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AUTOMORPHISMS OF SURFACES IN A CLASS OF WEHLER K3 SURFACES WITH PICARD NUMBER 4

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ABSTRACT. In this paper, we find the group of automorphisms (up to finite index) for K3 surfaces in a class of Wehler K3 surfaces with Picard number 4. In doing so, we demonstrate a variety of techniques, both general and ad hoc, that can be used to find the group of automorphisms of a K3 surface, particularly those with small Picard number.

Introduction. Given an algebraic K3 surface $\mathcal{X}$ defined over a number field k, what is its group of automorphisms $\mathcal{A} = \mathrm{Aut}(\mathcal{X}/k)$? In this paper, we offer some ideas of how to answer this natural question, and demonstrate these ideas by applying them to a particular example. This paper grew out of a talk given at the Banff International Research Station in December 2008.

There are three main general tools. (1) Every automorphism σ induces a linear action on the Picard lattice $\mathrm{Pic}(\mathcal{X})$ that preserves the intersection pairing; (2) the intersection pairing on the Picard lattice is a Lorentz product, so induces a hyperbolic structure on $\mathbb{H} = \mathcal{L}^+/\mathbb{R}^+$, where $\mathcal{L}^+$ is the light cone; and (3) a fundamental result due to Pjateckiĭ-Šapiro and Šafarevič, which establishes a correspondence between $\mathcal{A}$ and a particular subgroup of the lattice preserving isometries of $\mathbb{H}$.

We apply these results, together with some ad hoc results, to a class of K3 surfaces, and come up with a group of finite index in $\mathcal{A}$. Though not complete, we consider this answer to be sufficient; completing the problem likely depends on the arithmetic and not the geometry, i.e., it depends on $\mathcal{X}$ and not just on $\mathrm{Pic}(\mathcal{X})$.

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It is clear to the author that these techniques are applicable to many classes of K3 surfaces, particularly those with small Picard number. The hyperbolic space $\mathbb{H}$ of a surface with Picard number n is $n-1$ dimensional, hence the difficulty of dealing with surfaces with large Picard number includes our difficulty imagining hyperbolic spaces of large dimension.

1. Background. Let $\mathcal{X}/k$ be a K3 surface defined over a number field k. Let n be the dimension of the Picard lattice $\mathrm{Pic}(\mathcal{X})$, and let $\{D_1,\dots,D_n\}$ be a basis over $\mathbb{Z}$, so

$$\mathrm{Pic}(\mathcal{X}) = D_1\mathbb{Z} \oplus \cdots \oplus D_n\mathbb{Z}.$$

Let $J = [D_i \cdot D_j]$ be the intersection matrix for the basis $\mathcal{D}$. By the Hodge index theorem, J has signature $(1, n-1)$, i.e., it has one positive eigenvalue and $n-1$ negative eigenvalues. It therefore defines a Lorentz product, so $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$ is a Lorentz space. Let D be an ample divisor in $\mathrm{Pic}(\mathcal{X})$. We define the *light cone* to be the set

$$\mathcal{L}^+ = \{x \in \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R} : x \cdot x > 0, x \cdot D > 0\}.$$

The space $\mathbb{H} = \mathcal{L}^+/\mathbb{R}^+$, together with the distance $|AB|$ defined by

$$\|A\|\|B\| \cosh |AB| = A \cdot B,$$

is an $n-1$ dimensional hyperbolic geometry. (By $||x||$, we mean $\sqrt{x \cdot x}$.)

An automorphism $\sigma \in \mathcal{A} = \mathrm{Aut}(\mathcal{X}/k)$ acts linearly on the Picard lattice via the pull back map σ^*. We are therefore interested in the linear automorphisms of the Picard lattice, the group

$$\mathcal{O} = \{T \in M_{n\times n}(\mathbb{Z}) : T^t J T = J\}.$$

In this group is the subgroup of index two that preserves the light cone,

$$\mathcal{O}^+ = \{T \in \mathcal{O} : T\mathcal{L}^+ = \mathcal{L}^+\},$$

which is a discrete group of isometries on $\mathbb{H}$. It is an arithmetic group, and its fundamental domain has finite volume.

A divisor E is called *effective* if we can write

$$E = \sum_{i=1}^{k} a_i C_i,$$

where $a_i \geq 0$ and the C_i's are divisors represented by curves on $\mathcal{X}$. A divisor $D \in \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$ is called *ample* if $D \cdot E > 0$ for all effective divisors E. The ample cone $\mathcal{K} \subset \mathcal{L}^+$ is the set of all ample divisors in $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$.

