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ALMOST MULTIPLICATIVE LINEAR FUNCTIONALS AND ENTIRE FUNCTIONS

EHSAN ANJIDANI

ABSTRACT. Let T be a unital, continuous linear functional defined on complex Banach algebra A. First, we prove an approximate version of the Gleason-Kahane-Żelazko theorem: given $\epsilon > 0$, there exists an $M > 0$ such that, if

$$T(\exp x) \neq 0, \quad x \in A, \ \|x\| \leq M,$$

then T is ϵ-almost multiplicative. Then, we show that this result remains true if the exponential function is replaced by a nonsurjective entire function F with $F'(0) \neq 0$.

1. Introduction and preliminaries. Let A be a complex unital Banach algebra with unit e. The Gleason-Kahane-Żelazko (G-K-Z) theorem states that, if T is a unital linear functional on A such that

$$T(x) \neq 0, \quad \text{for all } x \in \text{Inv}A,$$

then T is multiplicative. By taking a close look at the standard proofs of the G-K-Z theorem, one can deduce the next, stronger result.

Theorem 1.1. *If T is a unital, linear functional on A such that*

$$T(\exp x) \neq 0, \quad \textit{for all } x \in A,$$

then T is multiplicative.

There are several extensions of the G-K-Z theorem, see [**6, 9, 12**]. In 1987, in the direction of the extension of the G-K-Z theorem, Arens [**3**] conjectured that, if F is a nonconstant entire function, then the condition

$$T \circ F(x) \neq 0, \quad x \in A,$$

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implies the multiplicativity of T. In 1997, Jarosz [**7**] proved the Arens conjecture as follows:

Theorem 1.2. *Let F be a nonconstant, entire function, let T be a linear functional on A, and let $T \circ F : A \to \mathbb{C}$ be a nonsurjective function. Then,*

(i) *if $T(e) \neq 0$, then $T/T(e)$ is multiplicative*;
(ii) *if $T(e) = 0$, then $T = 0$.*

In this paper, we prove the results which can be considered as approximate versions of Theorems 1.1 and 1.2.

A linear functional T on A is said to be ϵ-almost multiplicative if $\operatorname{mult}(T) \leq \epsilon$, where

$$\operatorname{mult}(T) = \sup\{\|T(xy) - T(x)T(y)\| : x, y \in A,\ \|x\| = \|y\| = 1\}.$$

Such functionals have been extensively studied, see [**5, 8, 10**] for more details and examples. Johnson [**10**] proved that a continuous linear functional ϕ on A is almost multiplicative if

$$d(\phi(a), \sigma(a)) < \epsilon, \quad a \in A,\ \|a\| = 1,$$

where $\sigma(a)$ is the spectrum of a. This and similar results can be considered as approximate versions of the G-K-Z theorem, see, for example, [**1,** Theorem 4.2] and [**11,** Theorem 5]. They are concerned with identifying the almost-multiplicative linear functionals among all linear functionals on Banach algebra A in terms of spectra.

In Section 2, we prove the result from which all such approximate versions of the G-K-Z theorem can be derived. This result is an analogue of Theorem 1.1: let T be a continuous unital linear functional on A. Given $\epsilon > 0$, there exists an $M > 0$ such that, if

$$T(\exp x) \neq 0, \quad x \in A,\ \|x\| \leq M,$$

then T is ϵ-almost multiplicative.

In Section 3, we show that this result remains true if the exponential function is replaced by a nonsurjective entire function F with $F'(0) \neq 0$.

Throughout this paper, let A be a complex unital Banach algebra with unit e, and let $A_{[r]}$ be the closed ball in A with center 0 and radius

$r > 0$. The open unit disc is denoted by $\mathbb{D}$. We denote the open disc of radius $r > 0$ around the origin in $\mathbb{C}$ by $\mathbb{D}(0, r)$.

2. Approximate version of the G-K-Z theorem. Our main result in this section is given by Corollary 2.2 and may be considered an approximate version of Theorem 1.1. First, we prove the next theorem by a similar method to the proof of [**2,** Theorem 3.2] or [**11,** Theorem 5].

