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SIGNED PERMUTATIONS AND THE BRAID GROUP

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ABSTRACT. We make a connection between the braid group and signed permutations. Using this link, we describe a commutative diagram which contains the fundamental sequence for the braid group.

1. The braid group. The braid group was explicitly defined in 1925 by Artin [**1**]. For a recent detailed description of braids see [**10**] and for a classical treatment of braids see [**3**]. In the context of our investigation, we will establish a very close connection between the braid group and Japanese ladders. This connection will enhance our understanding of a well-known, short exact sequence related to the braid group.

We begin by defining the braid group and then relating its elements to signed permutations. The braid group, $\mathcal{B}_n$, is generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ along with the defining relations:

(B1) $\sigma_i\sigma_j = \sigma_j\sigma_i \ , \ |i-j| \geq 2;$
(B2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \ 1 \leq i \leq n-2.$

Relation (B1) is commonly called *far-commutativity* and relation (B2) is referred to as the *braid* relation. There is also the *trivial* relation, $\sigma_i(\sigma_i)^{-1} = e = (\sigma_i)^{-1}\sigma_i$, where e is n straight strands with no crossings, namely, it is the identity. Pictorially, we may represent the braid σ_i as in Figure 1, the ith strand crossing over the $(i+1)$st.

Even in a small example like $\mathcal{B}_3$, the group can be complex; it is a nonabelian infinite group and corresponds to the fundamental group of the complement of the trefoil knot [**13**].

By enumerating each strand, it is fairly intuitive to associate permutations with braids. This can be accomplished by mapping each

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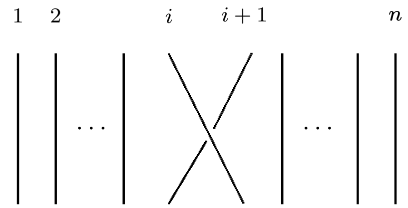

FIGURE 1. Generator of the braid group, σ_i.

generator, σ_i, to the transposition $(i, i+1)$. The kernel of this homomorphism is of particular interest.

Definition 1.1. The kernel of the homomorphism that maps $\mathcal{B}_n$ to $\mathcal{S}_n$ given by $\sigma_i \to (i, i+1)$ is $\mathcal{P}_n$, the pure braid group.

Pictorially, $\mathcal{P}_n$ consists of the braids in which each strand starts and ends in the same position. The pure braid group plays a significant role in a short exact sequence, which will be discussed in a later section.

1.1. Signed permutations and ladders. In the visual representation of a braid, instead of drawing two transversely crossing strands, one may draw vertical lines and a horizontal segment between them. This corresponds to a permutation in S_n, and is sometimes called a Japanese ladder, which is further described in [**6, 12**]. This representation is not unique, as exemplified in Figure 2.

Ladders lack the distinction between clockwise and counterclockwise crossings. We may equip ladders with this distinction by adding arrows to each rung. We will refer to these as *signed ladders*. Furthermore, signed ladders provide a visualization of so-called signed permutations. The braid σ_i corresponds to the signed ladder in Figure 3 and represents sending $i \to i+1$, $i+1 \to -i$ and $j \to j$ for $j \neq i$, $i+1$.

Definition 1.2. A *signed permutation* is a map $\alpha : \{1, 2, \ldots, n\} \to \{\pm 1, \pm 2, \ldots, \pm n\}$ such that if $\alpha(i) = k$ then $\alpha(i') \neq \pm k$ whenever $i \neq i'$.

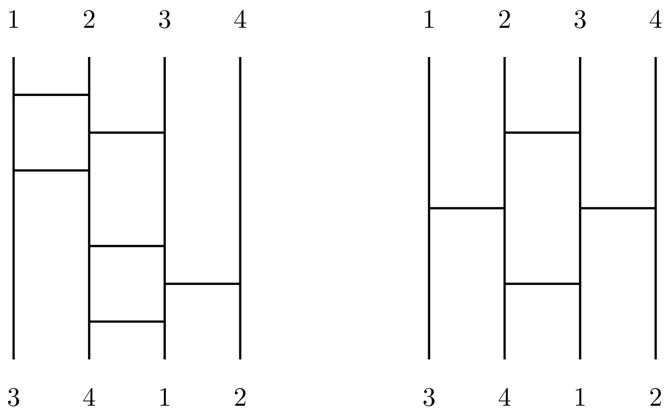

FIGURE 2. Two ladders representing the permutation $\pi = (1,3)(2,4)$.

