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GRAPH COMPOSITIONS OF SUSPENDED Y-TREES

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ABSTRACT. We count the number of graph compositions of suspended Y-trees via flats of the cycle matroid of the suspended Y-trees.

1. Introduction. The notion of graph composition was introduced by Knopfmacher and Mays in [**3**]. In this original work, various formulas, generating functions and recurrence relations for composition counting functions are given for several families of graphs. Graph compositions play an important role in the generalization of both ordinary compositions of positive integers and partitions of finite sets.

This work is extended further to bipartite graphs, as well as some operations on graphs, including unions of graphs and the 2-sum of graphs in [**1, 2, 7**]. In [**4**], the terminology of graph composition is explored further, but under a new term, *comppartition*. In [**5**], it is shown that the number of graph compositions is equal to the total number of flats of the cycle matroid.

The number of compositions for the following graphs are given in [**3**]:

(i) T_n: a tree on n vertices,
(ii) P_n: a path with n vertices,
(iii) S_n: a star graph on n vertices,
(iv) K_n: a complete graph on n vertices,
(v) C_n: a cycle graph on n vertices,
(vi) W_n: a wheel on n vertices,
(vii) L_n: a ladder graph on $2n$ vertices, and
(viii) $K_{m,n}$: a complete bipartite graph.

In particular, it is shown that $C(T_n) = C(P_n) = C(S_n) = 2^{n-1}$.

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In this paper, we study the number of graph compositions of certain suspended trees through their corresponding cycle matroids. In addition, we show that, although non-isomorphic trees have the same number of compositions, the resulting suspended trees do not have the same number of graph compositions.

2. Relationship between graph compositions and flats of cycle matroids. A *matroid* $M(E)$ is a set E, with rank function r, for which the following properties hold:

(i) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.
(ii) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.
(iii) If X and Y are subsets of E, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

The set E is called the *ground set* $M(E)$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the *rank function* of G is defined as $r(G) = V(G) - C(G)$, where $C(G)$ is the number of components of the graph G. Clearly, any graph G is a matroid, where $E(G)$ is the ground set. This is what is known as the *cycle matroid* of graph G. We denote the cycle matroid of G by $M(G)$.

The closure operator is one of the functions associated with a matroid. Let X be a rank r subset of E. Then the *closure* of X denoted by $\text{cl}(X)$ is defined as the the largest rank-r subset of E containing X. A flat of $M(E)$ is a set $A \subseteq E$ for which $\text{cl}(A) = A$. For a cycle matroid of G, it is clear that a flat is just a closed subgraph of G.

The relationship between graph compositions and flats of cycle matroids were established in [**5**]. We state the following theorems without proof; for details, we refer to [**5**].

Theorem 2.1. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let W be a subset of $V(G)$. A subgraph $\langle W\rangle$ is a vertex induced subgraph if and only if $\langle W\rangle$ is a flat of the cycle matroid $M(G)$.*

In what follows, we will denote the set of all distinct compositions of G by $\mathcal{C}_0(G)$. Recall that $C(G)$ denotes the number of compositions of the graph G. Thus, $C(G) = |\mathcal{C}_0(G)|$.

Theorem 2.2. *Let G be a labeled graph with vertex set $V(G)$, edge set $E(G)$ and cycle matroid $M(G)$. Then there is a bijection between the distinct compositions of G and the flats of $M(G)$.*

Theorem 2.3. *Let G be a labeled graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{C}_0(G)$ be the set of all distinct compositions of G, such that $C(G) = |\mathcal{C}_0(G)|$, and let $\mathcal{F}(M(G))$ be the set of all distinct flats of $M(G)$. Then, $C(G) = |\mathcal{F}(M(G))|$.*

