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AUGMENTED GENERALIZED HAPPY FUNCTIONS

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ABSTRACT. An augmented generalized happy function, $S_{[c,b]}$ maps a positive integer to the sum of the squares of its base b digits and a non-negative integer c. A positive integer u is in a *cycle* of $S_{[c,b]}$ if, for some positive integer k, $S^k_{[c,b]}(u) = u$, and, for positive integers v and w, v is *w-attracted* for $S_{[c,b]}$ if, for some non-negative integer ℓ, $S^{\ell}_{[c,b]}(v) = w$. In this paper, we prove that, for each $c \geq 0$ and $b \geq 2$, and for any u in a cycle of $S_{[c,b]}$: (1) if b is even, then there exist arbitrarily long sequences of consecutive u-attracted integers, and (2) if b is odd, then there exist arbitrarily long sequences of 2-consecutive u-attracted integers.

1. Introduction. Letting S_2 be the function that takes a positive integer to the sum of the squares of its (base 10) digits, a positive integer a is said to be a *happy number* if $S_2^k(a) = 1$ for some $k \in \mathbb{Z}^+$ [**5, 6**]. These ideas were generalized in [**2**]: fix an integer $b \geq 2$, and let

$$a = \sum_{i=0}^{n} a_i b^i,$$

where $0 \leq a_i \leq b-1$ are integers. For each integer $e \geq 2$, define the function $S_{e,b} : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S_{e,b}(a) = S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If $S^k_{e,b}(a) = 1$ for some $k \in \mathbb{Z}^+$, then a is called an *e-power b-happy number*.

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We further generalize these functions by allowing the addition of a constant after taking the sum of the powers of the digits. (Throughout this work, all parameters are assumed to be integers.)

Definition 1.1. Fix integers $c \geq 0$ and $b \geq 2$. Let $a = \sum_{i=0}^{n} a_i b^i$, where $0 \leq a_i \leq b-1$ are integers. For each integer $e \geq 2$, define the *augmented generalized happy function* $S_{e,b,c} : \mathbb{Z}^+ \to \mathbb{Z}^+$, by

$$S_{e,b,c}(a) = c + S_{e,b}(a) = c + \sum_{i=0}^{n} a_i^e.$$

In Section 2, we examine various properties of the function $S_{2,b,c}$, which, for ease of notation, we denote by $S_{[c,b]}$. In Section 3, we state and prove Theorem 3.2, an analogue to the existence of arbitrarily long sequences of consecutive happy numbers. Although this result is quite general, it leaves a particular case unresolved, which we make explicit in Conjecture 3.3. In Section 4, we prove that Conjecture 3.3 holds for small values of c and b.

2. Properties of $S_{[c,b]}$. In this section, we consider the function $S_{[c,b]} = S_{2,b,c}$. Note that $S_{[0,10]} = S_2$, as defined in Section 1.

Definition 2.1. Fix $c \geq 0$, $b \geq 2$ and $a \geq 1$. We say that a is a *fixed point* of $S_{[c,b]}$ if $S_{[c,b]}(a) = a$ and that a is in a *cycle* of $S_{[c,b]}$ if $S_{[c,b]}^k(a) = a$ for some $k \in \mathbb{Z}^+$. The smallest such k is called the *length* of the cycle.

As is well known, S_2 has exactly one fixed point and one nontrivial cycle. The standard proof uses a lemma similar to the following, which is a generalization for the function $S_{[c,b]}$.

Lemma 2.2. *Given $c \geq 0$ and $b \geq 2$, there exists a constant m such that, for each $a \geq b^m$, $S_{[c,b]}(a) < a$. In particular, this inequality holds for any $m \in \mathbb{Z}^+$ such that $b^m > b^2 - 3b + 3 + c$.*

