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A SIGN-CHANGING SOLUTION FOR THE SCHRÖDINGER-POISSON EQUATION IN $\mathbb{R}^3$

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ABSTRACT. We find a sign-changing solution for a class of Schrödinger-Poisson systems in $\mathbb{R}^3$ as an existence result by minimization in a closed subset containing all the sign-changing solutions of the equation. The proof is based on variational methods in association with the deformation lemma and Miranda's theorem.

1. Introduction. The interaction of a charge particle with an electromagnetic field can be described by a system of a nonlinear Schrödinger equations coupled with a Poisson equation of the type

$$\text{(NLSP)} \qquad \begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a domain, $V : \Omega \to \mathbb{R}$ and $2 < p < 2^* = 6$. Many recent studies of (NLSP) have focused on existence and nonexistence of solutions, multiplicity of solutions, ground states, radial and non-radial solutions, semiclassical limit and concentrations of solutions (see, for instance, [**1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 22, 23, 24, 25, 26**]).

In [**7**], Benci and Fortunato deal with the existence of eigensolutions of a linear version of (NLSP), under a Dirichlet condition in a bounded domain Ω in $\mathbb{R}^3$, and the potential V is constant. The system (NLSP)

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in a bounded domain has also been considered in the papers of Siciliano **[26]**, Ruiz and Siciliano **[24]** and Pisani and Siciliano **[22]**.

For $\Omega = \mathbb{R}^3$, D'Aprile and Mugnai proved **[11]** the existence of a nontrivial radial solution of (NLSP) for $4 \leq p < 6$, and V is a positive constant. In **[12]**, using a Pohozaev-type identity, D'Aprile and Mugnai proved that (NLSP) has no nontrivial solution when $p \leq 2$ or $p \geq 6$. This result was completed in **[23]**, where Ruiz showed that, if $p \leq 3$, then problem (NLSP) does not admit any nontrivial solution, and, if $3 < p < 6$, then there exists a nontrivial radial solution of (NLSP). In **[4]**, Azzollini and Pomponio proved the existence of ground state solutions of (NLSP) when $3 < p < 6$ and V is a positive constant. The case of the non-constant potential was also treated **[4]** for $4 < p < 6$ and V possibly unbounded below.

All of these papers concern positive solutions to (NLSP). There are few results about sign-changing solutions to (NLSP). The best references are **[1, 14, 17]**. Ianni **[14]** employed a dynamical (not variational) approach in order to show the existence of radial solutions to (NLSP) for V constant and $p \in [4, 6)$ with a prescribed number of nodal domains. To obtain this result, she first studied the existence of sign changing radial solutions for the corresponding (NLSP) in balls of $\mathbb{R}^3$ with Dirichlet boundary conditions. Kim and Seok **[17]** obtained results similar to **[14]** for $p \in (4, 6)$ by using an extension of the Nehari variational method **[21, 27]**. Alves and Souto **[1]** considered the problem (NLSP) in a bounded domain $\Omega \subset \mathbb{R}^3$ and $V \equiv 0$ and proved the existence of least energy sign-changing solutions for (NLSP) to change signs exactly once in Ω. The proof is based on variational methods. More precisely, it was proved that the associated energy functional assumes a minimum value on the nodal set, see definition in Section 2.

Motivated by the results just described, we are interested in finding sign-changing solutions for (NLSP) in $\mathbb{R}^3$, where potential V is not necessarily a radially symmetric function. The result will be stated for a class of more general problems:

$$\text{(SP)} \qquad \begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where f belongs to $C^1(\mathbb{R}, \mathbb{R})$ and satisfies:

(f_1)
$$\lim_{s\to 0}\frac{f(s)}{s}=0;$$

(f_2)
$$\lim_{|s|\to+\infty}\frac{f(s)}{|s|^5}=0;$$

(f_3)
$$\lim_{|s|\to+\infty}\frac{F(s)}{s^4}=+\infty,\quad \text{where } F(s)=\textstyle\int_0^s f(t)\,dt;$$

(f_4)
$$\frac{f(s)}{s^3}\quad \text{is non-decreasing in } |s|>0.$$

Remark 1.1. We observe that (f_2) is weaker than the usual subcritical condition. Conditions (f_1) and (f_4) imply that

$$H(s)=sf(s)-4F(s)$$

is a non-negative and non-decreasing function in $|s|$.

Here,
$$V:\mathbb{R}^3\longrightarrow\mathbb{R}$$

is locally Hölder continuous and satisfies the following assumptions:

(V_1) there exists $\alpha>0$ such that $V(x)\geq\alpha>0$, for all $x\in\mathbb{R}^3$;

(V_2)
$$V_\infty=\sup\{V(x):x\in\mathbb{R}^3\},$$

and
$$\lim_{|x|\to+\infty}V(x)=V_\infty;$$

(V_3) there exist $R_0>0$ and
$$\rho:(R_0,\infty)\longrightarrow(0,\infty)$$

a non-increasing function such that
$$\lim_{r\to\infty}\rho(r)e^{\delta r}=\infty$$

for all $\delta > 0$, and

$$V(x) \leq V_\infty - \rho(|x|), \quad \text{for all } |x| \geq R_0.$$

As an example, given $V_\infty > 1$ and $0 < \theta < 1$, let $V(x) = V_\infty - e^{-|x|^\theta}$.

Our main result is the next theorem.

Theorem 1.2. *Suppose that f satisfies (f_1)–(f_4) and V satisfies (V_1)–(V_3). Then problem* (SP) *possesses a least energy sign-changing solution, which changes sign exactly once in $\mathbb{R}^3$.*

Theorem 1.2 can be seen as a similar version for $\mathbb{R}^3$ of the result due to [**1**]. However, the reader is invited to observe that, in Sections 3, 4 and 5, we did a careful study involving some levels which do not appear in [**1**] because, in that paper, the problems were considered on a bounded domain. Here, we need to overcome the lack compactness involving the Sobolev imbedding in $\mathbb{R}^3$, which implies that energy functionals do not verify the well known Palais-Smale condition or Cerami condition.

