is defined by

$$L(\mathrm{sym}^j f,s):=\prod_p\prod_{m=0}^{j}(1-\alpha_p^{j-m}\beta_p^m p^{-s})^{-1} \tag{2.1}$$

for $\Re e\, s>1$. We may express it as a Dirichlet series: for $\Re e\, s>1$,

$$\begin{aligned} L(\mathrm{sym}^j f,s)&=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}\\ &=\prod_p\left(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}+\cdots\right). \end{aligned}\tag{2.2}$$

It is well known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. The Rankin-Selberg L-function $L(\mathrm{sym}^i f\otimes \mathrm{sym}^j f,s)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ is defined as

(2.3)
$$\begin{aligned} L(\mathrm{sym}^i f\otimes\mathrm{sym}^j f,s)&=\prod_p\prod_{m=0}^{i}\prod_{m'=0}^{j}\left(1-\frac{\alpha_p^{i-m}\beta_p^m\alpha_p^{j-m'}\beta_p^{m'}}{p^s}\right)^{-1}\\ &=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^i f\otimes\mathrm{sym}^j f}(n)}{n^s}. \end{aligned}$$

Lemma 2.1. *We have*

$$L(f\otimes f\otimes f,s)=L(f,s)^2L(\mathrm{sym}^3 f,s).$$

Proof. The proof of this lemma is immediate. In fact, by comparing the Euler products of both sides and recalling Deligne's famous result (1.2), we easily obtain this lemma. □

Lemma 2.2. *For $\Re e\, s > 1$, define*

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f\otimes f\otimes f}(n)^2}{n^s}.$$

Then we have

$$L(s) = \zeta(s)^5 L(\mathrm{sym}^2 f, s)^8 L(\mathrm{sym}^4 f, s)^4 L(\mathrm{sym}^4 f \otimes \mathrm{sym}^2 f, s) U(s), \tag{2.4}$$

where the function $U(s)$ is a Dirichlet series absolutely convergent in $\Re e\, s > 1/2$ and $U(s) \neq 0$ for $\Re e\, s = 1$.

Proof. Since $\lambda_{f\otimes f\otimes f}(n)^2$ is a multiplicative function and satisfies the trivial upper bound $O(n^{\varepsilon})$, we have that, for $\Re e\, s > 1$,

$$L(s) = \prod_p \left(1 + \frac{\lambda_{f\otimes f\otimes f}(p)^2}{p^s} + \frac{\lambda_{f\otimes f\otimes f}(p^2)^2}{p^{2s}} + \cdots\right).$$

In the half-plane $\Re e\, s > 1/2$, the corresponding coefficients of the term p^{-s} determine the analytic properties of $L(s)$. By Lemma 2.1, we easily find the identity

$$\begin{aligned}\lambda_{f\otimes f\otimes f}(p)^2 &= (\lambda_{\mathrm{sym}^3 f}(p) + 2\lambda_f(p))^2 \\ &= \lambda_{\mathrm{sym}^3 f}(p)^2 + 4\lambda_{\mathrm{sym}^3 f}(p)\lambda_f(p) + 4\lambda_f(p)^2.\end{aligned}$$

Then from (2.1)–(2.3), we have

$$\lambda_{f\otimes f\otimes f}(p)^2 = \lambda_{\mathrm{sym}^3 f\otimes \mathrm{sym}^3 f}(p) + 4\lambda_{\mathrm{sym}^3 f\otimes f}(p) + 4\lambda_{f\otimes f}(p).$$

Furthermore, one can easily find that

$$\begin{aligned}\lambda_{f\otimes f\otimes f}(p)^2 &= \left(1 + \lambda_{\mathrm{sym}^2 f\otimes \mathrm{sym}^4 f}(p)\right) \\ &\quad + 4\left(\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p)\right) \\ &\quad + 4\left(1 + \lambda_{\mathrm{sym}^2 f}(p)\right) \\ &= 5 + 8\lambda_{\mathrm{sym}^2 f}(p) + 4\lambda_{\mathrm{sym}^4 f}(p) + \lambda_{\mathrm{sym}^2 f\otimes \mathrm{sym}^4 f}(p).\end{aligned}$$

Now the lemma follows by standard arguments. □

As part of the far-reaching Langlands program, the study of the analytic properties of symmetric power L-functions $L(\mathrm{sym}^j f, s)$ is important in contemporary mathematics, and it will have a significant impact on modern number theory.