Proof. Let $m \in \mathbb{Z}^+$ be such that $b^m > b^2 - 3b + 3 + c$, and let $a \geq b^m$. Then

$$a = \sum_{i=0}^{m} a_i b^i,$$

for some $0 \leq a_i \leq b - 1$ with $a_m \neq 0$. Thus,

$$\begin{aligned} a - S_{[c,b]}(a) &= \sum_{i=0}^{n} a_i b^i - \left(c + \sum_{i=0}^{n} a_i^2\right) \\ &= a_n(b^n - a_n) + \sum_{i=1}^{n-1} a_i(b^i - a_i) + a_0(1 - a_0) - c \\ &\geq 1(b^m - 1) + 0 + (b-1)(1 - (b-1)) - c \\ &= b^m - b^2 + 3b - 3 - c \\ &> 0. \end{aligned}$$

Therefore, $S_{[c,b]}(a) < a$ for all $a \geq b^m$. □

It follows from Lemma 2.2 that, for any $c < 27$, for all $a \geq 100$, $S_{[c,10]}(a) < a$. We use this result to determine the fixed points and cycles of $S_{[c,10]}$ for $0 \leq c \leq 9$. The results are presented in Table 1.

As Table 1 illustrates, varying the constant greatly affects the behavior of $S_{[c,b]}$ under iteration. As expected, changing the base also changes the behavior, but, interestingly, there are patterns that occur when changing both the constant and the base. For example, we show that, if c and b are both odd, then $S_{[c,b]}$ has no fixed points. First, we present a key lemma that will be used repeatedly throughout the paper.

Lemma 2.3. *If b is odd, then $S^k_{[c,b]}(a) \equiv kc + a \pmod 2$.*

Proof. Let b be odd and, as usual, let

$$a = \sum_{i=0}^{n} a_i b^i.$$

Since b is odd,

$$a \equiv \sum_{i=0}^{n} a_i \pmod 2;$$

TABLE 1. Fixed points and cycles of $S_{[c,10]}$ for $0 \leq c \leq 9$.

c	Fixed points and cycles of $S_{[c,10]}$
0	$1 \to 1$ $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$
1	$6 \to 37 \to 59 \to 107 \to 51 \to 27 \to 54 \to 42 \to 21 \to 6$ $35 \to 35$ $75 \to 75$
2	$28 \to 70 \to 51 \to 28$ $29 \to 87 \to 115 \to 29$
3	$7 \to 52 \to 32 \to 16 \to 40 \to 19 \to 85 \to 92 \to 88 \to 131 \to 14 \to 20 \to 7$ $13 \to 13$ $93 \to 93$
4	$6 \to 40 \to 20 \to 8 \to 68 \to 104 \to 21 \to 9 \to 85 \to 93 \to 94 \to 101 \to 6$ $24 \to 24$ $45 \to 45$ $65 \to 65$ $84 \to 84$
5	$15 \to 31 \to 15$ $55 \to 55$
6	$16 \to 43 \to 31 \to 16$ $19 \to 88 \to 134 \to 32 \to 19$
7	$9 \to 88 \to 135 \to 42 \to 27 \to 60 \to 43 \to 32 \to 20 \to 11 \to 9$ $12 \to 12$ $36 \to 52 \to 36$ $66 \to 79 \to 137 \to 66$ $92 \to 92$
8	$26 \to 48 \to 88 \to 136 \to 54 \to 49 \to 105 \to 34 \to 33 \to 26$
9	$10 \to 10$ $11 \to 11$ $34 \to 34$ $46 \to 61 \to 46$ $74 \to 74$ $90 \to 90$ $91 \to 91$

therefore,

$$S_{[c,b]}(a) = c + \sum_{i=0}^{n} a_i^2 \equiv c + \sum_{i=0}^{n} a_i \equiv c + a \pmod 2.$$

A simple induction argument completes the proof. $\square$

Theorem 2.4. *If c and b are both odd, then $S_{[c,b]}$ has no fixed points and all of its cycles are of even length.*

Proof. By Lemma 2.3, since c is odd, $S^k_{[c,b]}(a) \equiv kc + a \equiv k + a \pmod 2$. Thus, if $S^k_{[c,b]}(a) = a$, then k is even. The result follows. $\square$

Lemma 2.2 also allows us to compute all fixed points and cycles for arbitrary values of c and b. Of particular interest in Section 4 is the case of where both c and b are odd. We provide lists of the fixed points and cycles of $S_{[c,b]}$ in this case, for small values of c and b, in Table 2. We label some of the sequences for ease of reference in the proof of Theorem 4.1.