As observed in [**5, 6**], the general procedure to find sign-changing solutions of an equation with a nonlinear term stalls upon the fact that the nodal set is not a submanifold of H^1 because the map $u \mapsto u^\pm$ lacks differentiability; thus, it is not evident that a minimizer of the associated energy functional on the nodal set is a solution of the equation. Furthermore, there is a worsening in the case considered here: since the associated energy functional has a nonlocal term, it follows that, even if u is a sign-changing solution of the problem, the functions $u^\pm$ do not both belong to the Nehari manifold, and so some arguments used to prove the existence of nodal solutions for semilinear local problems cannot be used in our arguments.

Our approach is based on some arguments presented in [**1, 6**] in association with the deformation lemma and Miranda's theorem. The contributions of our work are twofold: on one hand, it applies the construction of [**1**] in an unbounded domain like $\mathbb{R}^3$ and consequently deals with the difficulties it brings; on the other hand, it faces the subtle peculiarities of a nonlocal term.

We begin by establishing some estimates involving functions that change sign. We find a sign-changing solution as an existence result by minimization in a closed subset containing all the sign-changing solutions of the equation. At first, this may resemble the ideas found in [**5, 6**]. However, we need to choose a suitable minimizing sequence for the nodal level. This choice involves the corresponding equation in bounded domains (balls) and the problem is then to prove that the minimum of the energy on the corresponding closed subset containing all the sign-changing solutions of the equation in bounded domains is achieved by some function in the subset. In order to overcome the possible lack of regularity of this subset, it is crucial to apply a deformation lemma and detailed use of Miranda's theorem [**20**].

2. The variational framework and technical lemmas. In this section, we present the variational framework for dealing with problem (SP). The key observation is that equation (SP) can be transformed into a Schrödinger equation with a nonlocal term, see, for instance, [**4, 23, 26**]. This permits the use of variational methods. Effectively, by the Lax-Milgram theorem, given $u \in H^1(\mathbb{R}^3)$, there exists a unique

$$\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$$

such that

$$-\Delta\phi = u^2.$$

Using standard arguments, we have that ϕ_u verifies the following properties (for a proof, see [**11, 23**]):

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, we have*:

(i) *there exists $C > 0$ such that*

$$||\phi_u||_{D^{1,2}} \leq C||u||_{H^1}^2$$

and

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C||u||_{H^1}^4 \quad \text{for all } u \in H^1(\mathbb{R}^3),$$

where

$$||u||_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\, dx$$

and

$$||w||^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla w|^2 dx;$$

(ii) $\phi_u \geq 0$ *for all* $u \in H^1(\mathbb{R}^3)$;
(iii) $\phi_{tu} = t^2\phi_u$ *for all* $t > 0$ *and* $u \in H^1(\mathbb{R}^3)$;
(iv) *if* $a \in \mathbb{R}^3$ *and* $u_a(x) = u(x-a)$, *then*

$$\phi_{u_a}(x) = \phi_u(x-a)$$

and

$$\int_{\mathbb{R}^3} \phi_{u_a} u_a^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx;$$

(v) *if* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then*

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } H^1(\mathbb{R}^3)$$

and

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \geq \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

Therefore, $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of (SP) if, and only if, $\phi = \phi_u$ and $u \in H^1(\mathbb{R}^3)$ is a weak solution of the nonlocal problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + V(x)u + \phi_u u = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

Combining (f_1)–(f_2) with Lemma 2.1, the functional

$$J : H^1(\mathbb{R}^3) \longrightarrow \mathbb{R},$$

given by

$$J(u) = \frac{1}{2}||u||^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)\, dx,$$

where

$$||u||^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx, \qquad F(s) = \int_0^s f(t)\, dt,$$

belongs to $C^1(H^1(\mathbb{R}^3),\mathbb{R})$, and

$$J'(u)v = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\,dx + \int_{\mathbb{R}^3} \phi_u uv\,dx - \int_{\mathbb{R}^3} f(u)v\,dx,$$

for all u and v in $H^1(\mathbb{R}^3)$. Hence, critical points of J are the weak solutions for nonlocal problem (P).

In what follows, we denote the Nehari manifold associated with J by $\mathcal{N}$, that is,

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J'(u)u = 0\}.$$

A nontrivial critical point u of J is a ground state of (P) if

$$J(u) = c \doteq \inf_{\mathcal{N}} J.$$

Since we are searching for sign-changing solutions, our goal is to prove the existence of a critical point for J in the set

$$\mathcal{M} = \{u \in \mathcal{N} : J'(u)u^+ = J'(u)u^- = 0 \text{ and } u^\pm \neq 0\},$$

where $u^+ = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$. More precisely, our goal is to prove that there is a critical point of J, $w \in \mathcal{M}$, such that

$$J(w) = c_0 \doteq \inf_{u\in\mathcal{M}} J(u).$$

Since J has the nonlocal term

$$\int_{\mathbb{R}^3} \phi_u u^2\,dx,$$

if u is a sign-changing solution for (P), we have that

$$J'(u^+)u^+ = -\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 < 0$$

and

$$J'(u^-)u^- = -\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 < 0.$$

Consequently, although u was a sign-changing solution for (P), the functions $u^\pm$ do not belong both to $\mathcal{N}$. Hence, some arguments used to prove the existence of sign-changing solutions for a problem like

$$(\mathrm{P}_1) \qquad \begin{cases} -\Delta u + u = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

cannot be applied; thus, a careful analysis is necessary in many estimates.

Consider the Sobolev space $H^1(\mathbb{R}^3)$ endowed with the norm

$$||u||_*^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\, dx.$$

Let

$$J_\infty : H^1(\mathbb{R}^3) \longrightarrow \mathbb{R}$$

be the functional given by

$$J_\infty(u) = \frac{1}{2}||u||_*^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)\, dx,$$

and consider

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J'_\infty(u)u = 0\}, \qquad c_\infty = \inf_{\mathcal{N}_\infty} J_\infty.$$

The next lemma establishes that c_0 is a positive level. A similar result holds for c_∞.