Lemma 2.3. *Let $f(z) \in \mathrm{H}_k^*$ be a primitive cusp form. Let the jth symmetric power L-function $L(\mathrm{sym}^j f, s)$ be defined as in (2.1). For $j = 1, 2, 3, 4$, there exists an automorphic cuspidal self-dual representation, denoted by $\mathrm{sym}^j\pi_f = \bigotimes' \mathrm{sym}^j\pi_{f,v}$ of $\mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose local L-factors $L(\mathrm{sym}^j\pi_{f,p}, s)$ agree with the local L-factors $L_p(\mathrm{sym}^j f, s)$ in (2.1). In particular, for $j = 1, 2, 3, 4$, $L(\mathrm{sym}^j f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j + 1$.*

Proof. This lemma follows from [**7**] for $k = 2$ and from the recent works [**16, 17, 18**] when $k = 3, 4$. □

Lemma 2.4. *Let $f(z) \in \mathrm{H}_k^*$ be a primitive cusp form. Let $L(\mathrm{sym}^4 f \otimes \mathrm{sym}^2 f, s)$ be defined as in (2.3) with $i = 4$, $j = 2$. Then $L(\mathrm{sym}^4 f \otimes \mathrm{sym}^2 f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree* 15.

Proof. From Lemma 2.3, automorphic cuspidal self-dual representations exist, denoted by $\mathrm{sym}^4\pi_f$ of $\mathrm{GL}_5(\mathbb{A}_{\mathbb{Q}})$ and $\mathrm{sym}^2\pi_f$ of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$, whose local L-factors $L(\mathrm{sym}^4\pi_{f,p}, s)$ and $L(\mathrm{sym}^2\pi_{f,p}, s)$ agree with the local L-factors $L_p(\mathrm{sym}^4 f, s)$ and $L_p(\mathrm{sym}^2 f, s)$ respectively. From the works [**14, 15, 34, 35, 36, 37**] on the Rankin-Selberg theory associated to two automorphic cuspidal representations, we have this lemma. □

From Lemmas 2.3 and 2.4, we observe that $L(\mathrm{sym}^j f, s)$, $1 \le i \le 4$, $L(\mathrm{sym}^4 f \otimes \mathrm{sym}^2 f, s)$ are general L-functions in the sense of Perelli [**29**]. For general L-functions, we have the following averaged or individual convexity bounds.

Lemma 2.5. *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then, for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma + it)|^2 dt \ll T^{\max\{m(1-\sigma),1\}+\varepsilon}, \tag{2.5}$$

uniformly for $1/2 \le \sigma \le 1$ and $T > 1$; and

$$\mathfrak{L}(\sigma + it) \ll (|t| + 1)^{(m/2)(1-\sigma)+\varepsilon} \tag{2.6}$$

uniformly for $1/2 \le \sigma \le 1 + \varepsilon$ and $|t| \ge 1$.

For some L-functions with small degrees, we invoke either individual or average subconvexity bounds.

Lemma 2.6. *For any* $\varepsilon > 0$*, we have*

$$\int_0^T \left|\zeta(\tfrac{5}{7} + i\tau)\right|^{12} \mathrm{d}\tau \ll_\varepsilon T^{1+\varepsilon} \tag{2.7}$$

uniformly for $T \geq 1$*, and*

$$\zeta(\sigma + i\tau) \ll_\varepsilon (|\tau| + 1)^{\max\{(1/3)(1-\sigma),\, 0\}+\varepsilon} \tag{2.8}$$

uniformly for $1/2 \leq \sigma \leq 2$ *and* $|\tau| \geq 1$*.*

Proof. See, e.g., [**11,** Theorem 8.4 and (8.87)] and [**38,** Theorem II.3.6]. □