Recall that a positive integer a is a happy number if $S^k_{[0,10]}(a) = 1$ for some $k \in \mathbb{Z}^+$. We now generalize this idea to values of $c > 0$, noting that, in these cases, 1 is no longer a fixed point (nor in a cycle).

Definition 2.5. Fix $c \geq 0$ and $b \geq 2$. Let $U_{[c,b]}$ denote the set of all fixed points and cycles of $S_{[c,b]}$, that is,

$$U_{[c,b]} = \{a \in \mathbb{Z}^+ \mid S^m_{[c,b]}(a) = a \text{ for some } m \in \mathbb{Z}^+\}.$$

For $u \in U_{[c,b]}$, a positive integer a is a u-*attracted* number (for $S_{[c,b]}$) if $S^k_{[c,b]}(a) = u$, for some $k \in \mathbb{Z}_{\geq 0}$.

For example, referring to Table 1, for $S_{[4,10]}$, we see that 40 is 6-attracted, as are 20, 8 and the other numbers in that cycle. All of those numbers are also 40-attracted, etc. Additionally, since $S_{[4,10]}(2) = 8$, 2 is also 6-attracted. Similarly, 42 is 24-attracted, since $S_{[4,10]}(42) = 24$.

3. Consecutive u-attracted numbers. In this and the next section, we consider the existence of sequences of consecutive u-attracted numbers, for $u \in U_{[c,b]}$ for some fixed c and b. As seen in Table 1, for $S_{[3,10]}$, 19 and 20 are consecutive 7-attracted numbers (and direct calculation shows that 1 and 2 are as well). Does such a consecutive pair exist for every choice of $c \geq 0$, $b \geq 2$, and $u \in U_{[c,b]}$? Do there exist longer consecutive sequences of u-attracted numbers? The analogous questions were answered for happy numbers in **[1]** and for many cases of e-power b-happy numbers in **[3, 4]**.

TABLE 2. Cycles of $S_{[c,b]}$ for $1 \le c \le 9$ odd and $2 \le b \le 9$ odd.

$[c,b]$	Cycles of $S_{[c,b]}$	
$[1,3]$	$12 \to 20 \to 12$	
$[3,3]$	$22 \to 102 \to 22$	
$[5,3]$	$20 \to 100 \to 20$	C_1
	$21 \to 101 \to 21$	C_2
	$22 \to 111 \to 22$	C_3
$[7,3]$	$22 \to 120 \to 110 \to 100 \to 22$	
$[9,3]$	$112 \to 120 \to 112$	
$[1,5]$	$2 \to 10 \to 2$	
$[3,5]$	$23 \to 31 \to 23$	
$[5,5]$	$11 \to 12 \to 20 \to 14 \to 42 \to 100 \to 11$	
$[7,5]$	$23 \to 40 \to 43 \to 112 \to 23$	
$[9,5]$	$21 \to 24 \to 104 \to 101 \to 21$	
$[1,7]$	$13 \to 14 \to 24 \to 30 \to 13$	C_1
	$35 \to 50 \to 35$	C_2
$[3,7]$	$25 \to 44 \to 50 \to 40 \to 25$	C_1
	$26 \to 61 \to 55 \to 104 \to 26$	C_2
$[5,7]$	$6 \to 56 \to 123 \to 25 \to 46 \to 111 \to 11 \to 10 \to 6$	C_1
	$13 \to 21 \to 13$	C_2
	$34 \to 42 \to 34$	C_3
$[7,7]$	$26 \to 65 \to 125 \to 52 \to 51 \to 45 \to 66 \to 142 \to 40 \to 32 \to 26$	
$[9,7]$	$46 \to 115 \to 51 \to 50 \to 46$	
$[1,9]$	$3 \to 11 \to 3$	
$[3,9]$	$4 \to 21 \to 8 \to 74 \to 75 \to 85 \to 112 \to 10 \to 4$	
$[5,9]$	$25 \to 37 \to 70 \to 60 \to 45 \to 51 \to 34 \to 33 \to 25$	C_1
	$28 \to 81 \to 77 \to 124 \to 28$	C_2
	$46 \to 63 \to 55 \to 61 \to 46$	C_3
	$88 \to 157 \to 88$	C_4
$[7,9]$	$8 \to 78 \to 143 \to 36 \to 57 \to 100 \to 8$	C_1
	$25 \to 40 \to 25$	C_2
	$45 \to 53 \to 45$	C_3
	$48 \to 106 \to 48$	C_4
$[9,9]$	$12 \to 15 \to 38 \to 101 \to 12$	C_1
	$24 \to 32 \to 24$	C_2