Within $\mathcal{O}^+$, the *reflections through* -2 *divisors* play a special role. Let C be a divisor such that $C \cdot C = -2$. The map R_C defined by

$$R_C(\mathbf{x}) = \mathbf{x} + (C \cdot \mathbf{x})C,$$

which is a reflection through the hyperplane $C \cdot \mathbf{x} = 0$, is an isometry in $\mathcal{O}^+$. Let $\mathcal{O}'$ be the group generated by all such divisors. Note that, for $T \in \mathcal{O}^+$,

$$R_{TC} = T^{-1} R_C T,$$

so

$$\mathcal{O}' \lhd \mathcal{O}^+.$$

If $\sigma \in \mathcal{A}$ and E is effective, then $\sigma_* E = (\sigma^{-1})^* E$ is effective. Thus, $\sigma^* D \cdot E = D \cdot \sigma_* E > 0$ for all effective E and ample D, so $\sigma^* D$ is ample. We therefore define

$$\mathcal{O}'' = \{T \in \mathcal{O}^+ : T\mathcal{K} = \mathcal{K}\},$$

since the pullback map sends $\mathcal{A}$ into $\mathcal{O}''$. Pjateckiĭ-Šapiro and Šafarevič prove that the pullback map of $\mathcal{A}$ to $\mathcal{O}''$ has finite kernel and cokernel [**6**], and that $\mathcal{O}'' \cong \mathcal{O}^+/\mathcal{O}'$.

The interplay of $\mathcal{A}$ and the groups $\mathcal{O}^+$, $\mathcal{O}''$ and $\mathcal{O}'$ is quite pretty, as we will see in our example. We note that, given an arbitrary J, finding $\mathcal{O}^+$ is a non-trivial problem.

Remark 1.1. Let $\mathbf{n} \cdot \mathbf{x} = 0$ be a hyperplane through the origin in $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$. Reflection through this hyperplane is given by

$$R_{\mathbf{n}}(\mathbf{x}) = \mathbf{x} - 2\mathrm{proj}_{\mathbf{n}}\mathbf{x} = \mathbf{x} - 2\frac{(\mathbf{x} \cdot \mathbf{n})\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}.$$

Thus, if $\mathbf{n} \in \mathrm{Pic}(\mathcal{X})$ (so has integer entries) and $\mathbf{n} \cdot \mathbf{n} = \pm 1$ or ± 2, then $R_{\mathbf{n}} \in \mathcal{O}$. Since the intersection pairings for K3 surfaces are always even (see the adjunction formula), we only ever have $\mathbf{n} \cdot \mathbf{n} = \pm 2$. If $\mathbf{n} \cdot \mathbf{n} = -2$, then the plane intersects $\mathbb{H}$, and we get a reflection through a hyperline in $\mathbb{H}$. These are the reflections through -2 curves mentioned above. If $\mathbf{n} \in \mathrm{Pic}(\mathcal{X})$ and $\mathbf{n} \cdot \mathbf{n} = 2$, then the hyperplane does not intersect $\mathbb{H}$

and $-R_{\mathbf{n}}(\mathbf{x}) \in \mathcal{O}^+$. This is inversion through $\mathbf{n}$, which, when $n = 3$, is just rotation by π about $\mathbf{n}$. This can be used to find elements of $\mathcal{O}^+$.

2. A specific example. Let $\mathcal{X}$ be a surface described by a smooth $(2, 2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Such surfaces are sometimes known as Wehler K3 surfaces, so named since Wehler showed that they are K3 surfaces [**8**]. Such a surface can be expressed as the zero locus of

$$F(X, Y, Z) = X_0^2 F_0(Y, Z) + X_0 X_1 F_1(Y, Z) + X_1^2 F_2(Y, Z),$$

where $X = (X_0, X_1) \in \mathbb{P}^1$, and $F_i(Y, Z)$ is a $(2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1$ for all i. Since a smooth $(2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1$ is an elliptic curve, $\mathcal{X}$ is fibered by elliptic curves in each of the three directions.