Lemma 2.7. *Let* $f \in \mathrm{H}_k^*$ *and* $\varepsilon > 0$*. Then we have*

$$\int_0^T \left|L(f, \tfrac{5}{8} + i\tau)\right|^{4} \mathrm{d}\tau \ll_\varepsilon T^{1+\varepsilon} \tag{2.9}$$

uniformly for $T \geq 1$*, and*

$$L(f, \sigma + i\tau) \ll_{f,\varepsilon} (|\tau| + 1)^{\max\{(2/3)(1-\sigma),\, 0\}+\varepsilon} \tag{2.10}$$

uniformly for $1/2 \leq \sigma \leq 2$ *and* $|\tau| \geq 1$*.*

Proof. See, e.g., [**8,** Corollary] and [**12,** Theorem 2, (1.8)]. □

Lemma 2.8. *Let* $f \in \mathrm{H}_k^*$ *and* $\varepsilon > 0$*. Then we have*

$$L(\mathrm{sym}^2 f, \sigma + i\tau) \ll_{f,\varepsilon} (|\tau| + 1)^{\max\{(11/8)(1-\sigma),\, 0\}+\varepsilon} \tag{2.11}$$

uniformly for $1/2 \leq \sigma \leq 2$ *and* $|\tau| \geq 1$*.*

Proof. See, e.g., [**22,** Corollary 1.2]. □

We also need the Sato-Tate conjecture (now a theorem proved by Barnet-Lamb, Geraghty, Harris and Taylor [**1**]). For a prime number p, we write

$$\lambda_f(p) := 2\cos\theta_p, \quad 0 \leq \theta_p \leq \pi,$$

where $\lambda_f(p)$ is the pth normalized Fourier coefficient.

Lemma 2.9. *If $f(t)$, $t \in [0, \pi]$, is a continuous function, then the Sato-Tate law holds, namely,*

$$\sum_{p \leq x} f(\theta_p) \sim \left(\frac{2}{\pi} \int_0^{\pi} f(\theta) \sin^2 \theta \, d\theta \right) \frac{x}{\log x},$$

where p runs through the prime numbers, and $x \to \infty$.

Proof. This famous longstanding conjecture was proved by Barnet-Lamb, et al. [**1**]. See Theorem B and Corollary C therein. For similar facts related to the Sato-Tate law concerning elliptic curves, see, e.g., Mazur's expository article [**26**]. □

Lemma 2.10. *Let $g(n)$ be a non-negative multiplicative function satisfying*

$$0 \leq g(n) \leq A\, d(n)^B$$

for some constants A and B. If

$$\sum_{p \leq x} g(p) \sim a \frac{x}{\log x}, \quad a > 0,$$

Then there exists a suitable constant b such that

$$\sum_{n \leq x} g(n) \sim bx(\log x)^{a-1}.$$

Proof. See, e.g., [**28,** page 204, (1.1)–(1.3)]. □

3. Proof of Theorem 1.1. Firstly, we give the proof of Theorem 1.1. Recall that

$$L(f \otimes f \otimes f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \otimes f \otimes f}(n)}{n^s}, \tag{3.1}$$

for $\Re e\, s > 1$. From Lemmas 2.1 and 2.2, we learn that

$$L(f \otimes f \otimes f, s) = L(f, s)^2 L(\mathrm{sym}^3 f, s)$$

can be analytically continued to be an entire function in the whole complex plane.

By the Perron formula (see [**13,** Proposition 5.54]), we have

$$\sum_{n\leq x}\lambda_{f\otimes f\otimes f}(n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}L(f\otimes f\otimes f,s)\frac{x^s}{s}\,ds+O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr), \tag{3.2}$$

where $b=1+\varepsilon$ and $1\leq T\leq x$ is a parameter to be chosen later.