In light of Lemma 2.3, when c is even and b is odd, for each a and k, $S^k_{[c,b]}(a) \equiv a \pmod 2$. Thus, in these cases, there cannot exist a consecutive pair of (or longer sequences of consecutive) u-attracted numbers. With this in mind, we introduce the next definition, found in [**3**].

Definition 3.1. A sequence of positive integers is *d-consecutive* if it is an arithmetic sequence with constant difference d.

By [**3,** Corollary 2], given $b \geq 2$, and $d = \gcd(2, b-1)$, there exist arbitrarily long finite d-consecutive sequences of 1-attracted numbers for $S_{[0,b]}$. Adapting these ideas to augmented generalized happy functions, we prove the next theorem.

Theorem 3.2. *Let $c \geq 0$, $b \geq 2$ and $u \in U_{[c,b]}$ be fixed. Set $d = \gcd(2, b-1)$. Then, there exist arbitrarily long finite sequences of d-consecutive u-attracted numbers for $S_{[c,b]}$.*

As noted earlier, Lemma 2.3 shows that, if c is even and b is odd, then there do not exist any consecutive u-attracted numbers for $S_{[c,b]}$. However, for c and b both odd, we conjecture that there are arbitrarily long finite sequences of consecutive u-attracted numbers for $S_{[c,b]}$.

Conjecture 3.3. *Let $c > 0$ and $b > 2$ both be odd. Then, for each $u \in U_{[c,b]}$, there exist arbitrarily long finite sequences of consecutive u-attracted numbers for $S_{[c,b]}$.*

We now prove Theorem 3.2. In Section 4, we prove special cases of Conjecture 3.3, specifically, the cases with $c, b < 10$.

Our proof of Theorem 3.2 follows the general outline of the proofs in [**3**]. We note that, similar to the proofs in [**3**], our proofs lead to the somewhat stronger result in which $u \in U_{[c,b]}$ is replaced by u in the image of $S_{[c,b]}$.

We begin with a definition and two important lemmas.

Definition 3.4. A finite set, $T \subset \mathbb{Z}^+$, is $[c,b]$-*good* if, for each $u \in U_{[c,b]}$, there exists $n, k \in \mathbb{Z}_{\geq 0}$ such that, for all $t \in T$, $S^k_{[c,b]}(t+n) = u$.

Lemma 3.5. *Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the composition of a finite sequence of the functions $S_{[c,b]}$ and I, where $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $I(t) = t+1$. If $F(T)$ is $[c,b]$-good, then T is $[c,b]$-good.*

Proof. It suffices to show that, if $I(T)$ is $[c,b]$-good, then T is $[c,b]$-good, and if $S_{[c,b]}(T)$ is $[c,b]$-good, then T is $[c,b]$-good. First, suppose that $I(T)$ is $[c,b]$-good. Then, for each $u \in U_{[c,b]}$, there exist n', $k \in \mathbb{Z}_{\geq 0}$ such that $S^k_{[c,b]}(t+1+n') = u$ for all $t \in T$. Letting $n = n'+1$, it follows that T is $[c,b]$-good.