Lemma 2.2. *There exists $\rho > 0$ such that*

(i) $J(u) \geq ||u||^2/4$ *and* $||u|| \geq \rho$ *for all* $u \in \mathcal{N}$;
(ii) $||w^\pm|| \geq \rho$ *for all* $w \in \mathcal{M}$.

Proof. From Remark 1.1, for every $u \in \mathcal{N}$,

$$\begin{aligned} 4J(u) &= 4J(u) - J'(u)u \\ &= \|u\|^2 + \int_{\mathbb{R}^3} (f(u)u - 4F(u))\, dx \geq \|u\|^2, \end{aligned}$$

and (i) follows. For $\alpha > 0$ given by (V_1), we set $\epsilon \in (0, \alpha)$. Since f satisfies (f_1)–(f_2), there exists a $C = C(\epsilon) > 0$ such that

$$f(s)s \leq \epsilon s^2 + Cs^6 \quad \text{for all } s \in \mathbb{R}. \tag{2.1}$$

For every $w \in \mathcal{M}$, we have $J'(w^{\pm})w^{\pm} < 0$, which gives

$$\|w^{\pm}\|^2 \leq \|w^{\pm}\|^2 + \int_{\mathbb{R}^3} \phi_{w^{\pm}} (w^{\pm})^2 dx < \int_{\mathbb{R}^3} f(w^{\pm})w^{\pm} dx.$$

From equation (2.1), we obtain

$$\begin{aligned}
\|w^{\pm}\|^2 &\leq \epsilon \int_{\mathbb{R}^3} (w^{\pm})^2 dx + C \int_{\mathbb{R}^3} (w^{\pm})^6 dx \\
&\leq \frac{\epsilon}{\alpha} \int_{\mathbb{R}^3} V(x)(w^{\pm})^2 dx + C \int_{\mathbb{R}^3} (w^{\pm})^6 dx \\
&\leq \frac{\epsilon}{\alpha} \|w^{\pm}\|^2 + C\|w^{\pm}\|^6,
\end{aligned}$$

and (ii) is proved. □

The next lemma is a consequence of Miranda's theorem. Since the idea of the proof follows the same type of arguments explored in [**1,** Section 2], we will omit its proof.

Lemma 2.3. *Let $v \in H^1(\mathbb{R}^3)$ satisfy $v^{\pm} \neq 0$. Then, there are $t, s > 0$ such that*

$$J'(tv^+ + sv^-)v^+ = 0$$

and

$$J'(tv^+ + sv^-)v^- = 0.$$

Moreover, if $J'(v)(v^{\pm}) \leq 0$, we have $s, t \leq 1$.

3. The choice of the minimizing sequence. Given $R > 0$, let B_R be the ball of radius R centered at 0. Consider the problem:

$$(\mathrm{AP}_R) \qquad \begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } B_R, \\ -\Delta \phi = \widetilde{u}^2 & \text{in } \mathbb{R}^3, \\ \phi \in D^{1,2}(\mathbb{R}^3) & u \in H_0^1(B_R), \end{cases}$$

where

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in B_R, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus B_R. \end{cases}$$

By similar reasoning as used in **[1]**, we can prove that, for each $R > 0$, there exists a sign-changing solution $u = u_R$ of (AP_R) such that

$$c_R = \inf_{u \in \mathcal{M}_R} J_R(u) = J_R(u_R), \tag{3.1}$$

where

$$J_R : H_0^1(B_R) \longrightarrow \mathbb{R}$$

is the energy functional given by

$$J_R(u) = \frac{1}{2} \int_{B_R} |\nabla u|^2 + \frac{1}{4} \int_{B_R} \phi_u u^2 dx - \int_{B_R} F(u)\, dx,$$

and

$$\mathcal{M}_R = \{u \in H_0^1(B_R) : J_R'(u)u^+ = 0 = J_R'(u)u^- = 0, u^\pm \neq 0\}.$$

Lemma 3.1. *Let c_0 be the nodal level of J. Then*

$$\lim_{R \to +\infty} c_R = c_0.$$

Proof. Since $R \mapsto c_R$ is a non-increasing function and $c_R \geq c_0$ for all $R > 0$, if

$$\lim_{R \to +\infty} c_R = \widehat{c} > c_0,$$

then there exists a $\varphi \in \mathcal{M}$ such that $J(\varphi) < \widehat{c}$. From $\varphi \in \mathcal{M}$, $\varphi^\pm \neq 0$. Let $\varphi_n \in C_0^\infty(\mathbb{R}^3)$ be such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^3)$. We may assume that $\varphi_n^\pm \neq 0$. By Lemma 2.3, there exist $t_n, s_n > 0$ such that

$$J'(t_n\varphi_n^+ + s_n\varphi_n^-)\varphi_n^+ = 0$$

and

$$J'(t_n\varphi_n^+ + s_n\varphi_n^-)\varphi_n^- = 0.$$

In particular,

$$J'(t_n\varphi_n^+ + s_n\varphi_n^-)(t_n\varphi_n^+ + s_n\varphi_n^-) = 0.$$

Using

$$(t_n\varphi_n^+ + s_n\varphi_n^-)^+ = t_n\varphi_n^+ \neq 0$$

and

$$(t_n\varphi_n^+ + s_n\varphi_n^-)^- = s_n\varphi_n^- \neq 0,$$

we find that

$$t_n\varphi_n^+ + s_n\varphi_n^- \in \mathcal{M} \cap C_0^\infty(\mathbb{R}^3).$$