Let

$$\begin{aligned} p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ (X, Y, Z) &\longmapsto X \end{aligned}$$

be the projection onto the first component, and similarly define p_2 and p_3. Let $D_i = p_i^{-1} H$ be the pullback of a point $H \in \mathbb{P}^1$ for $i = 1, 2$ and 3.

A generic surface $\mathcal{X}$ in this class has Picard lattice $D_1\mathbb{Z} \oplus D_2\mathbb{Z} \oplus D_3\mathbb{Z}$ and intersection matrix

$$J = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

The generic Wehler K3 surfaces have been studied by Wang [**7**], Billard [**4**], and the author [**1**]. They contain no -2 curves, and their group of automorphisms is well understood [**8**]. Explicit examples are given in [**3**].

We will study the class of Wehler K3 surfaces with Picard number 4 and such that $F_2(Y, Z)$ factors into linear terms, i.e., $F_2(Y, Z) = L_1(Y, Z)L_2(Y, Z)$ where $L_i(Y, Z)$ is a $(1, 1)$ form in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathcal{X}$ contains the curve $((0, 1), Y, Z)$ such that $L_1(Y, Z) = 0$, which is rational and hence a -2 curve on $\mathcal{X}$. Let D_4 be the divisor class for this curve. The surface $\mathcal{X}$ also contains the -2 curve $((0, 1), Y, Z)$ such that $L_2(Y, Z) = 0$, and its divisor is $D_1 - D_4$. It is clear that $D_1 \cdot D_4 = 0$, and $D_2 \cdot D_4 = D_3 \cdot D_4 = 1$, so $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ is a basis of $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$ (since $\mathcal{X}$ has Picard number 4), and in this basis,

$$J = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

Since the elements of the basis $\mathcal{D}$ are in $\mathrm{Pic}(\mathcal{X})$, the lattice $D_1\mathbb{Z} \oplus D_2\mathbb{Z} \oplus D_3\mathbb{Z} \oplus D_4\mathbb{Z}$ is a sublattice of $\mathrm{Pic}(\mathcal{X})$.

Lemma 2.1. *The basis $\mathcal{D}$ of* $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R}$ *is a basis of* $\mathrm{Pic}(\mathcal{X})$ *over $\mathbb{Z}$.*

Proof. Suppose that there exists an element $C \in \mathrm{Pic}(\mathcal{X})$ such that C is not in the lattice generated by $\mathcal{D}$. Then there exists a $C' \in \mathrm{Pic}(\mathcal{X})$ such that C' is in the polytope generated by the elements of $\mathcal{D}$, i.e.,

$$C' = c_1 D_1 + c_2 D_2 + c_3 D_3 + c_4 D_4$$

with $0 \leq c_i < 1$. Let $a_i = C' \cdot D_i \in \mathbb{Z}$ and $\mathbf{a} = [a_1, a_2, a_3, a_4]$. Then $\mathbf{a} = J\mathbf{c}$ (where $\mathbf{c} = [c_1, c_2, c_3, c_4]$), so $0 \leq a_1 < 4$, $0 \leq a_2 < 5$, $0 \leq a_3 < 5$ and $-2 < a_4 < 2$. Thus, there are only a finite number of cases to check, which is easily done via computer. We find that the only other possibility for $\mathrm{Pic}(\mathcal{X})$ is the lattice spanned by $\{D_1/2, D_2, D_3, D_4\}$. Since D_1 represents an elliptic curve, it cannot be decomposed into elliptic curves, so $\mathcal{D}$ is a basis for $\mathrm{Pic}(\mathcal{X})$ over $\mathbb{Z}$. $\square$

Remark 2.2. In the above proof, we appealed to the geometry of the K3 surface. This could not be avoided since, by a result due to Morrison [**5**], there exists a K3 surface with Picard lattice isomorphic to the lattice spanned by $\{D_1/2, D_2, D_3, D_4\}$.

