ON THE RATIONALITY OF POINCARÉ SERIES
OF GORENSTEIN ALGEBRAS
VIA MACAULAY'S CORRESPONDENCE

GIANFRANCO CASNATI, JOACHIM JELISIEJEW AND ROBERTO NOTARI

ABSTRACT. Let A be a local Artinian Gorenstein algebra with maximal ideal $\mathfrak{M}$,

$$P_A(z) := \sum_{p=0}^{\infty} (\mathrm{Tor}_p^A(k,k)) z^p$$

its Poicaré series. We prove that $P_A(z)$ is rational if either $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \leq 4$ and $\dim_k(A) \leq 16$, or there exist $m \leq 4$ and c such that the Hilbert function $H_A(n)$ of A is equal to m for $n \in [2, c]$ and equal to 1 for $n = c + 1$. The results are obtained due to a decomposition of the apolar ideal $\mathrm{Ann}(F)$ when $F = G + H$ and G and H belong to polynomial rings in different variables.

1. Introduction and notation. Throughout the whole paper, k will denote an algebraically closed field of characteristic 0. By *ring* (respectively, *k-algebra*) we mean a Noetherian, associative, commutative and unitary ring A (respectively, algebra A over k). When A is local we will denote by $\mathfrak{M}$ its maximal ideal and we will assume that $k := A/\mathfrak{M}$.

Since J.P. Serre asked in 1955 if the Poincaré series of the local ring A, i.e.,

$$P_A(z) := \sum_{p=0}^{\infty} \dim_k(\mathrm{Tor}_p^A(k,k)) z^p,$$

is rational, also proving its rationality when A is regular, many authors have proved results confirming the rationality of P_A in several cases.

Thereafter, in 1982, D.J. Anick gave the first example of an Artinian local algebra A with transcendental P_A. Shortly afterwards, in 1983, the existence of an Artinian, Gorenstein, local ring with $\mathfrak{M}^4 = 0$ and transcendental P_A was proved by R. Bøgvad.

Nevertheless, several results show that large classes of local rings A have rational Poincaré series, e.g., J. Tate proved the rationality for complete intersections rings (see [16]), L. Avramov, A. Kustin, M. Miller and C. Jacobsson, A. Kustin and M. Miller for Gorenstein local rings with $\dim_k(\mathfrak{M}/\mathfrak{M}^2) \leq 4$ (see [1, 13] respectively), J. Sally and J. Elias, G. Valla for Gorenstein local rings with $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \leq 2$ (see [8, 14], respectively), the authors for Gorenstein local rings of multiplicity at most 10 (see [5]), the authors, jointly with J. Elias and M.E. Rossi for Gorenstein local k-algebras with $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 4$ and $\mathfrak{M}^4 = 0$ (see [4]).

All of the above results are based on the same smart combination of results on the Poincaré series due to L. Avramov and G. Levin (see [2]) and T.H. Gulliksen and G. Levin (see [11]) first used in [14] combined with suitable structure results on Gorenstein rings and k-algebras. In this last case, a fundamental role has been played by Macaulay's correspondence.

In Section 2, we give a quick resumé of the main results that we need later on in the paper about Macaulay's correspondence. In Section 3, we extend to arbitrary k-algebras a very helpful decomposition result already used in a simplified form by J. Elias and M.E. Rossi in [7] and in the aforementioned [4] for k-algebras with $\mathfrak{M}^4 = 0$. In Section 4, we explain how to relate the rationality of the Poincaré series of Gorenstein k-algebras with their representation in the setup of Macaulay's correspondence making use of the aforementioned decomposition result. Finally, in Section 5, we use that relationship in order to prove the following two results generalizing the aforementioned quoted results.

Theorem A. *Let A be an Artinian, Gorenstein local k-algebra with maximal ideal $\mathfrak{M}$. If there are integers $m \leq 4$ and $c \geq 1$ such that*

$$\dim_k(\mathfrak{M}^t/\mathfrak{M}^{t+1}) = \begin{cases} m & \text{if } t = 2, \ldots, c, \\ 1 & \text{if } t = c+1, \end{cases}$$

then P_A is rational.

Theorem B. *Let A be an Artinian, Gorenstein local k-algebra with maximal ideal $\mathfrak{M}$. If $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \leq 4$ and $\dim_k(A) \leq 16$, then P_A is rational.*

1.1. Notation. For each $N \in \mathbb{N}$ we set $S[N] := k[[x_1, \ldots, x_N]]$ and $P[N] := k[y_1, \ldots, y_N]$. We denote by $S[N]_q$ (respectively, $P[N]_q$) the homogeneous component of degree q of such a graded k-algebra, and we set

$$S[N]_{\leq q} := \bigoplus_{i=1}^{q} S[N]_i,$$

respectively,

$$P[n]_{\leq q} := \bigoplus_{i=1}^{q} P[n]_i.$$

Finally, we set $S[n]_+ := (x_1, \ldots, x_n) \subseteq S[n]$. The ideal $S[n]_+$ is the unique maximal ideal of $S[N]$.

A local ring R is Gorenstein if its injective dimension as an R-module is finite.

If $\gamma := (\gamma_1, \ldots, \gamma_N) \in \mathbb{N}^N$ is a multi-index, then we set $t^\gamma := t_1^{\gamma_1} \cdots t_N^{\gamma_N} \in k[t_1, \ldots, t_N]$.

For all other notation and results we refer to [10].

2. Preliminary results. In this section we list the main results on k-algebras we need in the next sections. Let A be a local, Artinian k-algebra with maximal ideal $\mathfrak{M}$. We denote by H_A the Hilbert function of the graded associated algebra

$$\mathrm{gr}(A) := \bigoplus_{t=0}^{+\infty} \mathfrak{M}^t/\mathfrak{M}^{t+1}.$$

We know that

$$A \cong S[n]/J$$

for a suitable ideal $J \subseteq S[n]_+^2 \subseteq S[n]$, where $n = \text{emdim}(A) := H_A(1)$. Recall that the *socle degree* $\text{sdeg}(A)$ of A is the greatest integer s such that $\mathfrak{M}^s \neq 0$.