Theorem 2.1. *Let ϕ be a unital, continuous linear functional on A. Suppose that there is an $M > 0$ such that $M \geq \ln \|\phi\|$ and*

$$\phi(\exp x) \neq 0, \quad x \in A_{[M]}. \tag{2.1}$$

Then,

$$\operatorname{jmult}(\phi) \leq \frac{1}{M}\left(\frac{6}{\ln 2} + \frac{1}{M}\right),$$

where

$$\operatorname{jmult}(\phi) = \sup\{\|\phi(a^2) - \phi(a)^2\| : a \in A,\ \|a\| = 1\}.$$

Proof. Let $a \in A$ with $\|a\| = 1$. Since ϕ is continuous and linear, the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) := \phi(\exp za) = \sum_{n=0}^{\infty} \frac{\phi(a^n)}{n!} z^n,$$

is entire such that, for all $z \in \mathbb{C}$, we have

$$|f(z)| \leq \|\phi\| \|\exp za\| \leq \|\phi\| e^{|z|}. \tag{2.2}$$

Therefore, f has growth order ≤ 1. Suppose that $\alpha_1, \alpha_2, \ldots$, are the zeros of f indexed with

$$|\alpha_1| \leq |\alpha_2| \leq \cdots .$$

Using Hadamard's factorization theorem [**13**], and, by the same method as in the proof of [**11,** Theorem 5], we obtain

$$\phi(a^2) - \phi(a)^2 = -\sum_j \frac{1}{\alpha_j^2}.$$

Now, let α_j be a zero of f. Since f is an entire function and $f(0) = 1$, by Jensen's formula **[13]**, we have

$$\sum_{k=1}^{N} \ln \frac{2|\alpha_j|}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(2|\alpha_j|e^{i\theta})| \, d\theta, \tag{2.3}$$

where $z_1, \ldots, z_N$, denote the zeros of f in the open disc of radius $2|\alpha_j|$ centered at the origin. Since $|\alpha_i| \leq |\alpha_j|$ for every $1 \leq i \leq j$, we obtain

$$j \ln 2 \leq \sum_{i=1}^{j} \ln \frac{2|\alpha_j|}{|\alpha_i|} \leq \sum_{k=1}^{N} \ln \frac{2|\alpha_j|}{|z_k|}. \tag{2.4}$$

Also, by equation (2.2), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(2|\alpha_j|e^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \ln(\|\phi\| e^{2|\alpha_j|}) \, d\theta = \ln \|\phi\| + 2|\alpha_j|. \tag{2.5}$$

Thus, by equations (2.3), (2.4) and (2.5), we obtain $j \ln 2 \leq \ln \|\phi\| + 2|\alpha_j|$. On the other hand, by equation (2.1), we have $|\alpha_j| > M \geq \ln \|\phi\|$. Therefore, we obtain

$$j \frac{\ln 2}{3} \leq |\alpha_j|, \quad \text{for all } j.$$

Let k be the greatest integer less than or equal to $(3/\ln 2)M$. Now, we find a bound for

$$\left| \sum_j \frac{1}{\alpha_j^2} \right|,$$

using $|\alpha_j| \geq M$ for $1 \leq j \leq k$, and $|\alpha_j| \geq (\ln 2/3)j$ for $j > k$. Then, we have

$$\begin{aligned}
\left| \sum_j \frac{1}{\alpha_j^2} \right| &\leq \sum_{j=1}^{k} \frac{1}{|\alpha_j|^2} + \sum_{j=k+1}^{\infty} \frac{1}{|\alpha_j|^2} \\
&\leq \sum_{j=1}^{k} \frac{1}{M^2} + \sum_{j=k+1}^{\infty} \left(\frac{3}{\ln 2} \right)^2 \frac{1}{j^2} \\
&\leq \frac{k}{M^2} + \left(\frac{3}{\ln 2} \right)^2 \left(\frac{1}{(k+1)^2} + \frac{1}{k+1} \right) \\
&\leq \frac{1}{M} \left(\frac{6}{\ln 2} + \frac{1}{M} \right).
\end{aligned}$$

Thus,

$$|\phi(a^2) - \phi(a)^2| \le \frac{1}{M}\left(\frac{6}{\ln 2} + \frac{1}{M}\right),$$

for all $a \in A$ with $\|a\| = 1$. Therefore, we obtain

$$\operatorname{jmult}(\phi) \le \frac{1}{M}\left(\frac{6}{\ln 2} + \frac{1}{M}\right). \qquad \square$$