Now we move the line of integration to $\Re e\, s=5/8$. In the rectangle formed by the line segments joining the points $b+iT$, $5/8+iT$, $5/8-iT$, $b-iT$, and $b+iT$, we note that $L(f\otimes f\otimes f,s)$ is an entire function. By Cauchy's theorem, we have

$$\begin{aligned}\sum_{n\leq x}\lambda_{f\otimes f\otimes f}(n)&=\frac{1}{2\pi i}\biggl\{\int_{5/8-iT}^{5/8+iT}+\int_{5/8+iT}^{b+iT}+\int_{b-iT}^{5/8-iT}\biggr\}\\&\qquad\cdot L(f\otimes f\otimes f,s)\frac{x^s}{s}\,ds+O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr)\\&:=J_1+J_2+J_3+O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr).\end{aligned} \tag{3.3}$$

For J_1, from Lemma 2.1, we have

$$J_1\ll x^{5/8}\int_1^T\biggl|L\biggl(f,\frac{5}{8}+it\biggr)^2L\biggl(\mathrm{sym}^3 f,\frac{5}{8}+it\biggr)\biggr|t^{-1}dt+x^{5/8+\varepsilon}. \tag{3.4}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned}J_1\ll x^{5/8+\varepsilon}\sup_{1\leq T_1\leq T}&\biggl(\int_{T_1}^{2T_1}\biggl|L\biggl(f,\frac{5}{8}+it\biggr)\biggr|^4dt\biggr)^{1/2}\\&\cdot\biggl(\int_{T_1}^{2T_1}\biggl|L\biggl(\mathrm{sym}^3 f,\frac{5}{8}+it\biggr)\biggr|^2dt\biggr)^{1/2}T_1^{-1}.\end{aligned}$$

By (2.5) in Lemma 2.5 with $m=4$ and $\sigma=5/8$, we have

$$\int_{T_1}^{2T_1}\biggl|L\biggl(\mathrm{sym}^3 f,\frac{5}{8}+it\biggr)\biggr|^2dt\ll T_1^{3/2+\varepsilon}.$$

This, together with (2.9) in Lemma 2.7, gives

$$J_1\ll x^{5/8+\varepsilon}T^{1/2+3/4-1+\varepsilon}\ll x^{5/8+\varepsilon}T^{1/4+\varepsilon}. \tag{3.5}$$

For the integrals over the horizontal segments, we use (2.6) of Lemma 2.5 with $m = 4$ and (2.10) of Lemma 2.7 to bound

$$\begin{aligned} J_2 + J_3 &\ll \max_{5/8 \le \sigma \le b} x^{\sigma} T^{(2\times 2/3+2)(1-\sigma)+\varepsilon} T^{-1} \\ &= \max_{(5/8)\le \sigma \le b} \left(\frac{x}{T^{10/3}}\right)^{\sigma} T^{7/3+\varepsilon} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{5/8+\varepsilon} T^{1/4+\varepsilon}. \end{aligned} \tag{3.6}$$

From (3.3), (3.5) and (3.6), we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n) \ll \frac{x^{1+\varepsilon}}{T} + x^{5/8+\varepsilon} T^{1/4+\varepsilon}. \tag{3.7}$$

On taking $T = x^{3/10}$ in (3.7), we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n) \ll x^{7/10+\varepsilon}. \tag{3.8}$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to that of Theorem 1.1. After applying the Perron formula to the generating function $L(s)$, and then shifting the line of integration to $\Re e\, s = 5/7$, we have

$$\begin{aligned} \sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^2 &= \frac{1}{2\pi i}\left\{ \int_{5/7-iT}^{5/7+iT} + \int_{5/7+iT}^{b+iT} + \int_{b-iT}^{5/7-iT} \right\} L(s)\frac{x^s}{s}\, ds \\ &\quad + xP(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right), \\ &:= xP(\log x) + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \end{aligned} \tag{4.1}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later, $P(t)$ is a polynomial of degree 4. Here, the main term $xP(\log x)$ comes from the residue of $L(s)x^s/s$ at the pole $s = 1$ of order 5.