We have an action of $S[n]$ over $P[n]$ given by partial derivation defined by identifying x_i with $\partial/\partial y_i$. Hence,

$$x^\alpha \circ y^\beta := \begin{cases} \alpha!\binom{\beta}{\alpha} y^{\beta-\alpha} & \text{if } \beta \geq \alpha, \\ 0 & \text{if } \beta \not\geq \alpha. \end{cases}$$

Such an action endows $P[n]$ with a structure of the module over $S[n]$. If $J \subseteq S[n]$ is an ideal and $M \subseteq P[n]$ is an $S[n]$-submodule we set

$$J^\perp := \{ F \in P[n] \mid g \circ F = 0, \text{ for all } g \in J\},$$
$$\text{Ann}(M) := \{ g \in S[n] \mid g \circ F = 0, \text{ for all } F \in M\}.$$

For the following results, see e.g., the paper [9] by J. Emsalem, the book [12] by A. Iarrobino and the references therein. Macaulay's inverse system theorem is based on the fact that constructions $J \mapsto J^\perp$ and $M \mapsto \text{Ann}(M)$ give rise to an inclusion-reversing bijection between ideals $J \subseteq S[n]$ such that $S[n]/J$ is a local Artinian k-algebra and finitely generated $S[n]$-submodules $M \subseteq P[n]$. In this bijection, Gorenstein k-algebras A with $\text{sdeg}(A) = s$ correspond to cyclic $S[n]$-submodules $\langle F \rangle_{S[n]} \subseteq P[n]$ generated by a polynomial F of degree s. We simply write $\text{Ann}(F)$ instead of $\text{Ann}(\langle F \rangle_{S[n]})$.

On the one hand, given an $S[n]$-module M, we define

$$\text{tdf}(M)_q := \frac{M \cap P[n]_{\leq q} + P[n]_{\leq q-1}}{P[n]_{\leq q-1}}$$

where

$$P[n]_{\leq q} := \bigoplus_{i=0}^{q} P[n]_i,$$

and

$$\text{tdf}(M) := \bigoplus_{q=0}^{\infty} \text{tdf}(M)_q.$$

The module $\text{tdf}(M)$ can be interpreted as the $S[n]$-submodule of $P[n]$ generated by the top degree forms of all polynomials in M.

On the other hand, for each $f \in S[n]$, the lowest degree of monomials appearing with non-zero coefficient in the minimal representation of f is called the *order of f* and it is denoted by $\mathrm{ord}(f)$. If

$$f = \sum_{i=\mathrm{ord}(f)}^{\infty} f_i, \quad f_i \in S[n]_i,$$

then $f_{\mathrm{ord}(f)}$ is called the *lower degree form of f*. It will be denoted in what follows with $\mathrm{ldf}(f)$.

If $f \in J$, then $\mathrm{ord}(f) \geq 2$. The *lower degree form ideal* $\mathrm{ldf}(J)$ associated to J is

$$\mathrm{ldf}(J) := (\mathrm{ldf}(f) \mid f \in J) \subseteq S[n].$$

We have $\mathrm{ldf}(\mathrm{Ann}(M)) = \mathrm{Ann}(\mathrm{tdf}(M))$ (see [**9**, Proposition 3], see also [**7**, formulas (2) and (3)]), whence

$$\mathrm{gr}(S[n]/\mathrm{Ann}(M)) \cong S[n]/\mathrm{ldf}(\mathrm{Ann}(M)) \cong S[n]/\mathrm{Ann}(\mathrm{tdf}(M)).$$

Thus,

(2.1)
$$H_{S[n]/\mathrm{Ann}(M)}(q) = \dim_k(\mathrm{tdf}(M)_q).$$

The module M is said to be *non-degenerate* if $H_{S[n]/\mathrm{Ann}(M)}(1) = \dim_k(\mathrm{tdf}(M)_1) = n$, i.e., if and only if the classes of $y_1, \ldots, y_n$ are in $\mathrm{tdf}(M)$. If $M = \langle F \rangle_{S[n]}$, then we write $\mathrm{tdf}(F)$ instead of $\mathrm{tdf}(M)$.

Let A be Gorenstein with $s := \mathrm{sdeg}(A)$, so that $\mathrm{Soc}(A) = \mathfrak{M}^s \cong k$. In particular, $A \cong S[n]/\mathrm{Ann}(F)$, where $F := \sum_{i=0}^{s} F_i$, $F_i \in P[n]_i$. For each $h \geq 0$ we set

$$F_{\geq h} := \sum_{i=h}^{s} F_i$$

(hence, $F_s = F_{\geq s}$).

Trivially, if $s \geq 1$, we can always assume that the homogeneous part of F of degree 0 vanishes, i.e., $F = F_{\geq 1}$. Moreover, due to [**6**, Lemma 2.2], we know that if $s \geq 2$ and $\mathrm{Ann}(F) \subseteq S[n]_+^2$, then we can also assume $F_1 = 0$, i.e., $F = F_{\geq 2}$; we will always make such an assumption in what follows.

We have a filtration with proper ideals (see [**12**, Definition 1.3]) of $\mathrm{gr}(A) \cong S[n]/\mathrm{ldf}(\mathrm{Ann}(F))$:

$$C_A(0) := \mathrm{gr}(A) \supset C_A(1) \supseteq C_A(2) \supseteq \cdots \supseteq C_A(s-2) \supseteq C_A(s-1) := 0.$$

Via the epimorphism $S[n] \twoheadrightarrow \mathrm{gr}(A)$ we obtain an induced filtration

$$\widehat{C}_A(0) \supset \widehat{C}_A(1) \supseteq \widehat{C}_A(2) \supseteq \cdots \supseteq \widehat{C}_A(s-2) \supseteq \widehat{C}_A(s-1).$$

By definition $\widehat{C}_A(0) = S[n]$ and $\widehat{C}_A(s-1) = \mathrm{ldf}(\mathrm{Ann}(F))$.