3. Graph compositions of suspended Y-trees. We define a Y-*tree* to be a graph with n vertices $\{a_1, a_2, \ldots, a_n\}$ and edge set:

$$\{\{a_1a_2\}, \{a_2a_3\}, \ldots, \{a_{n-2}a_{n-1}\}, \{a_{n-2}a_n\}\}.$$

A *suspended tree* is obtained from a tree graph by adding an extra vertex, which subsequently is connected to each leaf of the tree. Hence, we define the suspended tree of Y_n to be the graph with $n+1$ vertices $\{a_1, a_2, \ldots, a_n, x\}$ and edge set

$$\{\{a_1a_2\}, \{a_2a_3\}, \ldots, \{a_{n-2}a_{n-1}\}, \{a_{n-2}a_n\} \cup \{a_1x\}, \{a_{n-1}x\}, \{a_nx\}\}.$$

To ease notation, we shall denote a Y-tree on n vertices by Y_n and the suspended tree of Y_n by $\widetilde{Y_n}$.

It is clear from the definition that any suspended Y-tree has a subgraph with edge set

$$\{\{a_{n-2}a_{n-1}\}, \{a_{n-2}a_n\}, \{a_{n-1}x\}, \{a_nx\}\},$$

which is isomorphic to C_4. This special subgraph will be denoted by $\widetilde{C_4}$. Similarly, the subgraph

$$\{\{a_1a_2\}, \{a_2a_3\}, \ldots, \{a_{n-2}a_{n-1}\}, \{a_{n-1}x\}, \{a_1x\}\},$$

which is isomorphic to C_n, will be denoted by $\widetilde{C_{n_1}}$. The subgraph

$$\{\{a_1a_2\}, \{a_2a_3\}, \ldots, \{a_{n-2}a_n\}, \{a_nx\}, \{a_1x\}\},$$

which is also isomorphic to C_n, will be denoted by $\widetilde{C_{n_2}}$. Thus, for any positive integer i, we have $\widetilde{C_i} \cong C_i$.

Lemma 3.1. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then the intersection of the subgraphs will be*

$$\widetilde{C_4} \cap \widetilde{C_{n_1}} \cong P_3.$$

Lemma 3.2. *Let $\widetilde{Y_n}$ be a suspended Y-tree on n vertices, Y_n. Then $M(\widetilde{Y_n})$ cannot have a flat of size $k > (n+2)$.*

Proof. By definition, Y_n has three leaves and, since it is a tree, it has $(n-1)$ edges. By suspending this tree, we add three edges. Hence, $\widetilde{Y_n}$ has only $(n+2)$ edges. □

We denote the set of all flats of size k by f_k. Thus, the number of all flats of size k is represented by $|f_k|$. For a suspended Y-tree, different values of k give different formulas; hence, we have the following lemmas.

Lemma 3.3. *Let C_n be a cyclic graph of order n. There is no closed set of size $(n-1)$.*

Proof. Let C_n have edge set $\{a_1, a_2, \ldots, a_{n-1}, a_n\}$. It is clear that any $(n-1)$ edge set will not be a flat because the remaining edge can be added without changing the rank of this subset. □

Lemma 3.4. *Let $\widetilde{Y_n}$ be a suspended Y-tree on n vertices, Y_n, with $n > 4$. Let f_k represent the set of all flats of size k of $M(\widetilde{Y_n})$. Then, for $k \leq 3$, we have:*

$$|f_k| = \begin{cases} 1 & k = 0, \\ n+2 & k = 1, \\ \binom{n+2}{2} & k = 2, \\ \binom{n+2}{3} - \binom{4}{3} & k = 3. \end{cases}$$

Proof. Let $\widetilde{Y_n}$ represent a suspended Y-tree. We prove each case separately.

Case $(k = 0)$. It is clear that there is only one flat of size $k = 0$, which is the empty set.

Case $(k = 1)$. A flat of size $k = 1$ is just an edge. $\widetilde{Y_n}$ has $(n+2)$ edges.

Case ($k=2$). A flat of size $k = 2$ is just a pair of edges. Since there are no parallel edges, all possible combinations of pairs are flats. Hence, there are $\binom{n+2}{2}$ possible combinations.