3. The automorphisms. Because of its quadratic nature, $\mathcal{X}$ has several obvious automorphisms. Let us fix Y and Z, so that $F(X, Y, Z)$ is a quadratic in X with two roots X and (say) X'. Then the map

$$\sigma_1 : (X, Y, Z) \longmapsto (X', Y, Z)$$

is an automorphism of $\mathcal{X}$. The pull back σ_1^* has several obvious relations: $\sigma_1^* D_2 = D_2$ and $\sigma_1^* D_3 = D_3$ which, together with $\sigma_1^2 = \mathrm{Id}$, lead to some obvious intersections, such as:

$$\sigma_1^* D_1 \cdot D_2 = D_1 \cdot \sigma_{1*} D_2 = D_1 \cdot \sigma_1^* D_2 = D_1 \cdot D_2 = 2.$$

The only difficult intersections are $\sigma_1^* D_1 \cdot D_1 = 8$, $\sigma_1^* D_1 \cdot D_4 = 4$ and $\sigma_1^* D_4 \cdot D_4$. Let us fix a curve C in the divisor class D_1. The image $\sigma_1 C$ intersects C wherever $F(X, Y, Z) = 0$ has a double root, as well as at values of X where both $F = 0$ and $\partial_X F = 0$. This is the intersection of two $(2, 2)$ forms, so $\sigma_1^* D_1 \cdot D_1 = 8$.

Let us now consider the curve C in D_1 given by $X = (0, 1)$. As D_4 is a component of C, the image $\sigma_1 C$ intersects D_4 wherever $X = (0, 1)$ is a double root, as well as where $\partial_X F = 0$ and $L_1 = 0$, which is the intersection of a $(2, 2)$ form and a $(1, 1)$ form. Thus, $\sigma_1^* D_1 \cdot D_4 = 4$. The last intersection is more difficult still, so let us assign it a variable: $\sigma_1^* D_4 \cdot D_4 = a$. Then,

$$J\sigma_1^* = \begin{bmatrix} 8 & 2 & 2 & 4 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 4 & 1 & 1 & a \end{bmatrix}.$$

Using $\sigma_1^{*2} = \mathrm{Id}$, we get $a = 0$ or 4, so $\sigma_1^* = T_1$ or $S_1 T_1$, where

$$T_1 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since S_1 is the isometry that sends D_4 to $D_1 - D_4$, it is clearly a symmetry of the ample cone $\mathcal{K}$. Thus, both T_1 and $S_1 T_1$ are in $\mathcal{O}''$, and there is no need to resolve the ambiguity for a.

In a similar fashion, we can define $\sigma_2(X, Y, Z) = (X, Y', Z)$ and $\sigma_3(X, Y, Z) = (X, Y, Z')$, and their pullbacks σ_2^* and σ_3^*. We note that $\sigma_2^* D_1 = D_1$ and $\sigma_2^* D_3 = D_3$. As before, the intersection $\sigma_2^* D_2 \cdot D_2 = 8$. Looking at the action of σ_2 on the curve $X = (0, 1)$, which includes D_4, we see $\sigma_2^* D_4 = D_1 - D_4$, from which we get $\sigma_2^* D_2 \cdot D_4 = D_2 \cdot (D_1 - D_4) = 1$ and $\sigma_2^* D_4 \cdot D_4 = (D_1 - D_4) \cdot D_4 = 2$.

Thus,

$$J\sigma_2^* = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 8 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad \text{so} \quad \sigma_2^* = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

By symmetry, $\sigma_3^* = S_2\sigma_2^* S_2$ where

$$S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To see whether we have the full group of automorphisms of $\mathcal{X}$, we turn our attention to the action of $\mathcal{O}''$ on $\mathbb{H}$. We can send $\mathbb{H}$ to the Poincaré ball model in the following way. There exists an orthonormal basis with change of basis matrix Q that diagonalizes J. Let us write $J = Q^t A^t J_0 AQ$, where J_0 has $(-1,-1,-1,1)$ along the diagonal, A has (a_1, a_2, a_3, a_4) along the diagonal, and $-a_1^2$, $-a_2^2$, $-a_3^2$ and a_4^2 are the eigenvalues of J. Then, for a point $P \in \mathcal{L}^+$, the point $P' = AQP/||P||$ is a point on the surface $x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$. Let $\pi(P)$ be the stereographic projection of P' onto the plane $x_4 = 0$ through the point $(0,0,0,-1)$ (see Figure 1). Then π is a map from $\mathbb{H}$ to the Poincaré ball model of hyperbolic geometry.