The quotients $Q_A(a) := C_A(a)/C_A(a+1) \cong \widehat{C}_A(a)/\widehat{C}_A(a+1)$ are reflexively graded $\mathrm{gr}(A)$-modules whose Hilbert function is symmetric around $(s-a)/2$. In general, $\mathrm{gr}(A)$ is no longer Gorenstein, but the first quotient

$$(2.2) \qquad\qquad G(A) := Q_A(0) \cong S[n]/\mathrm{Ann}(F_s)$$

is characterized by the property of being the unique (up to isomorphism) graded Gorenstein quotient k-algebra of $\mathrm{gr}(A)$ with the same socle degree. Moreover, the Hilbert function of A satisfies

$$(2.3) \qquad H_A(i) = H_{\mathrm{gr}(A)}(i) = \sum_{a=0}^{s-2} H_{Q_A(a)}(i), \quad i \geq 0.$$

Since $H_A(0) = H_{G(A)}(0) = 1$, it follows that, if $a \geq 1$, then $Q_A(a)_0 = 0$, whence $Q_A(a)_i = 0$ when $i \geq s-a$ (see [**12**, Theorem 1.5]) for the same values of a. It follows that

$$(2.4) \qquad H_A(i) = \begin{cases} H_{Q_A(0)}(i) & \text{if } i = 0, s, \\ \displaystyle\sum_{a=0}^{s-i-1} H_{Q_A(a)}(i) & \text{if } 1 \leq i \leq s-1. \end{cases}$$

Moreover,

$$(2.5) \quad H_{\mathrm{gr}(A)/C_A(a+1)}(i) = H_{S[n]/\hat{C}_A(a+1)}(i) = \sum_{\alpha=0}^{a} H_{Q_A(\alpha)}(i), \quad i \geq 0.$$

We set

$$\begin{aligned} f_h &:= \sum_{\alpha=0}^{s-h} H_{Q_A(\alpha)}(1) = H_{S[n]/\hat{C}_A(s-h+1)}(1) \\ &= H_{\mathrm{gr}(A)/C_A(s-h+1)}(1) \end{aligned}$$

(so that $n = H_A(1) = f_2$).

Finally, we introduce the following new invariant.

Definition 2.1. Let A be a local, Artinian k-algebra with maximal ideal $\mathfrak{M}$ and $s := \mathrm{sdeg}(A)$. The *capital degree* of A, denoted by $\mathrm{cdeg}(A)$, is defined to be the maximum integer i, if any, such that $H_A(i) > 1$, and 0 otherwise. If $c = \mathrm{cdeg}(A)$, we also say that A is a c stretched k-algebra (for short, stretched if $c \leq 1$).

By definition $\mathrm{cdeg}(A) \geq 0$ and $\mathrm{cdeg}(A) \leq \mathrm{sdeg}(A)$, if A is Gorenstein, then we also have $\mathrm{cdeg}(A) < \mathrm{sdeg}(A)$.

The following result, proved in [**6**, Lemma 4.1 and Remark 4.2], will be repeatedly used in the paper.

Lemma 2.2. *Let A be a local, Artinian, Gorenstein, c stretched k-algebra. If $n := H_A(1)$, $m := H_A(2)$, $p := H_A(c)$, $s := \mathrm{sdeg}(A)$, then*

$$A \cong S[n]/\mathrm{Ann}(F),$$

where

$$F := y_1^s + \sum_{i=2}^{c+1} F_i + \sum_{j=m+1}^{n} y_j^2,$$

$$F_i \in P[f_i]_i, \quad i \geq 3, \qquad F_2 \in P[f_3]_2,$$

$$x_1^c \circ F_{c+1} = x_1^i \circ F_i = 0, \quad i = 3, \ldots, c+1$$

and

$$x_2 \circ F_{c+1}, \ldots, x_p \circ F_{c+1}$$

are linearly independent.

The rationality of the Poincaré series P_A of every stretched ring A is proved in [**14**]. The proof has been generalized to rings with $H_A(2) = 2$ in [**8**] and to rings with $H_A(2) = 3$ and $H_A(3) = 1$ in [**5**]. The rationality of P_A when A is a 2 stretched k-algebra has been studied in [**4**] with the restriction $\mathrm{sdeg}(A) = 3$.

3. Decomposition of the apolar ideal. In the present section, we explain how to decompose the ideal $\mathrm{Ann}(F)$ as the sum of two simpler ideals. Such a decomposition will be used in the next section in order to reduce the calculation of the Poincaré series of A to the one of a simpler k-algebra.

Lemma 3.1. *Let $m < n$, $G \in P[m]$, $H \in k[y_{m+1}, \ldots, y_n]$ be non-zero and $F = G+H$. Let us denote by $\mathrm{Ann}(G)$ and $\mathrm{Ann}(H)$ the annihilators of G and H inside $S[m]$ and $k[[x_{m+1}, \ldots, x_n]]$, respectively. Then,*

$$\mathrm{Ann}(F) = \mathrm{Ann}(G)S[n] + \mathrm{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \le i \le m \\ m+1 \le j \le n}}.$$

where $\sigma_G \in S[m]$ and $\sigma_H \in k[[x_{m+1}, \ldots, x_n]]$ are any series of order $\deg(G)$ and $\deg(H)$ such that $\sigma_G \circ G = \sigma_H \circ H = 1$.