Now, suppose that $S_{[c,b]}(T)$ is $[c,b]$-good. Then, for each $u \in U_{[c,b]}$, there exist n', $k' \in \mathbb{Z}_{\geq 0}$ such that $S^{k'}_{[c,b]}(S_{[c,b]}(t)+n') = u$ for all $t \in T$. Let $r \in \mathbb{Z}$ be the number of base b digits of the largest element of T, and set

$$n = \underbrace{11\ldots11}_{n'}\,\underbrace{00\ldots00}_{r},$$

in base b. Letting $k = k'+1$, we have

$$S^k_{[c,b]}(t+n) = S^{k'}_{[c,b]}(S_{[c,b]}(t+n)) = S^{k'}_{[c,b]}(S_{[c,b]}(t)+n') = u.$$

Hence, T is $[c,b]$-good. □

Lemma 3.6. *If $T = \{t\}$, then T is $[c,b]$-good.*

Proof. Let $u \in U_{[c,b]}$. Then, there exists some $v \in \mathbb{Z}^+$ such that $S_{[c,b]}(v) = u$. Let $r \in \mathbb{Z}_{\geq 0}$ be such that $t \leq b^r v$, and define $n = b^r v - t$ and $k = 1$. This yields $S^k_{[c,b]}(t+n) = S_{[c,b]}(v) = u$. Hence, T is $[c,b]$-good. □

Theorem 3.7. *Given $c \geq 0$ and $b \geq 2$, let $d = \gcd(2, b-1)$. A finite set T of positive integers is $[c,b]$-good if and only if all elements of T are congruent modulo d.*

Proof. Fix b, c, d, and nonempty T as in Theorem 3.7.

First, suppose that T is $[c,b]$-good, and let $t_1, t_2 \in T$. If b is even, then $d = 1$, and so, $t_1 \equiv t_2 \pmod{d}$, trivially. If b is odd, fix $u \in U_{[c,b]}$, and let $n, k \in \mathbb{Z}_{\geq 0}$ be such that $S^k_{[c,b]}(t_i+n) = u$ for $i = 1, 2$. Thus,

$$S^k_{[c,b]}(t_1+n) = S^k_{[c,b]}(t_2+n).$$

Applying Lemma 2.3, we obtain

$$kc + t_1 + n \equiv kc + t_2 + n \pmod 2,$$

so that $t_1 \equiv t_2 \pmod 2$. Thus, all of the elements of T must be congruent modulo $2 = \gcd(2, b-1) = d$.

Conversely, assume that all of the elements of T are congruent modulo d. Note that, if T has exactly one element, then, by Lemma 3.6, T is $[c, b]$-good. Therefore, we may assume that $|T| > 1$.

Letting $N = |T|$, assume by induction that any set of fewer than N elements, all of which are congruent modulo d, is $[c, b]$-good. Let $t_1, t_2 \in T$ be distinct, and assume, without loss of generality, that $t_1 > t_2$. We will construct a function F, a finite composition of the functions I and $S_{[c,b]}$, so that $F(t_1) = F(t_2)$.

Consider the cases:

1. If t_1 and t_2 have the same nonzero digits, we construct F_1 so that $F_1(t_1) = F_2(t_2)$.
2. If $t_1 \equiv t_2 \pmod{b-1}$, then we construct F_2 so that $F_2(t_1)$ and $F_2(t_2)$ have the same nonzero digits.
3. If it is neither the case that t_1 and t_2 have the same nonzero digits nor $t_1 \equiv t_2 \pmod{b-1}$, we construct F_3 so that $F_3(t_1) \equiv F_3(t_2) \pmod{b-1}$.

Composing some or all of these functions will yield the desired function F.

Case 1. If t_1 and t_2 have the same nonzero digits, it follows from the definition of $S_{[c,b]}$ that $S_{[c,b]}(t_1) = S_{[c,b]}(t_2)$. In this case, let $F = F_1 = S_{[c,b]}$.

Case 2. If $t_1 \equiv t_2 \pmod{b-1}$, then there is $v \in \mathbb{Z}^+$ so that $t_1 - t_2 = (b-1)v$. Let $r \in \mathbb{Z}^+$ be such that $b^r > bv + t_2 - v$ so that $b^r > bv$ and $b^r > t_2 - v$. Let $m = b^r + v - t_2$, and note that $m > 0$. Then,

$$I^m(t_1) = t_1 + b^r + v - t_2 = b^r + v + (b-1)v = b^r + bv$$

and

$$I^m(t_2) = t_2 + b^r + v - t_2 = b^r + v.$$

Since r was chosen so that $b^r > bv$, $I^m(t_1)$ and $I^m(t_2)$ must have the same nonzero digits, as in Case 1. In this case, let $F_1 = S_{[c,b]}$ and $F_2 = I^m$. Then let $F = F_1 \circ F_2$.