The Poincaré ball can, in turn, be unfolded into the Poincaré upper half space, using D_2 as the point at infinity. Note that $D_2 \cdot D_2 = 0$, so it is on $\partial\mathbb{H}$, the boundary of $\mathbb{H}$. Let $G = \langle T_1, S_1, S_2, \sigma_3^*, R_{D_4}\rangle \leq \mathcal{O}^+$. The eigenspace for S_1 and the eigenvalue 1 is spanned by $\{D_1, D_2, D_3\}$, so S_1 is a reflection in this plane. The map T_1 is a reflection in a plane that includes D_1 and D_2 and is perpendicular to the plane through which S_1 reflects. That plane includes the point $D_2 + D_4$. The map S_2 is reflection in a plane that includes D_1, and it sends D_2 to D_3. The plane through which it reflects is therefore represented by a hemisphere whose boundary is a circle through D_1 that is centered at D_3. The map R_{D_4} is a reflection, it sends D_2 to $D_2 + D_4$, and it fixes D_1, so it is a reflection through the hemisphere with boundary a circle centered at $D_2 + D_4$ and through D_1. This is enough information to sketch the fundamental domain for G, which is shown in Figure 2. Though our sketch does not need to be too precise, we note that the angle between

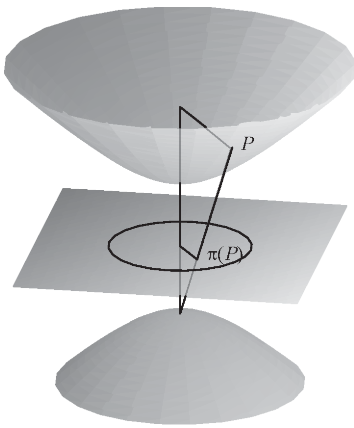

FIGURE 1. The projection of $\mathbb{H}$ to the Poincaré ball.

the circle that represents R_{D_4} and the line that represents T_1 is $\pi/4$. To see this, note that the normal $\mathbf{n}_1$ to the plane through which T_1 reflects is $\mathbf{n}_1 = -D_1 + D_2 + D_3$ (the eigenvector of T_1 associated to -1), and that

$$\frac{D_4 \cdot \mathbf{n}_1}{\|D_4\|\|\mathbf{n}_1\|} = \frac{\sqrt{2}}{2} = \cos(\pi/4).$$

As can be seen, the fundamental domain has infinite volume. Since the fundamental domain of $\mathcal{O}^+$ has finite volume, we know we are missing something.

To find another automorphism, we look at the second column of J, which is $[D_2 \cdot D_i] = [2, 0, 2, 1]$. Recall that D_2 is the divisor class given by the fibers over fixed Y, which are generically elliptic curves. Note that $D_2 \cdot D_4 = 1$, so the rational curve represented by D_4 intersects each of these elliptic curves exactly once, i.e., we have a fibration of elliptic curves with section. Let E be an elliptic curve in D_2, and let $O_E = E \cap D_4$ be its zero element.

We define a map σ_4 on $\mathcal{X}$ in the following way: For $P \in \mathcal{X}$, let E be the unique elliptic curve in D_2 that contains P. Define $\sigma_4(P) = -P$, using the group law on E with zero element O_E. Then σ_4 is an automorphism of $\mathcal{X}$. Since $\sigma_4(E) = E$, we know $\sigma_2^* D_2 = D_2$ and, since $\sigma_4(O_E) = O_E$, we get $\sigma_4^* D_4 = D_4$. We also note that σ_4^2 is the

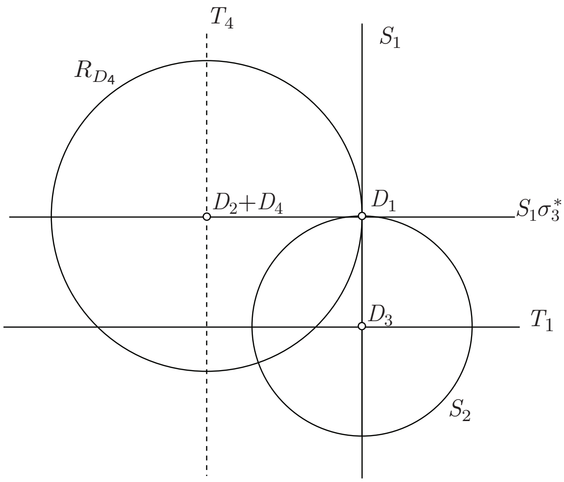

FIGURE 2. The upper half-space representation of isometries in $\mathcal{O}^+$. Each line or circle represents the plane or hemisphere above it and is labeled with the isometry that is a reflection through that (hyperbolic) plane. A fundamental domain for G is the region above the two hemispheres and bounded by the planes represented by the solid lines. This region has infinite volume. The fundamental domain for G^+ is the region above the hemispheres and bounded by the four planes represented by the three solid lines and the dotted line. This region has finite volume.

identity. Thus, σ_4^* is either the identity, rotation by π about the line with endpoints D_2 and $D_2 + D_4$, or reflection through a plane that includes that line.