Proof. The inclusions $\mathrm{Ann}(G)S[n], \mathrm{Ann}(H)S[n] \subseteq \mathrm{Ann}(F)$ are completely trivial. Also, the inclusion

$$(\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \le i \le m \\ m+1 \le j \le n}} \subseteq \mathrm{Ann}(F)$$

is easy to check. Thus,

$$\mathrm{Ann}(G)S[n] + \mathrm{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \le i \le m \\ m+1 \le j \le n}} \subseteq \mathrm{Ann}(F).$$

Conversely, let $p \in \mathrm{Ann}(F)$. Grouping the different monomials in p, we can write a decomposition $p = p_{\le m} + p_{>m} + p_{\mathrm{mix}}$, where $p_{\le m} \in S[m]$, $p_{>m} \in k[[x_{m+1}, \ldots, x_n]]$ and, finally, $p_{\mathrm{mix}} \in (x_i x_j)_{1 \le i \le m, m+1 \le j \le n} \subseteq S[n]$.

It is clear that

$$p_{\mathrm{mix}} \in (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \le i \le m \\ m+1 \le j \le n}} \subseteq \mathrm{Ann}(F),$$

hence it suffices to prove that

$$p_{\le m} + p_{>m} \in \mathrm{Ann}(G)S[n] + \mathrm{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \le i \le m \\ m+1 \le j \le n}}.$$

For this purpose, recall that $p_{\le m} + p_{>m} = p - p_{\mathrm{mix}} \in \mathrm{Ann}(F)$; thus, by definition,

$$0 = (p_{\le m} + p_{>m}) \circ F = p_{\le m} \circ G + p_{>m} \circ H.$$

Hence,
$$p_{\leq m} \circ G = u = -p_{>m} \circ H.$$

Since $p_{\leq m} \circ G \in P[m]$ and $p_{>m} \circ H \in k[y_{m+1}, \dots, y_n]$, it follows that $u \in k$. So $p_{\leq m} - u(\sigma_G - \sigma_H) \in \mathrm{Ann}(G)S[n]$, whence

$$p_{\leq m} \in (\sigma_G - \sigma_H) + \mathrm{Ann}(G)S[n] \subseteq \mathrm{Ann}(G)S[n] + \mathrm{Ann}(H)S[n]$$
$$+ (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq n}}.$$

A similar argument shows that

$$p_{>m} \in (\sigma_G - \sigma_H) + \mathrm{Ann}(H)S[n] \subseteq \mathrm{Ann}(G)S[n] + \mathrm{Ann}(H)S[n]$$
$$+ (\sigma_G - \sigma_H, x_i x_j)_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq n}},$$

and this concludes the proof. $\qquad\square$

Let F be as in the statement above. Then Lemma 3.1 with

$$G := \sum_{i=2}^{s} F_i, \qquad H := \sum_{j=m+1}^{n} y_j^2$$

yield the following corollary.

Corollary 3.2. *Let $m < n$, $G \in P[m]$ be non-zero, and let*

$$F = G + \sum_{j=m+1}^{n} y_j^2.$$

Let us denote by $\mathrm{Ann}(G)$ the annihilator of G inside $S[m]$. Then

$$\mathrm{Ann}(F) = \mathrm{Ann}(G)S[n] + (x_j^2 - 2\sigma, x_i x_j)_{\substack{1 \leq i < j \leq n \\ j \geq m+1}},$$

where $\sigma \in S[m]$ has order $\deg(G)$ and $\sigma \circ G = 1$.

Proof. It suffices to apply Lemma 3.1 with

$$H := \sum_{j=m+1}^{n} y_j^2,$$

taking into account that

$$\mathrm{Ann}(H) = (x_j^2 - x_{m+1}^2, x_i x_j)_{\substack{m+1 \le i < j \le n \\ j \ge m+1}},$$

and that $x_{m+1}^2 \circ H = 2$. $\hfill\square$

4. Rationality of Poincaré series. We now focus on the Poincaré series $P_A(z)$ of the k-algebra A defined in the introduction: we will generalize the aforementioned classical results [**5, 8, 14**].

Proposition 4.1. *Let A be a local, Artinian, Gorenstein, k-algebra with $n = H_A(1)$. Assume $A \cong S[n]/\mathrm{Ann}(F)$ where*

$$F = F_{\ge 2} := G + \sum_{j=m+1}^{n} y_j^2$$

for a suitable non-zero $G \in P[m]$. Then

$$P_A(z) = \frac{P_B(z)}{1 - (H_A(1) - H_B(1))z P_B(z)},$$

where $B := S[m]/\mathrm{Ann}(G)$.

Proof. Since $F = F_{\ge 2}$, it follows that the same is true for G; hence, $m = H_B(1)$. Now we can use Corollary 3.2 above.

Besides the decomposition result proved in the previous section, we will also use the two following fundamental facts about Poincaré series:

- for each local Artinian, Gorenstein ring C with $\mathrm{emdim}(C) \ge 2$,

$$(4.1) \qquad P_C(z) = \frac{P_{C/\mathrm{Soc}(C)}(z)}{1 + z^2 P_{C/\mathrm{Soc}(C)}(z)}$$

   (see [**2**, Theorem 2]);
- for each local Artinian ring C with maximal ideal $\mathfrak{N}$ and linearly independent elements $c_1, \dots, c_h \in \mathfrak{N} \setminus \mathfrak{N}^2$ of $\mathrm{Soc}(C)$, then

$$(4.2) \qquad P_C(z) = \frac{P_{C/(c_1,\dots,c_h)}(z)}{1 - hz P_{C/(c_1,\dots,c_h)}(z)}$$

   (see [**11**, Proposition 3.4.4]).

Due to Corollary 3.2, we have

$$\mathrm{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n) = \mathrm{Ann}(G)S[n] + (\sigma, x_{m+1}, \ldots, x_n),$$

where σ is as in Corollary 3.2. Thus,

$$\frac{S[n]}{\mathrm{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n)} \cong \frac{S[m]}{\mathrm{Ann}(G) + (\sigma)}.$$

Trivially, $S[m]/\mathrm{Ann}(G)$ is a local, Artinian, Gorenstein, k-algebra.