Case ($k=3$). For flats of size $k = 3$, there are $\binom{n+2}{3}$ possible combinations of 3-edge sets. By Lemma 3.3, any 3-edge set from the $\widetilde{C_4}$ subgraph of $\widetilde{Y_n}$ is not a flat. There are $\binom{4}{3}$ such possibilities; hence, we exclude these in order to get the number of flats to be:

$$\binom{n+2}{3} - \binom{4}{3}. \qquad \square$$

Lemma 3.5. *Let $\widetilde{Y_n}$ be a suspended Y-tree, with $n > 4$. Let f_k be the set of all flats of size k for $M(\widetilde{Y_n})$. Then, for $4 \leq k < (n-1)$,*

$$|f_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}.$$

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. By definition and Lemma 3.1, there exist subgraphs $\widetilde{C_4}$ and $\widetilde{C_n}$ with intersection P_3. For any k, such that $4 \leq k < (n-1)$, we have $\binom{n+2}{k}$ k-edge set combinations, of which some are flats and some are non-flats. Consider a general k-edge set, if three of the k-edges are contained in the $\widetilde{C_4}$ subgraph. Then this k-edge set is not a flat. But, there are $\binom{4}{3}$ ways that this could happen, and the other $(k-3)$-edges are chosen from the remaining $(n-2)$ edges. Hence, by the principle of inclusion/exclusion, we have $\binom{n+2}{k}$ total k-edge sets, of which some are flats and some are non-flats. Thus, we exclude $\binom{4}{3}\binom{n-2}{k-3}$ k-edge sets, which are not flats,

$$|f_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}. \quad \square$$

The case of $k = (n-1)$ gives a special formula, considering that there are two special subgraphs C_{n_1} and C_{n_2} that are isomorphic to C_n. By Lemma 3.3, we know that C_n has no flat of size $(n-1)$.

Lemma 3.6. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Let f_k be the set of all flats of size k of $M(\widetilde{Y_n})$. Then, for $k = (n-1)$,*

$$|f_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - 2\binom{n}{k}.$$

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. By definition, there exist subgraphs

$$\widetilde{C_4} \cong C_4, \qquad \widetilde{C_{n_1}} \cong C_n \quad \text{and} \quad \widetilde{C_{n_2}} \cong C_n.$$

There are $\binom{n+2}{n-1}$ possible $(n-1)$ edge sets, of which some are flats and some are non-flats.

Consider a general k-edge set where $k = (n-1)$. There are two cases of non-flats.

Case (i). If three of the edges are contained in the subgraph $\widetilde{C_4}$, then this set is not a flat by Lemma 3.3. There are $\binom{4}{3}$ ways of choosing the 3-edges in $\widetilde{C_4}$, and $\binom{n-2}{k-3}$ ways of choosing the remaining $k-3$. Thus, we have $\binom{4}{3}\binom{n-2}{k-3}$ sets, which are non-flats in this case.

Case (ii). If all of the k-edges where $k = (n-1)$ are contained in the subgraph $\widetilde{C_{n_1}}$ or subgraph $\widetilde{C_{n_2}}$, then by Lemma 3.3, there is a flat of size $(n-1)$ in C_n. Hence, there are $2\binom{n}{k}$ sets which are non-flats in this case.

Therefore, by the principle of inclusion/exclusion we have $\binom{n+2}{n-1}$ possible k-edge sets, of which some are flats and some are non-flats. Thus, we exclude the $\binom{n-2}{k-3} + 2\binom{n}{k}$ k-edge sets which are not flats, to get

$$|f_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - 2\binom{n}{k}. \qquad \square$$

Lemma 3.7. *Let $\widetilde{Y_n}$ be a suspended Y-tree, with $n > 4$. Let f_k be the set of all flats of size k in $M(\widetilde{Y_n})$. Then, for $n \leq k \leq (n+2)$,*

$$|f_k| = \begin{cases} \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - \binom{4}{1} & k = n, \\ 0 & k = (n+1), \\ 1 & k = (n+2). \end{cases}$$

Proof. We will prove each of these three cases separately.