We claim that there exists a subsequence, still denoted by

$$(t_n\varphi_n^+ + s_n\varphi_n^-),$$

such that

$$J(t_n\varphi_n^+ + s_n\varphi_n^-) \longrightarrow J(\varphi).$$

Suppose for the moment that the limit holds. Let n and $R > 0$ be such that

$$t_n\varphi_n^+ + s_n\varphi_n^- \in \mathcal{M}_R$$

and

$$J(t_n\varphi_n^+ + s_n\varphi_n^-) < \widehat{c}.$$

Hence,

$$c_R \leq J(t_n\varphi_n^+ + s_n\varphi_n^-) < \widehat{c},$$

and finally,

$$\widehat{c} = \lim_{R\to+\infty} c_R \leq J(t_n\varphi_n^+ + s_n\varphi_n^-) < \widehat{c},$$

which is impossible. To establish the last claim we begin with the observation that there exist subsequences (not renamed) such that $t_n \to 1$ and $s_n \to 1$. In fact, suppose, by contradiction, that $\limsup_{n\to\infty} t_n > 1$. Given $\delta > 0$, there exists a subsequence, still denoted by t_n, such that $t_n \geq \sigma$ for every n, for some $\sigma > 1$. Since

$$J'(\varphi_n) \longrightarrow J'(\varphi) = 0 \quad \text{and the function} \quad u \longmapsto u^+$$

is continuous, we have

$$\|\varphi_n^+\|^2 + \int_{\mathbb{R}^3} \phi_{\varphi_n^+}(\varphi_n^+)^2 dx \leq \int_{\mathbb{R}^3} f(\varphi_n^+)\varphi_n^+ dx + o_n(1). \tag{3.2}$$

However,

$$J'(t_n\varphi_n^+ + s_n\varphi_n^-)t_n\varphi_n^+ = 0,$$

that is,

$$\frac{1}{t_n^2}\|\varphi_n^+\|^2+\int_{\mathbb{R}^3}\phi_{\varphi_n^+}(\varphi_n^+)^2dx=\int_{\mathbb{R}^3}\frac{f(t_n\varphi_n^+)t_n\varphi_n^+}{t_n^4}\,dx. \tag{3.3}$$

Combining equation (3.2) with equation (3.3) gives

(3.4)

$$\left(1-\frac{1}{t_n^2}\right)\|\varphi_n^+\|^2\leq\int_{\mathbb{R}^3}\left[\frac{f(\varphi_n^+)\varphi_n^+}{(\varphi_n^+)^4}-\frac{f(t_n\varphi_n^+)t_n\varphi_n^+}{(t_n\varphi_n^+)^4}\right](\varphi_n^+)^4dx+o_n(1).$$

From (f_4) and Fatou's lemma, we have

$$0\leq\int_{\mathbb{R}^3}\left[\frac{f(\sigma\varphi^+)\sigma\varphi^+}{(\sigma\varphi^+)^4}-\frac{f(\varphi^+)\varphi^+}{(\varphi^+)^4}\right](\varphi^+)^4dx\leq\left(\frac{1}{\sigma^2}-1\right)\|\varphi^+\|^2<0,$$

which is impossible. Hence, $\limsup_{n\to\infty}t_n\leq 1$. Consequently, there exists a subsequence (not renamed) such that $\lim_{n\to\infty}t_n=t_0$. Taking to the limit as $n\to\infty$ in equation (3.4) and using (f_4) again, we get $t_0=1$. In exactly a similar way, there exists a subsequence (not renamed) such that $\lim_{n\to\infty}s_n=1$. Finally, considering that

$$\begin{aligned}J(t_n\varphi_n^++s_n\varphi_n^-)=&\frac{t_n^2}{2}\|\varphi_n^+\|^2+\frac{s_n^2}{2}\|\varphi_n^-\|^2+\frac{t_n^4}{4}\int_{\mathbb{R}^3}\phi_{\varphi_n^+}(\varphi_n^+)^2dx\\&+\frac{s_n^4}{4}\int_{\mathbb{R}^3}\phi_{\varphi_n^-}(\varphi_n^-)^2dx-\int_{\mathbb{R}^3}F(t_n\varphi_n^++s_n\varphi_n^-)\,dx,\end{aligned}$$

we obtain that

$$J(t_n\varphi_n^++s_n\varphi_n^-)\longrightarrow J(\varphi),$$

by Lemma 2.1 and the convergence $\varphi_n\to\varphi$ in $H^1(\mathbb{R}^3)$. □

4. The minimum level is achieved on $\mathcal{M}$. In this section, our main goal is to prove that the infimum c_0 of J on $\mathcal{M}$ is achieved. From Lemma 2.2 (i), we deduce that $c_0>0$. We begin with the next lemma.

Lemma 4.1. *Suppose that (u_n) is a sequence in $\mathcal{M}$ such that*

$$\limsup_{n\to\infty}J(u_n)<c+c_\infty.$$

Then (u_n) has a subsequence which converges weakly to some $w\in H^1(\mathbb{R}^3)$ such that $w^\pm\neq 0$.

Proof. From Lemma 2.2 (i), (u_n) is a bounded sequence. Hence, without loss of generality, we may suppose that there is a $w \in H^1(\mathbb{R}^3)$ verifying $u_n \rightharpoonup w$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to w(x)$ almost everywhere in $\mathbb{R}^3$. Observing that

$$J(u_n) = J(u_n^+) + J(u_n^-) + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx,$$

and

$$J'(u_n^+)u_n^+ = -\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx = J'(u_n^-)u_n^-,$$

we can suppose that

$$J(u_n^+) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx = \theta + o_n(1)$$

and

$$J(u_n^-) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx = \sigma + o_n(1),$$

where $\theta + \sigma < c + c_\infty$. We claim that $w^+ \neq 0$. Suppose, by contradiction, that $w^+ \equiv 0$. From condition (V_2) and the Sobolev compact imbedding, we obtain

$$\int_{\mathbb{R}^3} V(x)(u_n^+)^2 dx = \int_{\mathbb{R}^3} V_\infty (u_n^+)^2 dx + o_n(1),$$

which implies

$$J_\infty(u_n^+) = J(u_n^+) + o_n(1)$$

and

$$J'_\infty(u_n^+)u_n^+ = J'(u_n^+)u_n^+ + o_n(1).$$

Hence,

$$J_\infty(u_n^+) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx = \theta + o_n(1)$$

and

$$J'_\infty(u_n^+)u_n^+ = -\int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx + o_n(1).$$

We observe that $\theta \geq c_\infty$. In fact, let $t_n > 0$ be such that

$$J_\infty(t_n u_n^+) \geq J_\infty(t u_n^+),$$

for all $t > 0$. We have three possibilities for (t_n):

(i) $\limsup_{n\to\infty} t_n > 1$,
(ii) $\limsup_{n\to\infty} t_n = 1$,
(iii) $\limsup_{n\to\infty} t_n < 1$.