On the one hand, we know that $\mathrm{Soc}(A)$ is principal because A is Gorenstein. On the other hand, $\sigma \notin \mathrm{Ann}(F)$, because $\sigma \in S[m]$, whence $\sigma \circ F = \sigma \circ G = 1$, and $x_i\sigma \in \mathrm{Ann}(F)$ for each $i = 1, \ldots, n$, because $x_i\sigma \circ F = 0$. It follows that the class of σ generates $\mathrm{Soc}(A)$. Formula (4.1) implies that

$$P_A(z) = \frac{P_{S[n]/\mathrm{Ann}(F)+(\sigma)}(z)}{1 + z^2 P_{S[n]/\mathrm{Ann}(F)+(\sigma)}(z)}.$$

Notice that

$$x_i x_j \in \mathrm{Ann}(F) + (\sigma), \quad i = 1, \ldots, n, \ j = m+1, \ldots, n, \ i \le j.$$

In particular, $x_{m+1}, \ldots, x_n \in \mathrm{Soc}(S[n]/\mathrm{Ann}(F) + (\sigma))$. It follows from formula (4.2) that

$$P_{S[n]/\mathrm{Ann}(F)+(\sigma)}(z) = \frac{P_{S[n]/\mathrm{Ann}(F)+(\sigma,x_{m+1},\ldots,x_n)}(z)}{1 - (n-m)z P_{S[n]/\mathrm{Ann}(F)+(\sigma,x_{m+1},\ldots,x_n)}(z)}.$$

The inverse formula of (4.1) finally yields

$$P_{S[n]/\mathrm{Ann}(F)+(\sigma,x_{m+1},\ldots,x_n)} = P_{S[m]/\mathrm{Ann}(G)+(\sigma)}(z) = \frac{P_{S[m]/\mathrm{Ann}(G)}(z)}{1 - z^2 P_{S[m]/\mathrm{Ann}(G)}(z)}.$$

The statement now follows by combining the three equalities above. $\square$

A first immediate consequence of Proposition 4.1 is the following corollary.

Corollary 4.2. *Let A be a local, Artinian, Gorenstein, k-algebra with $n = H_A(1)$. Assume that $A := S[n]/\mathrm{Ann}(F)$, where*

$$F = F_{\ge 2} := G + \sum_{j=m+1}^{n} y_j^2,$$

for a suitable $G \in P[m]$. Then the series $P_A(z)$ is rational if and only if the same is true for $P_B(z)$ where $B := S[m]/\mathrm{Ann}(G)$.

Proof. If $G \neq 0$, then the result follows from Proposition 4.1 above.

If $G = 0$, then A is stretched; thus, $P_A(z)$ is rational by [**14, Theorem 2**]. Moreover, in this case, $B = 0$; thus, $P_B(z) = 0$ is also rational and the equivalence holds.   □

Corollary 4.3. *Let A be a local, Artinian, Gorenstein, k-algebra with $n = H_A(1)$. Assume that $A := S[n]/\mathrm{Ann}(F)$ where*

$$F = F_{\geq 2} := G + \sum_{j=m+1}^{n} y_j^2$$

for a suitable $G \in P[4]$. Then $P_A(z)$ is rational.

Proof. The statement follows from rationality of the Poincaré series of each local Artinian, Gorenstein ring with embedding dimension at most four (see [**13, 16, 17**]).   □

Let A be a local, Artinian, Gorenstein k-algebra with $n := H_A(1)$. Recall that we defined the numbers $f_h = H_{\mathrm{gr}(A)/C_A(s-h+1)}(1)$ in Section 2.

Corollary 4.4. *Let A be a local, Artinian, Gorenstein k-algebra such that $f_3 \leq 4$. Then $P_A(z)$ is rational.*

Proof. The statement follows from Corollary 4.3, taking into account Lemma 2.2.   □

5. Examples of algebras with rational Poincaré series. In this section, we give some examples of local, Artinian, Gorenstein k-algebras A with rational P_A using the results proved in the previous section and some other classical results that we now go to quickly recall.

Remark 5.1. For the following results, we refer to [**3,** Chapter 4].

If $i_0 \in \mathbb{Z}$ is positive, then each positive $a \in \mathbb{Z}$ can be uniquely written in the form

$$a = \sum_{i=1}^{i_0} \binom{k(i)}{i},$$

where $0 \leq k(i-1) < k(i)$ (see [**3**, Lemma 4.2.6]). We define

$$a^{\langle i_0 \rangle} := \sum_{i=1}^{i_0} \binom{k(i)+1}{i+1}.$$

Macaulay's growth theorem (see [**3**, Theorem 4.2.10]) states that the following conditions are equivalent for a function $h \colon \mathbb{N} \to \mathbb{N}$:

- there exists a graded k-algebra R with Hilbert function $H_R = h$;
- one has $h(0) = 1$ and $h(i+1) \leq h(i)^{\langle i \rangle}$ for each $i \geq 1$.

An immediate consequence of the above equivalence is the following. Let A be a local k-algebra. If $H_A(i) \leq i$ for some i, then $H_A(j) \leq H_A(i)$ for each $j \geq i$. Such a remark will be used intensively in the following proofs.

The following lemma generalizes a result due to Stanley (see [**15**, Section II.6 (c)]).

Lemma 5.2. *Let A be a local, Artinian, Gorenstein, 3 stretched k-algebra. If $H_A(3) \leq 5$, then*

$$\sum_{a=0}^{s-4} H_{Q_A(a)}(2) \geq H_A(3).$$

Proof. We set $m := H_A(3)$ and

$$p := \sum_{a=0}^{s-4} H_{Q_A(a)}(2).$$

We have to show that $p \geq m$: assume $p \leq m - 1$.

If $s = 4$, then

$$H_{Q_A(0)} = \sum_{a=0}^{s-4} H_{Q_A(a)}.$$

In particular, such a sum is the Hilbert function of a graded Gorenstein k-algebra; thus, it must be symmetric, say $(1, m, p, m, 1)$.