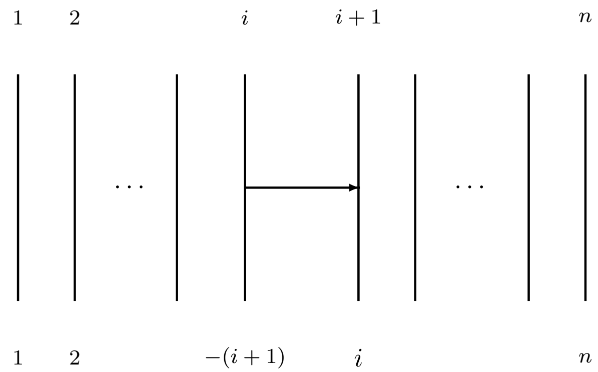

FIGURE 3. Signed ladder representation of σ_i.

For those more familiar with Coxeter groups, signed permutations can be viewed in terms of the Coxeter group of type B, see [**4, 5, 9**]. They can also be viewed as the group of signed permutation matrices and the hyperoctahedral group.

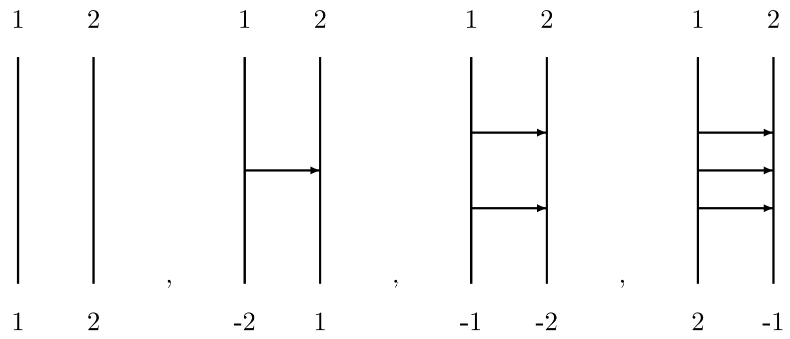

FIGURE 4. Realizable signed ladders on 2 elements.

Definition 1.3. Let π be a signed permutation. Consider the sequence*:*

$$\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n).$$

These numbers, at the bottom of the ladder, are the final state and are denoted by

$$[\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)].$$

If a ladder exists with this final state, then this final state is said to be *realizable.*

For each signed permutation, there are infinitely many signed ladders that correspond to it; however, not all signed permutations are realizable by signed ladders. We let $\mathcal{E}'_n$ be the set of all signed permutations, and let $\mathcal{E}_n$ be the group of signed permutations that are realizable as signed ladders. Figure 4 shows all realizable signed permutations for $n = 2$. There are four non-realizable signed permutations in $\mathcal{E}'_2$: $(1 \to -1,\ 2 \to 2)$, $(1 \to 1,\ 2 \to -2)$ $(1 \to 2,\ 2 \to 1)$, and $(1 \to -2,\ 2 \to -1)$.

In general, the signed permutations $\sigma_i, \sigma_i^2, \sigma_i^3$ and σ_i^4, which are $(i \to i+1,\ i+1 \to -i)$, $(i \to -i,\ i+1 \to -(i+1))$, $(i \to -(i+1),\ i+1 \to i)$ and $(i \to i,\ i+1 \to i+1)$, respectively, are realizable. The signed permutations $(i \to -i, i+1 \to i+1)$, $(i \to i, i+1 \to -(i+1))$, $(i \to i+1,\ i+1 \to i)$ and $(i \to -(i+1),\ i+1 \to -i)$ are not realizable.