For J_1, we have

$$J_1 \ll x^{5/7+\varepsilon} \sup_{1\le T_1\le T} I_1(T_1)^{5/12} I_2(T_1)^{1/2} I_3(T_1)^{1/12} T_1^{-1},$$

where

$$I_1(T_1) = \int_{T_1}^{2T_1} \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt,$$

$$I_2(T_1) = \int_{T_1}^{2T_1} \left|L\left(\mathrm{sym}^2 f, \frac{5}{7}+it\right)^8 L\left(\mathrm{sym}^4 f, \frac{5}{7}+it\right)^4\right|^2 dt,$$

and

$$I_3(T_1) = \int_{T_1}^{2T_1} \left|L\left(\mathrm{sym}^4 f\otimes \mathrm{sym}^2 f, \frac{5}{7}+it\right)\right|^{12} dt.$$

Then, by Lemmas 2.5, 2.6 and 2.8, we have

$$I_1(T_1) \ll T_1^{1+\varepsilon}, \qquad I_3(T_1) \ll T_1^{180/7+\varepsilon},$$

and

$$\begin{aligned} I_2(T_1) &\ll T_1^{16\times 11/8\times(1-5/7)+\varepsilon} \int_{T_1}^{2T_1} \left|L\left(\mathrm{sym}^4 f, \frac{5}{7}+it\right)^4\right|^2 dt \\ &\ll T_1^{12+\varepsilon}. \end{aligned}$$

Hence, we have

$$\begin{aligned} J_1 &\ll x^{5/7+\varepsilon} \sup_{1\le T_1\le T} I_1(T_1)^{5/12} I_2(T_1)^{1/2} I_3(T_1)^{1/12} T_1^{-1} \\ &\ll x^{5/7+\varepsilon} T^{635/84+\varepsilon}. \end{aligned} \tag{4.2}$$

For the integrals over the horizontal segments, we use (2.6) in Lemma 2.5 with $m = 35$, (2.8) in Lemma 2.6 and Lemma 2.8 to bound

$$\begin{aligned} J_2 + J_3 &\ll \max_{5/7\le\sigma\le b} x^\sigma T^{(5\times 1/3+8\times 11/8+35/2)(1-\sigma)+\varepsilon} T^{-1} \\ &= \max_{5/7\le\sigma\le b} \left(\frac{x}{T^{181/6}}\right)^\sigma T^{175/6+\varepsilon} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{5/7+\varepsilon} T^{160/21+\varepsilon}. \end{aligned} \tag{4.3}$$

From (4.1), (4.2) and (4.3), we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{5/7+\varepsilon} T^{160/21+\varepsilon}\right). \tag{4.4}$$

On taking $T = x^{6/181}$ in (4.4), we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x) + O(x^{175/181+\varepsilon}). \tag{4.5}$$

This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.4. By Deligne's bound (1.4), we can denote

$$\lambda_f(p) := 2\cos\theta_p. \tag{5.1}$$

Then, by the Hecke relation,

$$\lambda_{\mathrm{sym}^j f}(p) = \lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu$$

for all integers $\nu \ge 1$, we have

$$\lambda_{f\otimes f\otimes f}(p)^{2\ell} = (2\cos\theta_p)^{6\ell}. \tag{5.2}$$

By Lemma 2.9, it follows that

$$\sum_{p\le x} \lambda_{f\otimes f\otimes f}(p)^{2\ell} = \sum_{p\le x}(2\cos\theta_p)^{6\ell} \sim \left(\frac{2}{\pi}\int_0^\pi (2\cos\theta)^{6\ell}\sin^2\theta\, d\theta\right)\frac{x}{\log x},$$

where p runs over the prime numbers, and $x\to\infty$. Then, we have

$$\sum_{p\le x} \lambda_{f\otimes f\otimes f}(p)^{2\ell} = \frac{1}{3\ell+1}\binom{6\ell}{3\ell}(1+o(1))\frac{x}{\log x}, \tag{5.3}$$

as x tends to infinity. Since $\lambda_{f\otimes f\otimes f}(n)^{2\ell}$ is multiplicative and satisfies the inequality

$$\lambda_{f\otimes f\otimes f}(n)^{2\ell} \le d_8(n)^{2\ell},$$

then by Lemma 2.10, we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^{2\ell} \sim c_\ell x(\log x)^{\delta_\ell},$$

for a suitable positive constant c_ℓ with

$$\delta_\ell = \frac{1}{3\ell+1}\binom{6\ell}{3\ell} - 1. \quad \square$$

6. Proof of Theorem 1.5. The proof of Theorem 1.5 is similar to those of Theorems 1.1–1.4, so that we may be brief. By (2.3), we observe that

$$L(\mathrm{sym}^2 f \otimes f, s) = L(\mathrm{sym}^3 f, s) L(f, s). \tag{6.1}$$

This implies that, for any prime number p,

$$\lambda_{\mathrm{sym}^2 f \otimes f}(p) = \lambda_{\mathrm{sym}^2 f}(p)\lambda_f(p) = \lambda_{\mathrm{sym}^3 f}(p) + \lambda_f(p). \tag{6.2}$$