Let $s \geq 5$. Then formula (2.4) for $i \geq 4$ implies

$$1 = H_A(i) = \sum_{a=0}^{s-5} H_{Q_A(a)}(i).$$

Since A is 3-stretched and $H_{Q_A(0)}(i) \geq 1$ for $i \leq s$, it follows that $H_{Q_A(0)}(1) = H_{Q_A(a)}(s-1) = 1$. Thus, Remark 5.1 implies $H_{Q_A(0)}(i) = 1$ for $i \leq s$.

The same argument also proves $H_{Q_A(a)}(1) = H_{Q_A(a)}(s - a - 1) = 0$ in the range $1 \leq a \leq s - 5$. Due to formula (2.5) we deduce that

$$H_{\mathrm{gr}(A)/C_A(s-4)}(1) = \sum_{\alpha=0}^{s-5} H_{Q_A(\alpha)}(1) = 1,$$

whence

$$H_{\mathrm{gr}(A)/C_A(s-4)}(i) = \sum_{\alpha=0}^{s-5} H_{Q_A(\alpha)}(i) = 1 \quad \text{for } i \leq s.$$

We conclude that

$$H_{Q_A(a)} = \begin{cases} (1, 1, 1, 1, 1, \ldots, 1) & \text{if } a = 0, \\ (0, 0, 0, 0, 0, \ldots, 0) & \text{if } a = 1, \ldots, s - 5, \\ (0, m - 1, p - 1, m - 1, 0, \ldots, 0) & \text{if } a = s - 4. \end{cases}$$

In particular,

$$\sum_{a=0}^{s-4} H_{Q_A(a)} = H_{Q_A(0)} + H_{Q_A(s-4)}.$$

Notice that $f_4 = m$.

Recall that we have assumed $p < m$. Formula (2.5) and Remark 5.1 imply that if

$$p = H_{\mathrm{gr}(A)/C_A(s-4)}(2) = \sum_{a=0}^{s-4} H_{Q_A(a)}(2) \leq 2,$$

then $m = H_{\mathrm{gr}(A)/C_A(s-4)}(3) \leq p$, contradicting our assumption $p \leq m - 1$. We conclude that $3 \leq p$ necessarily. Moreover, if $p = 3$, then the bound of Macaulay's growth theorem (see Remark 5.1) gives $m \leq 4$. Since we assumed $m \leq 5$, we are left with two cases: $p = 3$ and $m = 4$ or $p = 4$ and $m = 5$. We examine the second case, the first one being analogous.

Lemma 2.2 with $c = 3$ asserts the existence of a polynomial $F := y_1^s + F_4 + F_3 + F_2$ such that $F_i \in P[f_i]_i$, $x_1^3 \circ F_4 = 0$, the derivatives $x_2 \circ F_4$, $x_3 \circ F_4$, $x_4 \circ F_4$ and $x_5 \circ F_4$ are linearly independent and $A \cong S[n]/\mathrm{Ann}(F)$. We set $B := S[n]/\mathrm{Ann}(F_{\geq 4})$.

We first check that

$$H_B = \sum_{a=0}^{s-4} H_{Q_A(a)} = (1, 5, 4, 5, 1, \ldots, 1).$$

On the one hand, [**12**, Lemma 1.10] implies that $\widehat{C}_A(a) = \widehat{C}_B(a)$, $a \leq s - 3$, whence

$$H_B(1) \geq \sum_{a=0}^{s-4} H_{Q_B(a)}(1) = \sum_{a=0}^{s-4} H_{Q_A(a)}(1) = 5.$$

On the other hand, $F_{\geq 4} \in P[f_4] = P[5]$, whence $5 = H_B(1) \leq 5$. It follows that equality holds; thus, $H_{Q_B(s-2)}(1) = H_{Q_B(s-3)}(1) = 0$. By symmetry, we finally obtain $H_{Q_B(s-2)} = H_{Q_B(s-3)} = 0$. This last vanishing completes the proof of the equality

$$H_B = \sum_{a=0}^{s-4} H_{Q_A(a)} = (1, 5, 4, 5, 1, \ldots, 1).$$

Let $I \subseteq k[x_1, \ldots, x_n] \subseteq S[n]$ be the ideal generated by the forms of degree at most 2 inside $\mathrm{Ann}(\mathrm{tdf}(F_{\geq 4})) = \mathrm{ldf}(\mathrm{Ann}(F_{\geq 4}))$. We obviously have $x_6, \ldots, x_n \in I$, because $F_{\geq 4} \in P[5]$.

Let

$$I^{\mathrm{sat}} := \bigcup_{h=1}^{\infty} (I : (x_1, \ldots, x_n)^h) :$$

it is easy to check that I^{sat} is an ideal. We set $R := k[x_1, \ldots, x_n]/I$, $R^{\mathrm{sat}} := k[x_1, \ldots, x_n]/I^{\mathrm{sat}}$. Due to the definition of I we know that $H_R(t) \geq H_B(t)$ for each $t \geq 0$, and equality holds true for $t \leq 2$. Moreover, we know that

$$H_B(2)^{\langle 2 \rangle} = H_B(3) \leq H_R(3) \leq H_R(2)^{\langle 2 \rangle} = H_B(2)^{\langle 2 \rangle};$$

hence,

$$H_R(3) = \binom{4}{3} + \binom{2}{2} = H_R(2)^{\langle 2 \rangle}.$$

The Gotzmann persistence theorem (see [**3,** Theorem 4.3.3]) implies that

$$H_R(t) = \binom{t+1}{t} + \binom{t-1}{t-1} = t + 2, \quad t \geq 2.$$

We infer $H_{R^{\mathrm{sat}}}(t) = t + 2$, $t \gg 0$.

When saturating, the ideal can only increase its size in each degree; hence, $H_{R^{\mathrm{sat}}}(t) \leq H_R(t)$ for each $t \geq 0$. Again Macaulay's bound (see Remark 5.1) thus forces $H_{R^{\mathrm{sat}}}(t) = H_R(t) = t + 2$ for $t \geq 2$. In particular, the components I_t and I_t^{sat} of degree $t \geq 2$ of I and I^{sat} coincide.