We shall now determine the cardinality of the group $\mathcal{E}'_n$ and show how $\mathcal{E}_n$ relates to it.

Theorem 1.4. *The cardinality of the group $\mathcal{E}'_n$ is $n!\,2^n$; moreover, $\mathcal{E}_n$ is normal in $\mathcal{E}'_n$.*

Proof. There are $n!$ permutations on n letters, and there are two choices of sign for each of the n letters. To find the size of $\mathcal{E}_n$ note that inserting σ_i^2 into a ladder results in multiplying the signs of the ith and $(i+1)$st positions by -1 but does not change the underlying permutation. For each permutation, there are $n-1$ places to put σ_i^2, giving $n!\,2^{n-1}$ elements in $\mathcal{E}_n$. The normality follows from the fact that $\mathcal{E}_n$ is of index 2 in $\mathcal{E}'_n$. $\square$

It can be shown how to transform one Japanese ladder into an equivalent ladder using two different representations of the permutation $(1,3)$, namely, $(1,3) = (1,2)(2,3)(1,2)$ and $(1,3) = (2,3)(1,2)(2,3)$. This is even more natural for braids. Consider the ladders given in Figure 5. They both represent the realizable signed permutation $(1 \to 3,\ 2 \to -2,\ 3 \to 1)$. Moreover, this represents one of the fundamental braid relations, namely, it is simply a visualization of property (B2), that is, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i \le n-2$.

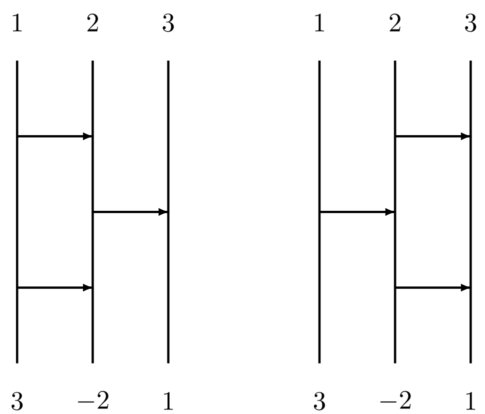

FIGURE 5. Equivalent signed permutations.

Recall the correspondence between a braid generator and a signed ladder by drawing σ_i from left to right as a rung. The inverse braid, σ_i^{-1}, would correspond to a ladder with the rung going from right to left. Note that, when viewed as signed ladders, $\sigma_i^{-1} = \sigma_i^3$, as illustrated in Figure 6. The ladders would also correspond to the same underlying permutation. This leads us to say that two braids are signed permutation equivalent if their corresponding signed permutations are equal.

Alexander's theorem states that there is an onto function from the set of braids to the set of links and knots. However, closing two braids might result in the same link so this map is not one-to-one. A natural question is whether two equivalent signed permutations can give two different knots. The answer is yes. For example, we have seen that, as signed permutations, $\sigma^{-1} = \sigma^3$; however, the closure of σ^{-1} is the unknot and the closure of σ^3 is the trefoil.

In a signed ladder, every occurrence of σ_i^{-1} can be replaced with σ_i^3, causing each rung in the ladder to run from left to right. Further, we say that a braid is positive if it can be expressed as a product of only positive powers of σ_i, $1 \leq i < n$. Figure 6 shows that every realizable signed permutation can be written as a positive braid.

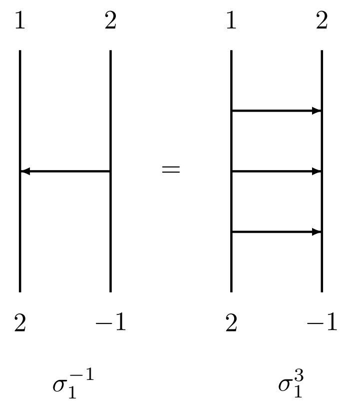

FIGURE 6. All realizable signed permutations can be written as positive braids.