For $\Re e\, s > 1$, define

$$D(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f \otimes f}(n)^2}{n^s}.$$

By (6.2), we have

$$\begin{aligned} D(s) &= L(f \otimes f, s) L(\mathrm{sym}^3 f \otimes f, s)^2 L(\mathrm{sym}^3 f \otimes \mathrm{sym}^3 f, s) V(s), \\ &= \zeta(s)^2 L(\mathrm{sym}^2 f, s)^3 L(\mathrm{sym}^4 f, s)^2 L(\mathrm{sym}^4 f \otimes \mathrm{sym}^2 f, s) V(s), \end{aligned} \tag{6.3}$$

where the function $V(s)$ is a Dirichlet series absolutely convergent in $\Re e\, s > 1/2$ and $V(s) \neq 0$ for $\Re e\, s = 1$.

From (6.1) and (6.3), we obtain that, by following the arguments in Sections 4 and 5,

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(n) \ll_{f,\varepsilon} x^{(2/3)+\varepsilon}.$$

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(n)^2 = xQ(\log x) + O_{f,\varepsilon}(x^{(17/18)+\varepsilon}),$$

where $Q(t)$ is a polynomial of degree 1 (note that the error terms mentioned just above need not be the best possible).

By (6.2) and the well-known equality $\lambda_f(p^2) = \lambda_f(p)^2 - 1$, we have

$$\lambda_{\mathrm{sym}^2 f \otimes f}(p)^{2\ell} = \lambda_{\mathrm{sym}^2 f}(p)^{2\ell} \lambda_f(p)^{2\ell} = (\lambda_f(p)^2 - 1)^{2\ell} \lambda_f(p)^{2\ell}. \tag{6.4}$$

Again, by Lemma 2.9, we have

$$\begin{aligned} \sum_{p \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(p)^{2\ell} &= \sum_{p \le x} \left((2\cos\theta_p)^3 - (2\cos\theta_p) \right)^{2\ell} \\ &\sim \left(\frac{2}{\pi} \int_0^{\pi} \left((2\cos\theta)^3 - (2\cos\theta) \right)^{2\ell} \sin^2\theta\, d\theta \right) \frac{x}{\log x}, \end{aligned}$$

where p runs over the prime numbers, and $x \to \infty$. Note that

$$\begin{aligned} &\frac{2}{\pi}\int_0^\pi \big((2\cos\theta)^3-(2\cos\theta)\big)^{2\ell}\sin^2\theta\,d\theta \\ &=\sum_{k=0}^{2\ell}\binom{2\ell}{k}(-1)^{2\ell-k}\frac{2}{\pi}\int_0^\pi (2\cos\theta)^{2\ell+2k}\sin^2\theta\,d\theta \\ &=\sum_{k=0}^{2\ell}\binom{2\ell}{k}(-1)^{2\ell-k}\frac{1}{\ell+k+1}\binom{2\ell+2k}{\ell+k} \\ &:=\gamma_\ell. \end{aligned} \tag{6.5}$$

Then, we have

$$\sum_{p\le x}\lambda_{\mathrm{sym}^2 f\otimes f}(p)^{2\ell}=\gamma_\ell(1+o(1))\,\frac{x}{\log x}, \tag{6.6}$$

as x tends to ∞. Since $\lambda_{\mathrm{sym}^2 f\otimes f}(n)^{2\ell}$ is multiplicative and satisfies the inequality

$$\lambda_{\mathrm{sym}^2 f\otimes f}(n)^{2\ell}\le d_6(n)^{2\ell},$$

then, by Lemma 2.10, we have

$$\sum_{n\le x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^{2\ell}\sim d_\ell x(\log x)^{\gamma_\ell-1}. \quad \square$$

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