Since $H_{R^{\mathrm{sat}}}$ is non-decreasing, it follows that

$$H_{R^{\mathrm{sat}}}(1) \leq H_{R^{\mathrm{sat}}}(2) = 4 < 5 = H_B(1) = H_R(1).$$

In particular, there exists a non-zero linear form $\ell \in I^{\mathrm{sat}} \setminus I$. The equality $I_2 = I_2^{\mathrm{sat}}$ forces $\ell x_j \in I_2 \subseteq \mathrm{Ann}(\mathrm{tdf}(F_{\geq 4}))$, $j = 1, \ldots, n$. Since $x_6, \ldots, x_n \in I$, it follows that we can assume $\ell \in S[5] \subseteq S[n]$, i.e.,

$$\ell = \sum_{i=1}^{5} \ell_i x_i, \quad \ell_i \in k.$$

Moreover, we also know that $y_1^s \in \mathrm{tdf}(F_{\geq 4})$ because $s \geq 5$. Hence, the condition $\ell x_1 \in \mathrm{Ann}(\mathrm{tdf}(F_{\geq 4}))$ implies $\ell_1 = 0$ and at least one of the remaining coefficients is non-zero.

If $j \geq 2$, then $x_j \circ F_{\geq 4} = x_j \circ F_4$, thus the condition $x_j \ell \in I_2 \subseteq \operatorname{Ann}(\operatorname{tdf}(F_{\geq 4}))$ implies that $\ell \circ F_4$ does not depend on such x_j, $j = 2, \ldots, 5$. It follows that $\ell \circ F_4$ is a multiple of y_1^3. Since we know that $x_1^3 \circ F_4 = 0$, we conclude that $\ell \circ F_4 = 0$. Such a vanishing contradicts the linear independence of the derivatives

$$x_2 \circ F_4, \quad x_3 \circ F_4, \quad x_4 \circ F_4, \quad x_5 \circ F_4.$$

The proof of the statement is now complete. $\square$

Using the results proved in the previous section and Lemma 5.2 we are able to handle the first example of this section, proving the following theorem generalizing [**4**, Corollary 2.2].

Theorem 5.3. *Let A be a local, Artinian, Gorenstein k-algebra with $H_A(2) \leq 4$ and $\operatorname{cdeg}(A) \leq 3$. Then P_A is rational.*

Proof. Let us examine the case $\operatorname{cdeg}(A) = 3$, the other ones being similar, but simpler. The proofs of these cases are left to the reader. Lemma 5.2 yields:

$$(5.1) \qquad H_A(2) \geq \sum_{a=0}^{s-4} H_{Q(a)}(2) \geq H_A(3).$$

If $\operatorname{sdeg}(A) \geq 5$, then decomposition (2.3) is

$$(1, 1, \ldots, 1) + (0, a_1, a_2, a_1, 0) + (0, b_1, b_1, 0) + (0, c_1, 0)$$

for some integers a_1, a_2, b_1, c_1. Inequality (5.1) is equivalent to $a_1 \leq a_2$. We know that $H_A(2) = a_2 + b_1 + 1 \leq 4$, so $f_3 = a_1 + b_1 + 1 \leq 4$, and the argument follows from Corollary 4.4. In the case $\operatorname{sdeg}(A) = 4$, the decomposition (2.3) changes, but the argument stays the same. $\square$

Now we skip the condition $\operatorname{cdeg}(A) = 3$, but we impose a restriction on the shape of H_A. The following theorem generalizes a well-known result proved when either $m = 1, 2$ (see [**8, 14**]) or $m \leq 4$ and $s = 3$ (see again [**4**]).

Theorem 5.4. *Let A be a local, Artinian, Gorenstein k-algebra such that $H_A(i) = m$, $2 \leq i \leq \operatorname{cdeg}(A)$. If $m \leq 4$, then P_A is rational.*

Proof. Let $c := \mathrm{cdeg}(A)$, $n := H_A(1)$, take a polynomial $F := y_1^s + F_{c+1} + \cdots$, $F_{c+1} \in P[f_{c+1}]_{c+1} = P[m]_{c+1}$ such that $A \cong S[n]/\mathrm{Ann}(F)$ (see Lemma 2.2) and set $B := S[n]/\mathrm{Ann}(F_{\geq c+1})$ so that $Q_A(a) = Q_B(a)$ for $a \leq s - c - 1$ (again by [**12**, Lemma 1.10]). In particular, $H_B(c) = m$, thus decomposition (2.3) implies $H_B(1) \geq m$. Since we know that $F_{\geq c+1} \in P[m]$, it follows that $H_B(1) \leq m$, hence equality must hold.

As in the proof of the previous lemma one immediately checks that either $s = c + 1$ and $H_{Q_A(0)} = (1, m, \ldots, m, 1)$, or $s \geq c + 2$ and

$$H_{Q_A(a)} = \begin{cases} (1, 1, \ldots, 1, 1, \ldots, 1) & \text{if } a = 0, \\ (0, 0, \ldots, 0, 0, \ldots, 0) & \text{if } a = 1, \ldots, s - c - 2, \\ (0, m - 1, \ldots, m - 1, 0, \ldots, 0) & \text{if } a = s - c - 1. \end{cases}$$

Assume that $H_B(i) \leq m - 1 \leq 3$ for some $i = 2, \ldots, c - 1$. Let i_0 be the maximal of such i's. If $i_0 \geq 3$, then $H_B(i_0) \leq i_0 < m$ so by Remark 5.1, we get that $H_B(i) < m$ for all $i > i_0$. In particular, $H_B(c) < m$, which is a contradiction. We conclude that $i_0 = 2$.