Though there are infinitely many reflections that include a given line, there are significant limitations on what they can be. Let $R_\mathbf{n}$ be a reflection through the plane $\mathbf{n} \cdot \mathbf{x} = 0$. Then $\mathbf{n}$ is an eigenvector of $R_\mathbf{n}$ associated to the eigenvalue -1. Since $R_\mathbf{n}$ has integer entries, we may choose $\mathbf{n}$ to have integer entries. Now suppose $R_\mathbf{m}$ is another reflection in $\mathcal{O}^+$. Then the angle θ between the two planes (if they intersect) is given by

$$\cos\theta = \frac{\pm\mathbf{n}\cdot\mathbf{m}}{\|\mathbf{n}\|\|\mathbf{m}\|}.$$

Thus,

$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta = \frac{(\mathbf{n}\cdot\mathbf{m})^2}{(\mathbf{n}\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{m})} \in \mathbb{Q}.$$

The composition of $R_{\mathbf{n}}$ and $R_{\mathbf{m}}$ is a rotation by 2θ (if the planes intersect), and since $\mathcal{O}^+$ is an arithmetic group, we know 2θ is a rational multiple of π. Since $\deg(\cos 2\pi/n) = \frac{1}{2}\phi(n)$, the only possibilities for $\cos 2\theta$ are $0, \pm\frac{1}{2}, \pm 1$.

If σ_4^* is a reflection with normal vector $\mathbf{n}$, then we can solve for $\mathbf{n}$ by noting $\mathbf{n}\cdot D_2 = 0$, $\mathbf{n}\cdot D_4 = 0$, and using the above argument with the reflection S_1. We get $\mathbf{n} = D_1 + D_2 - D_3$ or $\mathbf{n} = D_1 - 4D_2 - 2D_4$, which gives us $S_1\sigma_3^*$ and

$$T_4 = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 8 & 1 & 8 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 4 & 1 \end{bmatrix}.$$

The rotation by π about the line with endpoints D_2 and $D_2 + D_4$ is the map $S_1\sigma_3^*T_4$.

If $\sigma_4^* = \mathrm{Id}$ or $S_1\sigma_3^*$, then it would appear that the discovery of σ_4 has given us nothing new. However, it did lead us to the discovery of T_4, which is an element of $\mathcal{O}^+$. Furthermore, the group $G^+ = \langle T_1, S_1, S_2, \sigma_3^*, R_{D_4}, T_4\rangle$ has a fundamental domain with finite volume (see Figure 2), so G^+ has finite index in $\mathcal{O}^+$. (In fact, $G^+ = \mathcal{O}^+$, since this fundamental domain has no symmetries. We leave the proof to the reader, as the result is not a necessary component of the paper. First we show that such an isometry cannot interchange the cusps D_1 and D_2 and then we analyze where the other vertices of the fundamental domain can go.)

If $\sigma_4^* = T_4$ or $S_1\sigma_3^*T_4$, then $T_4 \in \mathcal{O}''$. It turns out the converse is also true.

Lemma 3.1. *Suppose $T_4 \in \mathcal{O}''$. Then $\sigma_4^* = T_4$ or $S_1\sigma_3^*T_4$.*

Proof. Select any five divisors $C_i = (S_1T_4)^i D_4$ from the infinite orbit of D_4 under the action of $\langle S_1T_4\rangle$. It is easy to see that this orbit is infinite since S_1T_4 is a translation on the boundary of the upper half-space. Since S_1T_4 fixes D_2, we get

$$C_i \cdot D_2 = (S_1T_4)^i D_4 \cdot D_2 = D_4 \cdot (T_4S_1)^i D_2 = D_4 \cdot D_2 = 1.$$

Note that $S_1\sigma_3^*$ commutes with both S_1 and T_4 (they are reflections that are perpendicular to each other), so

$$S_1\sigma_3^* C_i = S_1\sigma_3^*(S_1T_4)^i D_4 = (S_1T_4)^i(S_1\sigma_3^*)D_4 = (S_1T_4)^i D_4 = C_i.$$

And, of course, the image of C_i under the identity is also C_i.