Case 3. If neither of the above holds, let $w = t_1 - t_2$. We first show that there exists $0 \leq j < b-1$ such that

$$2j \equiv -S_{[c,b]}(w-1) + c - 1 \pmod{b-1}, \tag{3.1}$$

where, for convenience, we define $S_{[c,b]}(0) = 0$. If b is even, then 2 and $b-1$ are relatively prime; thus, 2 is invertible modulo $b-1$ and such a j clearly exists. If b is odd, then $d = 2$ and so w is even. By Lemma 2.3,

$$S_{[c,b]}(w-1) \equiv c + w - 1 \equiv c - 1 \pmod{2},$$

and so $-S_{[c,b]}(w-1) + c - 1$ is even. Thus, there exists a $j' \in \mathbb{Z}$ such that $2j' = -S_{[c,b]}(w-1) + c - 1$. Hence, there is a $0 \leq j < b$ satisfying equation (3.1).

Now choose $r' \in \mathbb{Z}^+$ such that $(j+1)b^{r'} > t_1$, and let $m' = (j+1)b^{r'} - t_2 - 1$. Note that $m' \geq 0$. Then,

$$\begin{aligned} S_{[c,b]}(t_1 + m') &= S_{[c,b]}((j+1)b^{r'} + w - 1) \\ &= (j+1)^2 + S_{[c,b]}(w-1) \\ &= (j^2 + 2j + 1) + S_{[c,b]}(w-1) \\ &\equiv j^2 + c \pmod{b-1} \end{aligned}$$

and

$$\begin{aligned} S_{[c,b]}(t_2 + m') &= S_{[c,b]}((j+1)b^{r'} - 1) \\ &= j^2 + (b-1)^2 r' + c \\ &\equiv j^2 + c \pmod{b-1}. \end{aligned}$$

Therefore, $S_{[c,b]} I^{m'}(t_1) \equiv S_{[c,b]} I^{m'}(t_2) \pmod{b-1}$, as in Case 2. In this case, let $F_1 = S_{[c,b]}$, $F_2 = I^m$, and $F_3 = S_{[c,b]} I^{m'}$ (with m chosen as in Case 2). Further, let $F = F_1 \circ F_2 \circ F_3$.

Thus, there exists a function F, a composition of a finite sequence of $S_{[c,b]}$ and I, such that $F(t_1) = F(t_2)$, and so, $|F(T)| < |T|$. By the induction hypothesis, it follows that $F(T)$ is $[c,b]$-good. Finally, by Lemma 3.5, T is $[c,b]$-good. $\square$

We are now ready to prove the main theorem of this paper.

Proof of Theorem 3.2. Let c, b, and u be given, and let $N \in \mathbb{Z}^+$ be arbitrary.

If b is even, set

$$T = \{t \in \mathbb{Z} \mid 1 \leq t \leq N\}.$$

By Theorem 3.7, since $d = 1$, T is $[c, b]$-good. Thus, there exist $k, n \in \mathbb{Z}_{\geq 0}$ such that, for each $t \in T$, $S^k_{[c,b]}(t + n) = u$. Hence, the set

$$\{t + n \in \mathbb{Z} \mid 1 \leq t \leq N\}$$

is a sequence of N consecutive u-attracted numbers.

If b is odd, then let

$$T = \{2t \in \mathbb{Z} \mid 1 \leq t \leq N\}.$$

Then, by Theorem 3.7, since $d = 2$, T is $[c, b]$-good and, as above, the set

$$\{2t + n \in \mathbb{Z} \mid 1 \leq t \leq N\}$$

is a sequence of N 2-consecutive u-attracted numbers. $\square$

4. Special cases of Conjecture 3.3. By Theorem 3.2, if b is odd, then there are arbitrarily long finite 2-consecutive sequences of u-attracted numbers. By Lemma 2.3, if, in addition, c is even, then there cannot exist any nontrivial consecutive sequences of u-attracted numbers. This leaves the existence of such sequences undetermined in the case of both c and b odd.