Corollary 2.2. *For each* ϵ, $k > 0$, *there exists an* $M > 0$ *such that, if* T *is a unital, continuous linear functional on* A *with* $\|T\| < k$ *and*

$$T(\exp x) \neq 0, \quad x \in A_{[M]},$$

then T *is* ϵ*-multiplicative.*

Proof. Let $\epsilon > 0$ and $k > 0$. By [**1,** Corollary 3.6], there is a $\delta > 0$ such that, if ϕ is a linear functional on A with $\operatorname{jmult}(\phi) < \delta$, then $\operatorname{mult}(\phi) < \epsilon$. Choose $M > 0$ with $M > \ln k$ and

$$\frac{1}{M}\left(\frac{6}{\ln 2} + \frac{1}{M}\right) < \delta.$$

Now, if T is a unital, continuous linear functional on A with $\|T\| < k$ and $T(\exp x) \neq 0$ for all $x \in A_{[M]}$, then, by Theorem 2.1, $\operatorname{jmult}(T) < \delta$. Thus, $\operatorname{mult}(T) < \epsilon$. $\square$

3. Almost multiplicative functionals and entire functions. In this section, we show that Corollary 2.2 remains true if the exponential function is replaced by a nonsurjective entire function F with $F'(0) \neq 0$. First, we give a sufficient condition for a linear functional to be continuous, compare with [**7,** Lemma 6].

Theorem 3.1. *Let* F *be a nonconstant entire function, and let* T *be a linear functional on* A. *Suppose that there is an* $r > 0$ *such that the function* $T \circ F : A_{[r]} \to \mathbb{C}$ *is nonsurjective. Then,* T *is continuous.*

Proof. Let $0 < r' < r$. Since F is a nonconstant entire function, there is a $z_0 \in \mathbb{C}$ with $|z_0| < r'$ such that $F'(z_0) \neq 0$. Let G be a function defined by

$$G(z) = F(z + z_0) - F(z_0), \quad z \in \mathbb{C}.$$

For every $x \in A$ with $\|x\| \leq r - r'$, we have $\|x + z_0 e\| < r$ and

$$T \circ G(x) = T \circ F(x + z_0 e) - T \circ F(z_0 e).$$

Thus, $T \circ G : A_{[r-r']} \to \mathbb{C}$ is nonsurjective since $T \circ F : A_{[r]} \to \mathbb{C}$ is nonsurjective. Hence, without loss of generality, we may assume that $F(0) = 0$ and $F'(0) \neq 0$. Since $F'(0) \neq 0$, there are neighborhoods U and V of 0 such that F is a homeomorphism of U onto V. The function $(F|_U)^{-1}$ is holomorphic on V. Hence, there are $\epsilon > 0$ and a complex sequence $\{\beta_n\}$ such that

$$(F|_U)^{-1}(\omega) = \sum_{n=0}^{\infty} \beta_n \omega^n,$$

for all $\omega \in \mathbb{D}(0, \epsilon)$. Suppose that F has the power series expansion

$$\sum_{n=0}^{\infty} \alpha_n z^n.$$

For every $\omega \in \mathbb{D}(0, \epsilon)$, we have

$$\begin{aligned} \omega = F((F|_U)^{-1}(\omega)) &= \sum_{n=0}^{\infty} \alpha_n ((F|_U)^{-1}(\omega))^n \\ &= \sum_{n=0}^{\infty} \alpha_n \left(\sum_{k=0}^{\infty} \beta_k \omega^k \right)^n \\ &= \alpha_1 \beta_1 \omega + \cdots . \end{aligned}$$

Hence, $\alpha_1 \beta_1 = 1$ and, for every $n > 1$, the coefficient of ω^n is 0. Therefore, $F((F|_U)^{-1}(x)) = x$ for all $x \in A$ with $\|x\| < \epsilon$. Since $(F|_U)^{-1} : A_{[\epsilon]} \to A$ is continuous at 0, there is a $0 < \delta < \epsilon$ such that $\|(F|_U)^{-1}(x)\| < r$ for all $x \in A$ with $\|x\| \leq \delta$. Thus, we have

$$T(A_{[\delta]}) = T \circ F((F|_U)^{-1}(A_{[\delta]})) \subseteq T \circ F(A_{[r]}) \subsetneqq \mathbb{C},$$

and T is nonsurjective on $A_{[\delta]}$; hence, T is continuous. $\square$

The main result of this section follows.