We now show that (i) cannot occur and (ii) or (iii) imply $\theta \geq c_\infty$. From

$$J'_\infty(t_n u_n^+) t_n u_n^+ = 0,$$

we have

$$t_n^2 ||u_n^+||_\infty^2 + t_n^4 \int_{\mathbb{R}^3} \phi_{u_n^+} (u_n^+)^2 dx = \int_{\mathbb{R}^3} f(t_n u_n^+) t_n u_n^+ dx, \tag{4.1}$$

and from $J'(u_n)u_n^+ = 0$, it follows that

$$||u_n^+||^2 + \int_{\mathbb{R}^3} \phi_{u_n^+} (u_n^+)^2 dx + \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx = \int_{\mathbb{R}^3} f(u_n^+) u_n^+ dx,$$

which implies

$$\begin{aligned} ||u_n^+||_\infty^2 + \int_{\mathbb{R}^3} \phi_{u_n^+} (u_n^+)^2 dx + \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx \\ = \int_{\mathbb{R}^3} f(u_n^+) u_n^+ dx + o_n(1). \end{aligned} \tag{4.2}$$

Combining equations (4.1) and (4.2), we obtain

$$\begin{aligned} \left(1 - \frac{1}{t_n^2}\right) ||u_n^+||_\infty^2 + \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx \\ = \int_{\mathbb{R}^3} \left[\frac{f(u_n^+)}{(u_n^+)^3} - \frac{f(t_n u_n^+)}{(t_n u_n^+)^3} \right] (u_n^+)^4 dx + o_n(1). \end{aligned} \tag{4.3}$$

If (i) holds, there exists an $a > 1$ such that $t_n \geq a$ for infinitely many n. By Lemma 2.2 (ii), the left hand side in equation (4.3) is bounded from below by a positive number. On the other hand, by (f_4), the integral on the right hand side of equation (4.3) is non-positive. This yields a contradiction. Hence, (i) does not hold. Suppose that (iii) holds.

Then, $t_n \leq 1$ and Remark 1.1 imply

$$\begin{aligned}
4c_\infty &\leq 4J_\infty(t_n u_n^+) \\
&= 4J_\infty(t_n u_n^+) - J'_\infty(t_n u_n^+)(t_n u_n^+) \\
&= t_n^2 ||u_n^+||_*^2 + \int_{\mathbb{R}^3} [f(t_n u_n^+) t_n u_n^+ - 4F(t_n u_n^+)] dx \\
&\leq ||u_n^+||_*^2 + \int_{\mathbb{R}^3} [f(u_n^+) u_n^+ - 4F(u_n^+)] dx \\
&= 4J_\infty(u_n^+) - J'_\infty(u_n^+)(u_n^+) \\
&= 4J_\infty(u_n^+) + \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx + o_n(1) \\
&= 4\theta + o_n(1).
\end{aligned}$$

Taking the limit $n \to \infty$, we find $\theta \geq c_\infty$. If (ii) occurs, there exists a subsequence (still denoted by t_n) such that $\lim_{n\to\infty} t_n = 1$. As a consequence,

$$4J_\infty(t_n u_n^+) - J'_\infty(t_n u_n^+)(t_n u_n^+) = 4J_\infty(u_n^+) - J'_\infty(u_n^+)(u_n^+) + o_n(1).$$

Thus,

$$\begin{aligned}
4c_\infty &\leq 4J_\infty(t_n u_n^+) \\
&= 4J_\infty(t_n u_n^+) - J'_\infty(t_n u_n^+)(t_n u_n^+) \\
&= 4J_\infty(u_n^+) - J'_\infty(u_n^+)(u_n^+) + o_n(1) \\
&= 4J_\infty(u_n^+) + \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx + o_n(1) \\
&= 4\theta + o_n(1).
\end{aligned}$$

Taking the limit $n \to \infty$, we also obtain $\theta \geq c_\infty$. Since $\theta + \sigma < c + c_\infty$ and $\theta \geq c_\infty$, we have $\sigma < c$. Let $s_n > 0$ be such that $J(s_n u_n^-) \geq J(t u_n^-)$ for all $t > 0$. Using that $J'(u_n^-)(u_n^-) < 0$, we get $s_n < 1$. Hence,

$$\begin{aligned}
4c &\leq 4J(s_n u_n^-) \\
&= 4J(s_n u_n^-) - J'(s_n u_n^-)(s_n u_n^-) \\
&= s_n^2 \|u_n^-\|^2 + \int_{\mathbb{R}^3} [f(s_n u_n^-) s_n u_n^- - 4F(s_n u_n^-)] dx \\
&\leq \|u_n^-\|^2 + \int_{\mathbb{R}^3} [f(u_n^-) u_n^- - 4F(u_n^-)] dx
\end{aligned}$$

$$
\begin{aligned}
&= 4J(u_n^-) - J'(u_n^-)(u_n^-) \\
&= 4J(u_n^-) + \int_{\mathbb{R}^3} \phi_{u_n^-}(u_n^+)^2 dx, \\
&= 4\sigma + o_n(1),
\end{aligned}
$$

which implies $c \leq \sigma$, contrary to $\sigma < c$. Hence, $w^+ \neq 0$, as claimed. Similar arguments to those above show that $w^- \neq 0$, and the proof is complete. □