Finally, since $T_4 \in O''$, each of the divisors C_i are nodal, i.e., they each represent a rational curve. Let E be an elliptic curve in D_2 that does not include any of the finite number of intersections given by the five C_i's taken in pairs, and let $P_i = C_i \cap E$, where we are now using C_i to represent both a divisor class and (in this usage) the unique curve in that divisor class. Then, of course, $\sigma_4(P_i) = -P_i$. On the other hand, $\sigma_4(P_i) = \sigma_4(C_i \cap E)$, so is in $\sigma_4^* C_i \cap E$. But $\sigma_4^* C_i = C_i$, and since $C_i \cdot D_2 = 1$, there is only one point in this image, namely, P_i, i.e., $-P_i = \sigma_4(P_i) = P_i$ so $2P_i = 0$. But E has at most four points P such that $2P = 0$, giving us a contradiction. Thus, σ_4^* cannot be the identity or $S_1\sigma_3^*$, so it must be either T_4 or $S_1\sigma_3^* T_4$. $\square$

We, therefore, have an incentive to prove T_4 is in $\mathcal{O}''$.

The groups $\mathcal{O}'$ and $\mathcal{O}''$ intersect at just the identity. Thus, an element $T \in \mathcal{O}^+$ is in $\mathcal{O}''$ if and only if TD is ample for any (and all) ample D. This gives us an incentive to find more ample divisors.

Lemma 3.2. *Suppose $D \in \mathrm{Pic}(\mathcal{X})$ and C_0 is a nodal curve. Let Q be the perpendicular projection (with respect to the intersection pairing) of D onto the hyperplane $C_0 \cdot \mathbf{x} = 0$. If D is ample, then every point on the open line segment from D to Q is ample. If D is on the boundary of the ample cone, $C_0 \cdot D \neq 0$, and the line DQ is not in a hyperplane $C \cdot \mathbf{x} = 0$ for any nodal curve C, then every point on the open line segment from D to Q is ample.*

Proof. The ample cone is a polyhedral region bounded by the hyperplanes $C \cdot \mathbf{x} = 0$ where C ranges over all nodal curves. If two bounding hyperplanes $C_1 \cdot \mathbf{x} = 0$ and $C_2 \cdot \mathbf{x} = 0$ intersect, then the angle θ of intersection, as measured inside the ample cone, is given by $2\cos\theta = C_1 \cdot C_2$. Thus, $\theta = \pi/2$ or $\pi/3$ (corresponding to $C_1 \cdot C_2 = 0$ or 1). If not all points between D and Q are ample, then the segment must cross the boundary of the ample cone, i.e., there must be a point P on the line segment such that P is on the hyperplane $C \cdot \mathbf{x} = 0$ for some nodal curve C. To arrive at a contradiction, we will construct a triangle whose angle sum is greater than π.

Let U be the subspace spanned by D, C_0, and C, so the intersection of U with $\mathbb{H}$ is a hyperbolic plane. Since Q is in the space spanned by D and C, Q is in U. Note that the hyperplanes $C \cdot \mathbf{x} = 0$ and $C_0 \cdot \mathbf{x} = 0$ must intersect, for if they do not, then $C_0 \cdot \mathbf{x} = 0$ cannot bound the ample cone anywhere since the ample cone is on the other side of the hyperplane $C \cdot \mathbf{x} = 0$. This contradicts C_0 being nodal. Let R be the point of intersection of $C_0 \cdot \mathbf{x} = 0$, $C \cdot \mathbf{x} = 0$, U, and $\mathbb{H}$. Consider ΔPQR. Note that $\angle PQR = \pi/2$, and $\angle QRP$ is the supplement of the angle θ described above, since P is outside the ample cone. Hence, $\angle QRP \geq \pi/2$, and the angle sum in ΔPQR is greater than π, a contradiction, so P could not exist.