Theorem 3.2. *Let F be an entire function such that $F'(0) \neq 0$, and let there be a $z_0 \in \mathbb{C}$ such that $F(z) \neq z_0$ for every $z \in \mathbb{C}$. Then, for each ϵ, $k > 0$, there is an $M > 0$ such that, if T is a unital, continuous* *linear functional on A with $\|T\| < k$ and $T \circ F(x) \neq z_0$ for all $x \in A_{[M]}$, then T is ϵ-multiplicative.*

In order to prove this result, we first prove the next theorem.

Theorem 3.3. *Let g be an entire function with $g'(0) \neq 0$, and let T be a unital, continuous linear functional on A. Suppose that there is an $M > 0$ with*

$$M > \frac{144}{|g'(0)|} \ln(2\|T\|), \quad \textit{such that } (T \circ \exp g)(x) \neq 0,$$

for all $x \in A_{[M]}$. Then, $\operatorname{jmult}(T) < \epsilon$*, where*

$$\epsilon = \frac{576}{M|g'(0)|}\left(\frac{6}{\ln 2} + \frac{144}{M|g'(0)|}\right).$$

Proof. By a similar method to [**7,** Proof of Theorem 3], we first prove the result on the disc algebra $A(\mathbb{D})$. Let ϕ be a unital, continuous linear functional on $A(\mathbb{D})$, and suppose that there is an $R > 0$ with $R > (72/|g'(0)|) \ln \|\phi\|$ such that $(\phi \circ \exp g)(x) \neq 0$ for all $x \in A(\mathbb{D})_{[R]}$. The function

$$G(z) = \frac{g(Rz) - g(0)}{Rg'(0)}$$

is entire, satisfying $G(0) = 0$ and $G'(0) = 1$. By Bloch's theorem [**4**], there is a disc $\mathcal{S} \subseteq \mathbb{D}$ such that G is one-to-one on $\mathcal{S}$ and $G(\mathcal{S})$ contains a disc of radius $1/72$. Hence, $R\mathcal{S} \subseteq \mathbb{D}(0, R)$, g is one-to-one on $R\mathcal{S}$ and $g(R\mathcal{S})$ contains a disc of radius $R|g'(0)|/72$ and center at some point ω_0. Let $a \in A(\mathbb{D})$ with $\|a\| = 1$, and let $\lambda \in \mathbb{C}$ with $|\lambda| < R|g'(0)|/72$. Define the function

$$\psi : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$$

by

$$\psi(z) = (g|_{R\mathcal{S}} - \omega_0)^{-1}(\lambda a(z)).$$

It is clear that $\psi \in A(\mathbb{D})$. Since

$$\mathbb{D}\left(0, \frac{R|g'(0)|}{72}\right) \subseteq (g - \omega_0)(R\mathcal{S}),$$

we have

$$\psi(z) \in R\mathcal{S} \subseteq \mathbb{D}(0, R),$$

for all $z \in \overline{\mathbb{D}}$. Hence, $\|\psi\| \le R$, and so $(\phi \circ \exp g)(\psi) \neq 0$. On the other hand, we have

$$g \circ \psi(z) = g((g|_{R\mathcal{S}} - \omega_0)^{-1}(\lambda a(z))) = \lambda a(z) + \omega_0,$$

for all $z \in \overline{\mathbb{D}}$. Hence, $g \circ \psi = \lambda a + \omega_0$. Thus,

$$\begin{aligned} \phi(\exp \lambda a) = \phi(\exp(g \circ \psi - \omega_0)) &= e^{-\omega_0} \phi(\exp g(\psi)) \\ &= e^{-\omega_\circ} (\phi \circ \exp g)(\psi) \neq 0. \end{aligned}$$

Now, by Theorem 2.1, we have

$$\text{jmult}(\phi) \le \frac{72}{R|g'(0)|}\left(\frac{6}{\ln 2} + \frac{72}{R|g'(0)|}\right). \tag{3.1}$$