In this section, we prove that Conjecture 3.3 holds for values of b and c both less than 10.

Theorem 4.1. *Let both* $1 \leq c \leq 9$ *and* $3 \leq b \leq 9$ *be odd, and let* $u \in U_{[c,b]}$. *Then, there exist arbitrarily long finite sequences of consecutive* u*-attracted numbers for* $S_{[c,b]}$.

By Theorem 3.7, no set containing even two consecutive integers can be $[c, b]$-good. Hence, in order to prove Theorem 4.1, we need a new, similar property, which we define next.

Definition 4.2. A finite set, T, is $[c,b]$-*cycle-good* if, for each cycle C of $S_{[c,b]}$, there exist $k, n \in \mathbb{Z}^+$ such that, for all $t \in T$, $S^k_{[c,b]}(t+n) \in C$.

Note that any $[c,b]$-good set is necessarily $[c,b]$-cycle-good, but not the converse. We need the next analog of Lemma 3.5, with $[c,b]$-cycle-good in place of $[c,b]$-good. Its proof completely parallels that of Lemma 3.5 and so is omitted.

Lemma 4.3. *Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the composition of a finite sequence of the functions, $S_{[c,b]}$ and I, where $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $I(t) = t+1$. If $F(T)$ is $[c,b]$-cycle-good, then T is $[c,b]$-cycle-good.*

Proof of Theorem 4.1. First note that, if $S_{[c,b]}$ has only one cycle, then for each $u \in U_{[c,b]}$, every positive integer is u-attracted. Hence, the theorem holds in these cases. Thus, using Table 2, it remains to prove the conjecture for each $[c,b]$ in the set

$$A = \{[5,3],[1,7],[3,7],[5,7],[5,9],[7,9],[9,9]\}.$$

Fix $[c,b] \in A$, and let T be a nonempty finite set of positive integers. We now prove that T is $[c,b]$-cycle-good.

TABLE 3. Values for the proof of Theorem 4.1.

$[c,b]$	v	V_j
$[5,3]$	20	$V_1 = \{20,100\}$, $V_2 = \{20,21\}$, $V_3 = \{20,111\}$
$[1,7]$	13	$V_1 = \{13,14\}$, $V_2 = \{13,30\}$, $V_3 = \{13,50\}$
$[3,7]$	44	$V_1 = \{44,25\}$, $V_2 = \{44,50\}$, $V_3 = \{44,61\}$, $V_4 = \{44,104\}$
$[5,7]$	6	$V_1 = \{6,10\}$, $V_2 = \{6,21\}$, $V_3 = \{6,25\}$, $V_4 = \{6,34\}$, $V_5 = \{6,56\}$, $V_6 = \{6,111\}$
$[5,9]$	37	$V_1 = \{37,25\}$, $V_2 = \{37,34\}$, $V_3 = \{37,45\}$, $V_4 = \{37,61\}$, $V_5 = \{37,63\}$, $V_6 = \{37,70\}$ $V_7 = \{37,81\}$, $V_8 = \{37,124\}$, $V_9 = \{37,157\}$
$[7,9]$	8	$V_1 = \{8,25\}$, $V_2 = \{8,36\}$, $V_3 = \{8,45\}$, $V_4 = \{8,78\}$, $V_5 = \{8,100\}$, $V_6 = \{8,106\}$
$[9,9]$	15	$V_1 = \{15,12\}$, $V_2 = \{15,32\}$, $V_3 = \{15,38\}$

TABLE 4. $S_{[c,b]}^{k_i}(V_1 + n_i) \subseteq C_i$.