Case (i). ($k = n$). Let $\widetilde{Y_n}$ be a suspended Y-tree. We have $\binom{n+2}{k}$ k-edge set combinations, of which some are flats and some are non-flats. Consider a general k-edge set where $k = (n-1)$. There are two cases of non-flats.

Case (ii). If three of the edges are contained in the subgraph $\widetilde{C_4}$, then this set is not a flat by Lemma 3.3. There are $\binom{4}{3}$ ways of choosing the three edges in $\widetilde{C_4}$, and $\binom{n-2}{k-3}$ ways of choosing the remaining $(k-3)$. Hence, we have

$$\binom{4}{3}\binom{n-2}{k-3}$$

sets which are non-flats in this case.

Case (iii). If $(n-1)$ of the edges are contained in the subgraph $\widetilde{C_{n_1}}$ or subgraph $\widetilde{C_{n_2}}$, then this set is not a flat by Lemma 3.3. There are $\binom{n}{n-1}$ ways of choosing the $(n-1)$ edges in $\widetilde{C_{n_1}}$ or $\widetilde{C_{n_2}}$, and $\binom{2}{1}$ ways of choosing the remaining edge. Hence, we have

$$\binom{n}{n-1}\binom{2}{1}$$

sets which are non-flats in this case. It follows that there are

$$2\binom{n}{n-1}\binom{2}{1}$$

sets which are non-flats in this case.

There are $\binom{n+2}{n}$ possible combinations of edge sets of size ($k = n$) which include both closed and non closed sets. If three of these edges are contained in the C_4 subgraph, then we do not have a closed set. There are $\binom{4}{3}$ ways that this could happen. In each case of the remaining $(n-2)$ edges, $(k-3)$ are now unavailable for our potential closed set, hence we exclude these. Furthermore, there are $\binom{4}{1}$, two for each C_n subgraph, where the closed set would have $(k-1)$ of its edges contained in a cyclic graph. Since, in this case, $(k-1) = (n-1)$, we cannot have such closed sets, as they are not maximums.

Case $k = (n+1)$. Let $\widetilde{Y_n}$ represent a suspended Y-tree. Any subgraph made from a combination of $(n+1)$ edges will not be a closed set, as the final edge can be added without the rank of this subgraph being changed.

Case $k = (n+2)$. Let $\widetilde{Y_n}$ represent a suspended Y-tree. There is only one subgraph with $(n+2)$ edges, and that is the entire graph itself, which happens to be a closed set and the only set of size $k = (n+2)$. □

Lemma 3.8.

$$\sum_{k=4}^{n-2}\left[\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}\right] = 3(2^n) + 4 - \left[\frac{70n + 2n^3}{6}\right].$$

Proof. Let

$$A = \sum_{k=4}^{n-2}\binom{n+2}{k},$$

and let

$$B = \sum_{k=4}^{n-2}\binom{4}{3}\binom{n-2}{k-3}.$$

Then,

$$\begin{aligned}
A &= \sum_{k=4}^{n-2}\binom{n+2}{k} \\
&= \sum_{k=1}^{n+2}\binom{n+2}{k} - \left[\binom{n+2}{0} + \binom{n+2}{1} + \binom{n+2}{2} + \binom{n+2}{3}\right] \\
&\quad - \left[\binom{n+2}{n-1} + \binom{n+2}{n} + \binom{n+2}{n+1} + \binom{n+2}{n+2}\right] \\
&= 2^{n+2} - \left[2\binom{n+2}{n-1} + 2\binom{n+2}{n} + 2\binom{n+2}{n+1} + 2\binom{n+2}{n+2}\right] \\
&= 2^{n+2} - \left[2\frac{(n+2)(n+1)n}{3!} + 2\frac{(n+2)(n+1)}{2} + 2(n+2) + 2\right] \\
&= 2^{n+2} - \left[\frac{2n^3 + 12n^2 + 34n}{6} + 8\right].
\end{aligned}$$