Lemma 4.2. *If $c_0 < c + c_\infty$, there exists a $w \in \mathcal{M}$ which minimizes J on $\mathcal{M}$.*

Proof. We begin by recalling (equation (3.1)) that there exists a least energy sign-changing solution u_n to (AP_R) for $R = n$, that is,

$$J(u_n) = c_n = \inf_{\mathcal{M}_n} J,$$

where $c_n = c_R$ and $\mathcal{M}_n = \mathcal{M}_R$. By Lemma 3.1, $c_n \to c_0$ as $n \to \infty$. Moreover, $J'(u_n)v = 0$ for all $v \in H_0^1(B_n)$. Since $c_0 < c + c_\infty$, u_n converges weakly to some $w \in H^1(\mathbb{R}^3)$ such that $w^\pm \neq 0$ by Lemma 4.1. Using $J'(u_n)v = 0$ for all $v \in H_0^1(B_n)$, we get $J'(w) = 0$, and consequently, $w \in \mathcal{M}$.

We claim that $J(w) = c_0$. In fact, combining Fatou's lemma with Remark 1.1, we have

$$c_0 \leq J(w) - \frac{1}{4}J'(w)w \leq \liminf_{n\to\infty} \left(J(u_n) - \frac{1}{4}J'(u_n)u_n \right) = c_0,$$

which implies that $c_0 = J(w)$. □

Up until now, we have proved that, under condition $c_0 < c + c_\infty$, there exists a $w \in \mathcal{M}$ such that $J(w) = c_0$ and $J'(w) = 0$.

5. Estimate on the level c_0. This section is devoted to showing that $c_0 < c + c_\infty$. The proofs herein are based upon ideas found in **[18]**. From now on, set $u, v \in H^1(\mathbb{R}^3)$ to be *ground state solutions* of (P) and (P_∞) given by **[2,** Theorems 1.3 and 1.5], respectively. We know that u and v should have defined signs. Without loss of generality, we will suppose that u and v are positive functions in $\mathbb{R}^3$, $J(u) = c$, $J_\infty(v) = c_\infty$, $J'(u) = 0$ and $J'_\infty(v) = 0$. Using Moser's and De Giorgi's iterations, we can show that u and v have exponential decay, and consequently, ϕ_v and ϕ_u have the same behavior. Using this information, a direct computation gives the next result:

Lemma 5.1. *There exist $C > 0$ and $\delta > 0$ such that, for all $R > 0$,*

$$\int_{|x|\geq R} (|\nabla u|^2 + u^2)\, dx \leq Ce^{-\delta R},$$
$$\int_{|x|\geq R} (|\nabla v|^2 + v^2)\, dx \leq Ce^{-\delta R},$$
$$\int_{|x|\geq R} (F(u) + uf(u) + F(v) + vf(v))\, dx \leq Ce^{-\delta R},$$
$$\int_{|x|\geq R} \phi_u v^2 dx + \int_{|x|\geq R} \phi_v u^2 dx \leq Ce^{-\delta R}.$$

For each $n \in \mathbb{N}$, set $v_n(x) = v(x + ne_1)$, where $e_1 = (1, 0, 0) \in \mathbb{R}^3$. The same conclusion for Lemma 5.1 is satisfied by function v_n and

$$\int_{\mathbb{R}^3} \phi_u v_n^2 dx = \int_{\mathbb{R}^3} \phi_{v_n} u^2 dx = O(e^{-n\delta}). \tag{5.1}$$

Lemma 5.2. *Suppose that V satisfies (V_2)–(V_3) and f satisfies (f_2) and (f_5). Then,*

$$\sup_{(\alpha,\beta)\in\mathbb{R}^2} J(\alpha u + \beta v_n) < c + c_\infty,$$

provided n is sufficiently large.

Proof. We begin by proving that there is an $r_0 > 0$ such that

$$J(\alpha u + \beta v_n) \leq 0 \quad \text{for all } (\alpha, \beta) \in \mathbb{R}^2$$

such that $\alpha^2 + \beta^2 \geq r_0$ and $n \geq r_0$. Since $J(v) \leq J_\infty(v)$ for all v, it is sufficient to show that

$$J_\infty(\alpha u + \beta v_n) \leq 0 \quad \text{for all } \alpha^2 + \beta^2 \geq r_0, \quad n \geq r_0.$$

In fact, suppose that J_∞ does not satisfy this claim. Thus, for each n, there are $(\alpha_n, \beta_n) \in \mathbb{R}^2$ such that $J_\infty(\alpha_n u + \beta_n v_n) > 0$ and

$\alpha_n^2 + \beta_n^2 \to \infty$, that is,

$$(5.2) \quad \frac{1}{2}||\alpha_n u + \beta_n v_n||_*^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{(\alpha_n u+\beta_n v_n)}(\alpha_n u + \beta_n v_n)^2 dx \geq \int_{\mathbb{R}^3} F(\alpha_n u + \beta_n v_n)\, dx.$$

We have $||v_n||_* = ||v||_*$, and from Lemma 5.1,

$$(5.3) \qquad \int_{\mathbb{R}^3} (\nabla u \nabla v_n + V_\infty u v_n)\, dx = O(e^{-n\delta}).$$

It follows that

$$(5.4) \qquad ||\alpha_n u + \beta_n v_n||_*^2 = \alpha_n^2 ||u||_*^2 + \beta_n^2 ||v||_*^2 + O(e^{-n\delta}),$$

and then $\sigma_n = ||\alpha_n u + \beta_n v_n||_* \to +\infty$. Set

$$z_n = \frac{\alpha_n u + \beta_n v_n}{||\alpha_n u + \beta_n v_n||_*},$$

and suppose that $z_n \rightharpoonup z$. Dividing (5.2) by σ_n^4, we have

$$\frac{1}{2\sigma_n^2} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{z_n} z_n^2 dx \geq \int_{\mathbb{R}^3} \frac{F(\alpha_n u + \beta_n v_n)}{(\alpha_n u + \beta_n v_n)^4} z_n^4\, dx.$$