Due to the symmetry of $H_{Q_B(s-c-1)}$, we deduce that $c = 3$. If $H_{Q_B(s-3)}(2) = q$, the symmetry of $H_{Q_B(s-3)}$ implies $H_{Q_B(s-3)}(1) = q$; hence decomposition (2.3) implies

$$m = H_B(1) = \sum_{a=0}^{s-2} H_{Q_B(a)}(1) = m + q + H_{Q_B(s-2)}(1).$$

It follows that $q = H_{Q_B(s-2)}(1) = 0$, whence $H_B = (1, m, p, m, 1, \ldots, 1)$ where $p \leq m - 1$ which cannot occur by Lemma 5.2.

We conclude that $H_{Q_A(s-c-1)}(i) = H_{Q_B(s-c-1)}(i) = m - 1$ for each $i = 2, \ldots, c$, then the hypothesis on $H_A(i)$ and decomposition (2.3) yield

$$H_{Q_A(a)} = \begin{cases} (0, 0, 0, \ldots, 0, 0, \ldots, 0) & \text{if } a = s - c, \ldots, s - 3, \\ (0, n - m, 0, \ldots, 0, 0, \ldots, 0) & \text{if } a = s - 2, \end{cases}$$

whence

$$f_3 = \sum_{a=1}^{s-3} H_{Q(a)}(1) = m \leq 4. \qquad \square$$

As a third example, we skip the condition on the shape of H_A but we put a limit on $\dim_k(A)$, slightly extending the result proved in [**6**].

Theorem 5.5. *Let A be a local, Artinian, Gorenstein k-algebra with $\dim_k(A) \leq 16$ and $H_A(2) \leq 4$. Then P_A is rational.*

Proof. Due to [**13**], we can restrict our attention to k-algebras A with $H_A(1) \geq 5$.

Rationality of the Poincaré series of stretched k-algebras is proved in [**14**]. For k-algebras A with $H_A(2) = 2$, which are called *almost stretched*, see [**8**]. For the case of k-algebras A with $\text{sdeg}(A) = 3$ and $H_A(2) \leq 4$, see [**4**]. Finally, the case $H_A(i) = m$, $2 \leq i \leq \text{cdeg}(A)$ with $m \leq 4$ is covered by Theorem 5.4 above.

There are several cases which are not covered by the aforementioned results. In each of these cases one can check that the condition $f_3 \leq 4$ of Corollary 4.4 is fulfilled.

Due to the above summary of known results, we can assume $H_A(2) \geq 3$. The restriction $H_A(2) \leq 4$ implies $H_A(3) \leq 5$ again by Remark 5.1.

Theorem 5.3 deals with the case $\text{sdeg}(A) = 4$. Let us analyze the cases $\text{sdeg}(A) = 5$ and $\dim_k A \leq 16$. The decomposition is

$$(1, a_1, a_2, a_2, a_1, 1) + (0, b_1, b_2, b_1, 0) + (0, c_1, c_1, 0) + (0, d_1, 0),$$

for some integers a_1, a_2, b_1, b_2, c_1, d_1. If $a_1 = 1$, then the k-algebra is 3-stretched, so we may suppose $a_1 \geq 2$. We know that $H_A(2) = a_2 + b_2 + c_1 \leq 4$, and we would like to prove $a_1 + b_1 + c_1 \leq 4$. Suppose $a_1 + b_1 + c_1 \geq 5$. Then the inequality on the dimension of A shows that $2 \cdot a_2 + b_2 \leq 4$, in particular, $a_2 \leq 2$ and from Remark 5.1, it follows that $a_1 = a_2 = 2$. It follows that $b_2 = 0$ and once again, from the same remark, $b_1 = 0$. This forces $a_1 + b_1 + c_1 = 2 + c_1 = a_2 + b_2 + c_1 \leq 4$, a contradiction.

Let us now suppose that $\text{sdeg}(A) = 6$. Look at the first row of the symmetric decomposition (2.3): $(1, a_1, a_2, a_3, a_2, a_1, 1)$.

- If $a_1 \geq 3$, then $a_2, a_3 \geq 3$ and the sum of the row is at least 17.
- If $a_1 = 2$, then $a_2 = a_3 = 2$ and the sum of the row is 12. If we suppose that $f_3 \geq 5$, then the sum of the first column of the

remaining part of the decomposition will be at least 3, so the sum of whole remaining part will be at least $2 \cdot 3 = 6$, and the dimension will be at least $12 + 6 > 16$.

- Suppose $a_1 = 1$, and look at the second row $(0, b_1, b_2, b_2, b_1, 0)$. If $b_1 = 0$, then the k-algebra is 3-stretched so the result follows from Theorem 5.3. From $H_A(2) \leq 4$, it follows that $b_2 \leq 3$. If $b_2 = 3$, then $b_1 \geq 2$ so the dimension is at least $7 + 10 > 16$. If $b_2 \leq 2$, then $b_1 \leq b_2$ from Remark 5.1. Hence, the same argument as before applies.

Let us finally suppose that $\text{sdeg}(A) \geq 7$. Take the first row, beginning with $(1, a_1, a_2, \ldots)$. If $a_1 \geq 3$, then its sum is at least $3 \cdot \text{sdeg}(A) - 1 > 16$. If $a_1 = 2$, the sum of this row is $2 \cdot \text{sdeg}(A) \geq 14$. Then one can argue as in the case $\text{sdeg}(A) = 6$, $a_1 = 2$. A similar reasoning shows that, when $a_1 = 1$, the k-algebra has decomposition $(1, 1, \ldots, 1) + (0, 4, 4, 0)$, and so $H_A(2) \geq 5$. $\qquad\square$

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Dipartimento di Scienze Matematiche, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy
Email address: **gianfranco.casnati@polito.it**

Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
Email address: **jjelisiejew@mimuw.edu.pl**

Dipartimento di Matematica, Politecnico di Milano, via Bonardi 9, 20133 Milano, Italy
Email address: **roberto.notari@polimi.it**