$$
\begin{aligned}
B &= \sum_{k=4}^{n-2} \binom{4}{3}\binom{n-2}{k-3} \\
&= 4\left(\sum_{k=0}^{n-2}\binom{n-2}{k} - \left[\binom{n-2}{0} + \binom{n-2}{n-4} + \binom{n-2}{n-3} + \binom{n-2}{n-2}\right]\right) \\
&= 4\left(2^{n-2} - \left[\frac{n^2-3n+6}{2}\right]\right) \\
&= 2^n - \left[2n^2 - 6n + 12\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{k=4}^{n-2}\left[\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}\right] &= A - B \\
&= 2^{n+2} - \left[\frac{2n^3+12n^2+34n}{6} + 8\right] \\
&\quad - \left(2^n - \left[2n^2-6n+12\right]\right) \\
&= 3(2^n) + 4 - \left[\frac{70n+2n^3}{6}\right]. \qquad \square
\end{aligned}
$$

The following two lemmas can be easily verified.

Lemma 3.9.

$$
\binom{n+2}{n-1} - \binom{4}{3}\binom{n-2}{n-4} - 2\binom{n}{n-1} = \frac{1}{6}(n^3 - 9n^2 + 50n) - 12.
$$

Lemma 3.10.

$$
\binom{n+2}{n} - \binom{4}{3}\binom{n-2}{n-3} - \binom{4}{1} = \frac{1}{6}(3n^2 + 9n) - 4n + 5.
$$

Now, we are in a position to state and prove the main theorem. Recall that we denote the set of all flats of size k by f_k. Thus, the number of all flats of size k is represented by $|f_k|$. Recall that we denote the set of all distinct compositions of G by $\mathcal{C}_o(G)$. Also recall that $C(G)$ denotes the number of compositions of the graph G. Thus, $C(G) = |\mathcal{C}_o(G)|$.

Theorem 3.11. *Let $\widetilde{Y_n}$ be a suspended Y-tree on n vertices. The number of graph compositions of $\widetilde{Y_n}$ are*

$$C(\widetilde{Y_n}) = 3(2^n - n) - 2.$$

Proof. By Theorem 2.3, we know that

$$C(\widetilde{Y_n}) = |\mathcal{F}(M(\widetilde{Y_n}))|,$$

where $\mathcal{F}(M(\widetilde{Y_n}))$ is the set of all distinct flats of $M(\widetilde{Y_n})$. Thus,

$$C(\widetilde{Y_n}) = |\mathcal{F}(M(\widetilde{Y_n}))| = \sum_{k=0}^{n+2} |f_k|.$$

By Lemmas 3.4, 3.5, 3.6 and 3.7, we obtain:

$$\begin{aligned}\sum_{k=0}^{n+2} |f_k| = 1 + \Big[n+2\Big] + \left[\binom{n+2}{2}\right] + \left[\binom{n+2}{3} - \binom{4}{3}\right] \\ + \sum_{k=4}^{n-2}\left[\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}\right] \\ + \left[\binom{n+2}{n-1} - \binom{4}{3}\binom{n-2}{n-4} - 2\binom{n}{n-1}\right] \\ + \left[\binom{n+2}{n} - \binom{4}{3}\binom{n-2}{n-3} - \binom{4}{1}\right] + 1.\end{aligned}$$

Hence, simplifying and applying Lemmas 3.8, 3.9 and 3.10, we get

$$\begin{aligned}\sum_{k=0}^{n+2} |f_k| &= \left(\frac{n^3+3n^2+17n}{6}\right) + \left(3(2^n) + 4 - \left[\frac{70n+2n^3}{6}\right]\right) \\ &\quad + \left(\frac{1}{6}(n^3 - 9n^2 + 50n) - 12\right) + \left(\frac{1}{6}(3n^2+9n) - 4n + 5\right) + 1 \\ &= 3(2^n) - 2 \\ &\quad + \left(\frac{n^3+3n^2+17n-70n-2n^3+n^3-9n^2+50n+3n^2+9n-24n}{6}\right) \\ &= 3(2^n) - 3n - 2.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n+2} |f_k| = C(\widetilde{Y_n}) = 3(2^n - n) - 2. \qquad \square$$

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