Now, fix $x \in A$ with $\|x\| = 1$. For every

$$a(z) = \sum_{n=0}^{\infty} \gamma_n z^n \in A(\mathbb{D}),$$

define

$$\mathbf{a}(x) := \sum_{n=0}^{\infty} \gamma_n \left(\frac{x}{2}\right)^n.$$

It is clear that $\mathbf{a}(x)$ is an element of A, and the function $a \mapsto \mathbf{a}(x)$, an algebraic homomorphism, maps from $A(\mathbb{D})$ into A. We define the linear functional

$$\widetilde{T} : A(\mathbb{D}) \longrightarrow \mathbb{C}$$

by

$$\widetilde{T}(a) = T(\mathbf{a}(x)). \tag{3.2}$$

It is easy to see that

$$\widetilde{T}(\exp g(a)) = T \circ \exp g(\mathbf{a}(x)), \tag{3.3}$$

for all $a \in A(\mathbb{D})$. Let $a \in A(\mathbb{D})$ with $\|a\| \le M/2$. If

$$\sum_{n=0}^{\infty} \gamma_n z^n$$

is the power series expansion of the analytic function a, then, by the Cauchy estimate, we have $|\gamma_n| \leq M/2$ for all $n \in \mathbb{N} \cup \{0\}$. Thus,

$$\|\mathbf{a}(x)\| \leq \sum_{n=0}^{\infty} \frac{|\gamma_n|}{2^n} \leq \frac{M}{2} \sum_{n=0}^{\infty} 2^{-n} = M.$$

Hence, by equation (3.3) and our assumption, for any $a \in A(\mathbb{D})$ with $\|a\| \leq M/2$, we have

$$\widetilde{T} \circ \exp g(a) \neq 0.$$

Also, for every $a \in A(\mathbb{D})$ with $\|a\| \leq 1$, we have

$$|\widetilde{T}(a)| = |T(\mathbf{a}(x))| \leq \|T\| \|\mathbf{a}(x)\| \leq 2\|T\|,$$

and so $\widetilde{T}$ is continuous, $\|\widetilde{T}\| \leq 2\|T\|$. Since

$$M > \frac{144}{|g'(0)|} \ln \frac{2\|T\|}{|T(e)|},$$

we obtain

$$\frac{72}{|g'(0)|} \ln \|\widetilde{T}\| \leq \frac{72}{|g'(0)|} \ln(2\|T\|) < \frac{M}{2}.$$

Therefore, setting $R = M/2$ and $\phi = \widetilde{T}$, from equation (3.1), we obtain

$$\operatorname{jmult}(\widetilde{T}) \leq \frac{144}{M|g'(0)|} \left(\frac{6}{\ln 2} + \frac{144}{M|g'(0)|} \right). \tag{3.4}$$

The identity function on $\mathbb{C}$ is denoted by $\mathbf{Z}$. By equation (3.2), we have

$$T(x) = 2\widetilde{T}(\mathbf{Z}), \qquad T(x^2) = 4\widetilde{T}(\mathbf{Z}^2).$$

Hence, by equation (3.4), we obtain

$$\begin{aligned} |T(x^2) - T(x)^2| &= |4\widetilde{T}(\mathbf{Z}^2) - 4\widetilde{T}(\mathbf{Z})^2| \\ &\leq \frac{576}{M|g'(0)|} \left(\frac{6}{\ln 2} + \frac{144}{M|g'(0)|} \right) \\ &:= \epsilon. \end{aligned}$$

Thus, we have $\operatorname{jmult}(T) < \epsilon$. □

Remark 3.4. According to Theorem 3.3, if A is a commutative Banach algebra, then it follows from [**11,** Lemma 4] that T is a 2ϵ-multiplicative linear functional.

Proof of Theorem 3.2. By the Weierstrass factorization theorem **[4]**, there is an entire function g such that $F - z_0 = \exp g$. Since $F'(0) \neq 0$, we have $g'(0) \neq 0$. Now, by the same reasoning as in the proof of Corollary 2.2, it is sufficient to consider $M > 0$ with $M > (144/|g'(0)|)\ln(2k)$ and

$$\frac{576}{M|g'(0)|}\left(\frac{6}{\ln 2} + \frac{144}{M|g'(0)|}\right) < \delta,$$

and to use Theorem 3.3. □

Finally, consider the linear functional T with $T(e) = 0$; suppose that $T \circ F$ is nonsurjective on $A_{[M]}$. In this case, we show that T is close to 0.