1.2. A short exact sequence. The usual short exact sequence used when studying the braid group is:

$$1 \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{B}_n \longrightarrow \mathcal{S}_n \longrightarrow 1.$$

The kernel in this sequence represents those braids whose underlying permutation is the identity. From the point of view presented here, instead of mapping $\mathcal{B}_n$ to $\mathcal{S}_n$, it makes more sense to consider the mapping of $\mathcal{B}_n$ to $\mathcal{E}_n$. Then the kernel, $\mathcal{K}_n$, consists only of those braids whose underlying signed permutation is the identity. This means that, not only is the underlying permutation the identity, but all signs are positive. Our new short exact sequence is:

$$1 \longrightarrow \mathcal{K}_n \longrightarrow \mathcal{B}_n \longrightarrow \mathcal{E}_n \longrightarrow 1.$$

To find the kernel $\mathcal{K}_n$, we get our hands dirty and consider the smallest example, $n = 2$. According to earlier calculations, $|\mathcal{E}_2| = 4$, see Figure 4, and $B_2 = \mathbb{Z}$, so $\mathcal{K}_2$ should be $4\mathbb{Z}$. In fact, $\mathcal{K}_2 = \langle \sigma_1^4 \rangle$, see Figure 7. This can also be seen as Dirac's belt trick [**8, 11**], or Feynman's plate trick [**7**]. More specifically, it is a physical representation of the fact that $SU(2)$ double covers $SO(3)$. The knot theorist Kauffman has called it the quaternionic handshake.

Let us delve into the group $\mathcal{E}_2$. The element σ_1^2 is the signed permutation $(1 \to -1,\ 2 \to -2)$. Exploring this further, we notice that, while its underlying permutation is the identity, the signs are not all positive. Therefore, it is not the identity as a signed permutation. Note that σ_1^4 is the identity. This explains why we want σ_i^4 to be in the kernel, but we do not want σ_i^2 to be in the kernel. In terms of Kauffman's quaternionic handshake, the first rung of σ_1^2 is like multiplying by the quaternion $\mathbf{i}$. Then, the next rung is like multiplying by the quaternion $\mathbf{j}$ and then by the quaternion $\mathbf{k}$ since not only is the rung going over but being multiplied by -1 since it is going across in the wrong direction. In this sense, everything has returned to its original position but has been multiplied by -1.

In general, the group $\mathcal{K}_n$ is a bit more complicated. In Figure 8, we have an element of $\mathcal{K}_3$ that is not generated by a combination of $(\sigma_i)^4$ nor its conjugates.

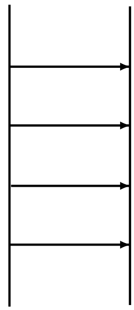

FIGURE 7. Generator of $\mathcal{K}_2$.

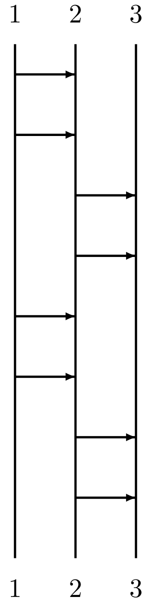

FIGURE 8. Element of $\mathcal{K}_3$.

There are infinitely many distinct elements in $\mathcal{P}_n$ and $\mathcal{K}_n$ for $n > 1$. For example, consider $(\sigma_1^4)^k$ for any k. As braids, they are distinct but they represent only one element as signed permutations.

Juxtaposing the two earlier short exact sequences, we obtain:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{K}_n & \longrightarrow & \mathcal{B}_n & \longrightarrow & \mathcal{E}_n & \longrightarrow & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{P}_n & \longrightarrow & \mathcal{B}_n & \longrightarrow & \mathcal{S}_n & \longrightarrow & 1 \end{array}$$

Given this beautiful commutative diagram, it is natural to wonder how $\mathcal{K}_n$ sits inside $\mathcal{P}_n$. There are $n-1$ possible spaces between posts to put σ_i^2 which corresponds to 2^{n-1} cosets of $\mathcal{K}_n$ in $\mathcal{P}_n$. Theorem 1.5 is an immediate consequence.