$[c,b]$	k_1, n_1	k_2, n_2	k_3, n_3	k_4, n_4
$[5,3]$	$0,0$	$0,1$	$2,11120200$	-
$[1,7]$	$0,0$	$1,1111111$	-	-
$[3,7]$	$0,0$	$0,30$	-	-
$[5,7]$	$0,0$	$1,3$	$1,121303$	-
$[5,9]$	$0,0$	$2,81$	$5,11$	$1,156155$
$[7,9]$	$3,2$	$6,131$	$4,135$	$3,13$
$[9,9]$	$0,0$	$3,218$	-	-

To fix notation, let T_e be the set of all even elements of T, and let T_o be the set of all odd elements of T. We assume that neither T_e nor T_o is empty since, otherwise, by Theorem 3.7, T is $[c,b]$-good and thus $[c,b]$-cycle-good. Let the constant v and the sets V_j be as given in Table 3, and let ℓ be the length of C_1, the cycle of $S_{[c,b]}$ containing v, as seen in Table 2.

By Theorem 3.7, the set T_e is $[c,b]$-good. Thus, there exist positive integers k_1 and n_1 such that, for each $t \in T_e$, $S_{[c,b]}^{k_1}(t + n_1) = v$. Let

$$T' = \{S_{[c,b]}^{k_1}(t + n_1) \mid t \in T_o\}.$$

It follows from Lemma 2.3 that the elements of T' are all congruent modulo 2. Hence, by Theorem 3.7, the set T' is also $[c,b]$-good. Thus, there exist positive integers k_2 and n_2 such that, for each $t \in T'$, $S_{[c,b]}^{k_2}(t + n_2) = v$. Combining these results, we find that

$$S_{[c,b]}^{k_2}\left(S_{[c,b]}^{k_1}(T + n_1) + n_2\right) = \left\{v, S_{[c,b]}^{k_2}(v + n_2)\right\},$$

where, again using Lemma 2.3, since v is even, $S_{[c,b]}^{k_2}(v+n_2)$ is odd. Let k_3 be a multiple of ℓ, sufficiently large so that $S_{[c,b]}^{k_2+k_3}(v + n_2) \in U_{[c,b]}$. Then,

$$S_{[c,b]}^{k_2+k_3}\left(S_{[c,b]}^{k_1}(T + n_1) + n_2\right) = \left\{v, S_{[c,b]}^{k_2+k_3}(v + n_2)\right\} = V_j,$$

for some j.

By Lemma 4.3, to prove that T is $[c,b]$-cycle-good, it suffices to prove that each of the V_j is $[c,b]$-cycle-good. To each C_i, we associate a pair

TABLE 5. $S_{[c,b]}^{k_j'}(V_j + n_j') \subseteq V_1$.

$[c, b]$	k_2', n_2'	k_3', n_3'	k_4', n_4'	k_5', n_5'	k_6', n_6'	k_7', n_7'	k_8', n_8'	k_9', n_9'
$[5, 3]$	$2, 1202$	$2, 12112$						
$[1, 7]$	$1, 0$	$5, 14$						
$[3, 7]$	$3, 0$	$5, 131$	$3, 3$					
$[5, 7]$	$9, 1$	$5, 16$	$6, 114$	$7, 0$	$6, 114$			
$[5, 9]$	$9, 212$	$4, 7$	$6, 22$	$5, 147$	$7, 0$	$1, 212$	$6, 147$	$7, 32$
$[7, 9]$	$9, 3$	$6, 150$	$4, 16$	$11, 31$	$9, 31$			
$[9, 9]$	$7, 6$	$3, 0$						

(k_i, n_i), as given in Table 4, and to each V_j, we associate a pair (k_j', n_j'), as in Table 5.

To show that V_1 is $[c, b]$-cycle-good, fix a cycle C_i of $S_{[c,b]}$, and note that, for each $t \in V_1$, $S_{[c,b]}^{k_i}(t + n_i) \in C_i$. Thus, V_1 is $[c, b]$-cycle-good. Now, fix j such that $V_j \neq V_1$. Direct calculation shows that, for each $t \in V_j$, $S_{[c,b]}^{k_j'}(t + n_j') \in V_1$. Since V_1 is $[c, b]$-cycle-good, so is V_j. Thus, by Lemma 4.3, T is $[c, b]$-cycle-good.

Hence, for each c and b as in the theorem, every finite set is $[c, b]$-cycle-good. Considering the specific sets $T_N = \{1, 2, \ldots, N\}$ completes the proof. $\square$

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