The boundedness of (z_n) together with the above inequality and (f_3) shows that $z \equiv 0$. Passing to the limit as $n \to \infty$ in the equality,

$$\begin{aligned} o_n(1) &= \int_{\mathbb{R}^3} (\nabla u \nabla z_n + V_\infty u z_n)\, dx \\ &= \alpha_n ||\alpha_n u + \beta_n v_n||_*^{-1} ||u||_*^2 \\ &\quad + \beta_n ||\alpha_n u + \beta_n v_n||_*^{-1} \int_{\mathbb{R}^3} (\nabla u \nabla v_n + V_\infty u v_n)\, dx, \end{aligned}$$

we obtain from equations (5.3) and (5.4) that $\alpha_n ||\alpha_n u + \beta_n v_n||_*^{-1}$ converges to 0. By Lemma 2.1 (iv),

$$J_\infty(\alpha_n u + \beta_n v_n) = J_\infty(\alpha_n u_n + \beta_n v),$$

where $u_n(x) = u(x - ne_1)$. Proceeding exactly as in the previous argument we can show that $\beta_n ||\alpha_n u + \beta_n v_n||_*^{-1}$ converges to 0. From equation (5.4), $z_n \to 0$, which contradicts $||z_n||_* = 1$. Hence, the claim holds for J_∞, and, in consequence, for J.

We now consider $n \geq r_o$, $\alpha^2 + \beta^2 \leq r_0$. From equation (5.1) and Lemma 5.1, there are $\delta > 0$ and $C = C(u, v, r_0)$ such that

$$\left| \int_{\mathbb{R}^3} [F(\alpha u + \beta v_n) - F(\alpha u) - F(\beta v_n)] \, dx \right| \leq Ce^{-n\delta},$$

$$\left| \int_{\mathbb{R}^3} \left[\phi_{(\alpha u + \beta v_n)} (\alpha u + \beta v_n)^2 - \alpha^4 \phi_u u^2 - \beta^4 \phi_{v_n} v_n^2 \right] \, dx \right| \leq Ce^{-n\delta},$$

$$\left| \|\alpha u + \beta v_n\|^2 - \alpha^2 \|u\|^2 - \beta^2 \|v_n\|^2 \right| \leq Ce^{-n\delta}.$$

Hence,

$$J(\alpha u + \beta v_n) \leq J(\alpha u) + J(\beta v_n) + Ce^{-n\delta}. \tag{5.5}$$

Let $t_n > 0$ be such that

$$J(t_n v_n) = \max_{t \geq 0} J(t v_n).$$

We observe that

$$J(t_n v_n) = J_\infty(t_n v_n) + \int_{\mathbb{R}^3} (V(x) - V_\infty) t_n^2 v_n^2 dx$$

and

$$\int_{\mathbb{R}^3} (V(x) - V_\infty) t_n^2 v_n^2 \, dx \leq \int_{|x - ne_1| \leq 1} (V(x) - V_\infty) t_n^2 v_n^2 \, dx.$$

For $R_0 > 0$ and ρ, the non-increasing function given by (V_3), we have

$$\int_{|x - ne_1| \leq 1} (V(x) - V_\infty) t_n^2 v_n^2 dx \leq -\rho(n+1) \int_{|x - ne_1| \leq 1} t_n^2 v_n^2 \, dx,$$

for every $n \geq R_0 + 1$. Hence,

$$J(\beta v_n) \leq J(t_n v_n) \leq J_\infty(t_n v_n) - \rho(n+1) t_n^2 |v|^2_{L^2(B_1(0))}, \tag{5.6}$$

for every $n \geq R_0 + 1$. By the definition of t_n we have

$$t_n \|v_n\|^2 + t_n^3 \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 dx = \int_{\mathbb{R}^3} f(t_n v_n) v_n dx = \int_{\mathbb{R}^3} f(t_n v) v \, dx. \tag{5.7}$$

Combining equation (5.7) with the fact

$$\|v_n\|^2 = \|v_n\|_*^2 + o_n(1) = \|v\|_*^2 + o_n(1),$$

by (V_2)-(V_3), and using Lemma 2.1 (iv) and equation (2.1), we get

$$\begin{aligned} t_n^2(\|v\|_*^2 + o_n(1)) &\leq t_n^2(\|v\|_*^2 + o_n(1)) + t_n^4 \int_{\mathbb{R}^3} \phi_v v^2 dx \\ &= t_n \int_{\mathbb{R}^3} f(t_n v) v dx \\ &\leq \epsilon t_n^2 \int_{\mathbb{R}^3} v^2 dx + C t_n^6 \int_{\mathbb{R}^3} v^6 dx, \end{aligned}$$

for some positive constant C. Therefore, there exists $\tau > 0$ such that $t_n^2 \geq \tau$ for every n. Using equations (5.5), (5.6) and the fact that

$$J_\infty(t_n v_n) = J_\infty(t_n v) \leq J_\infty(v) = c_\infty,$$

we have

$$\begin{aligned} J(\alpha u + \beta v_n) &\leq J(\alpha u) + J_\infty(t_n v_n) + Ce^{-n\delta} - t_n^2 \rho(n+1)|v|_{L^2(\mathbb{R}^3)}^2 \\ &\leq c + c_\infty + e^{-n\delta}\left(C - \tau |v|_{L^2(\mathbb{R}^3)}^2 e^{n\delta}\rho(n+1)\right), \end{aligned}$$

and the proof follows by the limit condition on ρ in (V_3). □

We now have the next lemma.