Theorem 1.5. *Let $\mathcal{P}_n$ be a pure braid group, and let $\mathcal{K}_n$ be the set of those braids whose underlying signed permutation is the identity. Then, we have $[\mathcal{P}_n : \mathcal{K}_n] = 2^{n-1}$.*

1.3. An explicit description of $\mathcal{K}_n$. The pure braid group, $\mathcal{P}_n$, is generated by

$$a_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},$$

where $1 \leq i < j \leq n$, [**2**]. We aim to expose a similar construction of $\mathcal{K}_n$ and to fully understand the manner in which it sits inside $\mathcal{P}_n$. In order to do so we must first consider the signs created by all possible final states of a signed permutation.

Theorem 1.6. *The only allowable final states for a signed permutation π, adhere to the property*

$$\prod_{i=1}^{n} \mathrm{Sign}(\pi(i)) = (-1)^{\mathrm{par}(\pi)},$$

where $\mathrm{par}(\pi)$ *is the parity of the underlying permutation, π.*

Proof. Each time a transposition is made one element is multiplied by -1. This implies that the product of the signs of the elements in the final state must equal $(-1)^r$ where r is the number of rungs in the signed ladder corresponding to π. □

Since every element in $\mathcal{P}_n$, and hence in $\mathcal{K}_n$, is the identity permutation, which is even and the product of the signs in the final state is $+1$, that is, there must be an even number of rungs in the ladder corresponding to each element of $\mathcal{P}_n$. We observe that

$$\sigma_m(k) = \begin{cases} k & k \neq m; \\ k+1 & m = k; \\ -k & m = k+1. \end{cases}$$

Hence, the the signs of the final state of $a_{i,j}$ may be viewed in sequence as

$$[+,+,\ldots,+,-,+,+,\ldots,+,+,-,+,\ldots,+],$$

where the negative signs are in the ith and the jth positions. Therefore, $a_{i,j} \notin \mathcal{K}_n$. However, multiplying $a_{i,j}$ by σ_i^2 simply multiplies the signs of the ith and $(i+1)$st positions by -1. So, by multiplying $a_{i,j}$ by

$$\prod_{k=i}^{j-1} \sigma_k^2,$$

all signs in the final state will become positive.

Theorem 1.7. *The group $\mathcal{K}_n$ is generated by $r_{i,j} = a_{i,j}\sigma_i^2\sigma_{i+1}^2 \cdots \sigma_{j-1}^2$, $1 \leq i < j \leq n$.*

Proof. Recall that $\mathcal{P}_n$ is generated by $a_{i,j}$ for $1 \leq i < j \leq n$. By the preceding argument, $r_{i,j} \in \mathcal{K}_n$. Thus, $\langle r_{i,j}\rangle \leq \mathcal{K}_n$. Moreover, the cosets of $\langle r_{i,j}\rangle$ in $\mathcal{P}_n$ are

$$\prod_{\substack{A\subset\{1,\ldots,n\} \\ i\in A}} \sigma_i^2,$$

which implies that $[\mathcal{P}_n : \langle r_{i,j}\rangle] = 2^{n-1}$. However, $[\mathcal{P}_n : \mathcal{K}_n] = 2^{n-1}$ by Theorem 1.5. Therefore $\mathcal{K}_n$ is generated by $\{r_{i,j}\}$. $\square$

Note that we have a canonical map $\mathcal{P}_n \to \mathbb{Z}_2^{n-1}$, since every strand in $\mathcal{P}_n$ ends where it begins, and $\mathbb{Z}_2^{n-1}$ corresponds to possible changes in sign. Further, we have the diagram:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{K}_n \arrow[r] \arrow[d, equal] & \mathcal{P}_n \arrow[r] \arrow[d] & \mathbb{Z}_2^{n-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{K}_n \arrow[r] & \mathcal{B}_n \arrow[r] \arrow[d] & \mathcal{E}_n \arrow[r] \arrow[d] & 1 \\
 & & \mathcal{S}_n \arrow[r, equal] \arrow[d] & \mathcal{S}_n \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

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