Theorem 3.5. *Let F be an entire function with $F'(0) \neq 0$. Let T be a linear functional on A with $T(e) = 0$. Suppose that there is an $M > 0$ such that $T \circ F$ is nonsurjective on $A_{[M]}$. Then,*

$$\|T\| \leq \frac{288r}{M|F'(0)|},$$

where

$$r = \inf\{|\alpha| : \alpha \notin T \circ F(A_{[M]})\}.$$

Proof. Since $T(e) = 0$ and $F(ze) = F(z)e$ for every $z \in \mathbb{C}$, we have $T \circ F(ze) = 0$ for all $z \in \mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \{0\}$ be such that $T \circ F(x) \neq \alpha$, for all $x \in A_{[M]}$.

First, we prove that, if ϕ is a linear functional on $A(\mathbb{D})$ with $\phi(e) = 0$ and $\phi \circ F(x) \neq \alpha$ for all $x \in A(\mathbb{D})_{[M]}$, then ϕ is continuous and $\|\phi\| \leq 72|\alpha|/M|F'(0)|$. By Bloch's theorem, there is a disc $\mathcal{S} \subseteq \mathbb{D}$ such that F is one-to-one on $M\mathcal{S}$ and $F(M\mathcal{S})$ contains a disc of radius $M|F'(0)|/72$ and center at some point ω_0. Let $a \in A(\mathbb{D})$ with $\|a\| = 1$, and let $\lambda \in \mathbb{C}$ with

$$|\lambda| < \frac{M|F'(0)|}{72}.$$

As in the proof of Theorem 3.3, we define the function $\psi : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\psi(z) = (F|_{M\mathcal{S}} - \omega_0)^{-1}(\lambda a(z)).$$

We see that $\psi \in A(\mathbb{D})_{[M]}$, and so $\phi \circ F(\psi) \neq \alpha$. Since $F(\psi) = F \circ \psi = \lambda a + \omega_0$, we have

$$\phi \circ F(\psi) = \lambda\phi\,(a) + \omega_0\phi\,(e) = \lambda\phi\,(a).$$

Hence, $\lambda\phi\,(a) \neq \alpha$, for all $\lambda \in \mathbb{C}$ with $|\lambda| < M|F'(0)|/72$. Thus,

$$|\phi\,(a)| \leq \frac{72|\alpha|}{M|F'(0)|},$$

for all $a \in A(\mathbb{D})$ with $\|a\| = 1$. Hence, ϕ is continuous and $\|\phi\| \leq 72|\alpha|/M|F'(0)|$.

Now, let $x \in A$ with $\|x\| = 1$. Define the linear functional $\widetilde{T}$ on $A(\mathbb{D})$ similarly to that given in the proof of Theorem 3.3, that is, $\widetilde{T}(a) = T(\mathbf{a}(x))$ for every $a \in A(\mathbb{D})$. We have

$$\widetilde{T} \circ F(a) = T \circ F(\mathbf{a}(x)).$$

Hence, by the same reasoning as in the proof of Theorem 3.3, we have $\widetilde{T} \circ F(a) \neq \alpha$, for all $a \in A(\mathbb{D})$ with $\|a\| \leq M/2$ and $\widetilde{T}(e) = T(e) = 0$. Thus, $\widetilde{T}$ is continuous and

$$\|\widetilde{T}\| \leq \frac{144|\alpha|}{M|F'(0)|}.$$

Hence, we have

$$|T(x)| = 2|\widetilde{T}(\mathbf{Z})| \leq \frac{288|\alpha|}{M|F'(0)|},$$

for all $x \in A$ with $\|x\| = 1$; thus,

$$\|T\| \leq \frac{288|\alpha|}{M|F'(0)|}. \qquad \square$$

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University of Neyshabur, Department of Mathematics, P.O.Box 91136-899, Neyshabur, Iran

Email address: **anjidani@neyshabur.ac.ir**