Lemma 5.3. *The number c_0 verifies the inequality*:

$$c_0 < c + c_\infty. \tag{5.8}$$

Proof. Let u and v_n be functions as in the proof of Lemma 5.2. Let

$$D = \left[\frac{1}{2}, \frac{3}{2}\right] \times \left[\frac{1}{2}, \frac{3}{2}\right]$$

and

$$\Psi(\xi, \tau) = \left(J'((\xi u - \tau v_n)^+)(\xi u - \tau v_n)^+, J'((\xi u - \tau v_n)^-)(\xi u - \tau v_n)^-\right).$$

Using $J'(u)u = 0$ and (f_4), we obtain

$$J'\left(\frac{1}{2}u\right)\frac{1}{2}u > 0 \quad \text{and} \quad J'\left(\frac{3}{2}u\right)\frac{3}{2}u < 0. \tag{5.9}$$

Lemma 2.1 (iv), condition (V_2) and $J'_\infty(v)v = 0$ imply that there exists $n_0 \in \mathbb{N}$ such that

$$J'\left(\frac{1}{2}v_n\right)\frac{1}{2}v_n > 0 \quad \text{and} \quad J'\left(\frac{3}{2}v_n\right)\frac{3}{2}v_n < 0, \tag{5.10}$$

for all $n \geq n_0$. Since $v(x) \to 0$ as $|x| \to \infty$, it follows from (5.9)–(5.10), by increasing n_0 if necessary, that

$$J'\left(\left(\frac{1}{2}u - \tau v_n\right)^+\right)\left(\frac{1}{2}u - \tau v_n\right)^+ > 0, \tag{5.11}$$

$$J'\left(\left(\frac{3}{2}u - \tau v_n\right)^+\right)\left(\frac{3}{2}u - \tau v_n\right)^+ < 0, \tag{5.12}$$

for every $n \geq n_0$ and $\tau \in [1/2, 3/2]$, and

$$J'\left(\left(\xi u - \frac{1}{2}v_n\right)^-\right)\left(\xi u - \frac{1}{2}v_n\right)^- > 0, \tag{5.13}$$

$$J'\left(\left(\xi u - \frac{3}{2}v_n\right)^-\right)\left(\xi u - \frac{3}{2}v_n\right)^- < 0, \tag{5.14}$$

for every $n \geq n_0$ and $\xi \in [1/2, 3/2]$. Noting that the function Ψ is continuous in D and considering inequalities (5.11)–(5.14), we can apply Miranda's theorem **[20]** and conclude that there exists $(\xi_0, \tau_0) \in D$ such that $\Psi(\xi_0, \tau_0) = (0, 0)$. This gives $\xi_0 u - \tau_0 v_n \in \mathcal{M}$ for every $n \geq n_0$. Consequently,

$$c_0 \leq J(\xi_0 u - \tau_0 v_n),$$

which implies

$$c_0 \leq \sup_{(\alpha,\beta)\in\mathbb{R}^2} J(\alpha u + \beta\omega_n).$$

The lemma follows by combining the last inequality with Lemma 5.2. □

6. Proof of Theorem 1.2. In this section we establish a proof of Theorem 1.2. From Sections 5 and 6, there exists a critical point w of J, which is a sign-changing solution for problem (SP). The proof is completed by showing that w has exactly two nodal domains. Arguing by contradiction, we suppose that

$$w = u_1 + u_2 + u_3, \quad \text{with } u_i \neq 0, u_1 \geq 0,\, u_2 \leq 0$$

and

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset \quad \text{for } i \neq j,\, i,j = 1,2,3,$$

with $\operatorname{supp}(u_i)$ denoting the support of u_i. Setting $v = u_1 + u_2$, we see that $v^{\pm} \neq 0$. Moreover, using the fact that $J'(w) = 0$, it follows that

$$J'(v)(v^{\pm}) \leq 0.$$

By Lemma 2.3, there are $t, s \in (0,1]$ such that $tv^+ + sv^- \in \mathcal{M}$, or equivalently, $tu_1 + su_2 \in \mathcal{M}$, and so,

$$J(tu_1 + su_2) \geq c_0. \tag{6.1}$$

Since $w = v + u_3$, we have $w^2 = v^2 + u_3^2$ and $\phi_w = \phi_v + \phi_{u_3}$. Hence,

$$J(w) = J(v) + J(u_3) + \frac{1}{2}\int_{\mathbb{R}^3} \phi_v u_3^2\, dx. \tag{6.2}$$

Supposing that $u_3 \neq 0$, we claim that

$$J(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2\, dx > 0. \tag{6.3}$$

In fact, by Remark 1.1 and using $J'(w)u_3 = 0$ combined with $u_3 \neq 0$, we obtain

$$\begin{aligned} J(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2\, dx &= J(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2\, dx - \frac{1}{4}J'(w)u_3 \\ &= \frac{1}{4}\|u_3\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} (f(u_3)u_3 - 4F(u_3))\, dx > 0. \end{aligned}$$

Similar arguments to those above show that

$$J(v) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2\, dx = \frac{1}{4}\|v\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} (f(v)v - 4F(v))\, dx. \tag{6.4}$$

From (6.1)–(6.4), for every $t, s \in (0,1]$, we have

$$\begin{aligned} c_0 &\leq J(tu_1 + su_2) \\ &= J(tu_1 + su_2) - \frac{1}{4}J'(tu_1 + su_2)(tu_1 + su_2) \\ &= \frac{t^2}{4}\|u_1\|^2 + \frac{s^2}{4}\|u_2\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{4}\int_{\mathbb{R}^3} [f(tu_1+su_2)(tu_1+su_2) - 4F(tu_1+su_2)]\,dx \\
&\le \frac{1}{4}\|u_1\|^2 + \frac{1}{4}\|u_2\|^2 \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^3} [f(u_1+u_2)(u_1+u_2) - 4F(u_1+u_2)]\,dx \\
&= J(v) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2 dx \\
&< J(v) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2 dx + J(u_3) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v u_3^2 dx \\
&= J(v) + J(u_3) + \frac{1}{2}\int_{\mathbb{R}^3} \phi_v u_3^2 dx \\
&= J(w) = c_0,
\end{aligned}
$$

which is a contradiction. Therefore, $u_3 = 0$ and w has exactly two nodal domains. □

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