If D is on the boundary of the ample cone and $C_0 \cdot D \neq 0$, then $C_0 \cdot D > 0$. If $D + \epsilon Q \notin \mathcal{K}$ for sufficiently small $\epsilon > 0$, then there exists a nodal C such that $C \cdot (D + \epsilon Q) \leq 0$ for all $\epsilon > 0$ sufficiently small. If the line segment DQ is not on $C \cdot \mathbf{x} = 0$, then the inequality is strict. In particular, since $D \in \partial\mathcal{K}$, we have $C \cdot D = 0$. If the hyperplanes $C \cdot \mathbf{x} = 0$ and $C_0 \cdot \mathbf{x} = 0$ do not intersect, then $C \cdot \mathbf{x} < 0$ for all $\mathbf{x}$ such that $C_0 \cdot \mathbf{x} = 0$, i.e., the hyperplane $C_0 \cdot \mathbf{x} = 0$ is not a bounding plane of $\mathcal{K}$, which cannot be the case. Thus, they must intersect and we can construct R as before. If $R \neq Q$, then we arrive at a contradiction as before using the triangle ΔDQR. If $R = Q$, then the line segment DQ lies on $C' \cdot \mathbf{x} = 0$.

Thus, $D + \epsilon Q \in \mathcal{K}$ for sufficiently small $\epsilon > 0$, and hence the open line segment joining $D + \epsilon Q$ and Q lies in the ample cone. Since $\epsilon > 0$ is arbitrary, the open line segment DQ lies in the ample cone. $\square$

Since D_2 represents elliptic curves, it is on the boundary of the ample cone. Since $R_{D_4} D_2 = D_2 + D_4$, the projection of D_2 onto the plane $D_4 \cdot \mathbf{x} = 0$ has the form $Q = D_2 + aD_4$. Solving for a in $D_4 \cdot (D_2 + aD_4) = 0$, we get $a = 1/2$. Thus, $D_2 + cD_4$ is ample

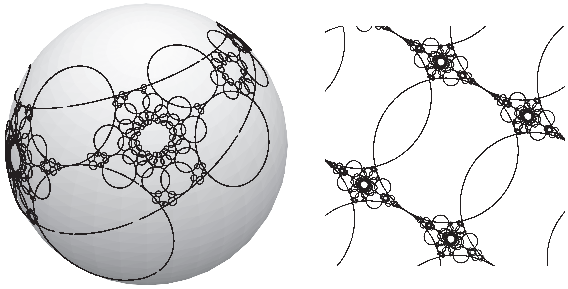

FIGURE 3. The Poincaré ball and upper half-space models of $\mathcal{K}/\mathbb{R}^+$. Each circle represents a hyperbolic plane that bounds the region. The region is also a fundamental domain for $\mathcal{O}'$.

for $c \in (0, 1/2)$. In particular, $D_2 + D_4/4 \in \mathcal{K}$, and since it is in the eigenspace of T_4 with eigenvalue 1, we get $T_4 \in O''$. Hence, $\sigma_4^* = T_4$ or $S_1\sigma_3^*T_4$.

We therefore conclude $\langle \sigma_1^*, \sigma_2^*, \sigma_3^*, \sigma_4^* \rangle$ has finite index (of 1 or 2) in $\mathcal{O}'' = \langle T_1, \sigma_2^*, \sigma_3^*, \sigma_4^*, S_2 \rangle = \langle T_1, \sigma_2^*, \sigma_4^*, S_1, S_2 \rangle$, and that $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$ has finite index in $\mathrm{Aut}(\mathcal{X})$. We also get

$$\mathcal{O}' = \langle R_{TD_4} : T \in \mathcal{O}'' \rangle.$$

4. The ample cone. The ample cone is the conal region bounded by the planes $C \cdot \mathbf{x} = 0$ where C ranges over all nodal curves, i.e., over the O'' orbit of D_4. Modulo $\mathbb{R}^+$, the ample cone $\mathcal{K}$ lies in $\mathbb{H}$, and is depicted in Figure 3 in the Poincaré ball and upper half-space models. The region $\mathcal{K}/\mathbb{R}^+$ is also a fundamental domain for $\mathcal{O}'$.

Associated to the ample cone is a fractal. Consider the sphere that represents the boundary of $\mathbb{H}$ at infinity in the Poincaré ball model. Every plane that bounds the ample cone $\mathcal{K}$ slices this sphere in two pieces. Remove the piece that represents the half space that does not include $\mathcal{K}$. After doing this for all bounding planes of $\mathcal{K}$, what is left is a fractal. The fractal is also known as the limit set of $\mathcal{O}''$ and can be thought of as the set of all points $x \in \partial\mathbb{H}$ such that, for any plane in $\mathbb{H}$ that does not have x on its boundary, and any $P \in \mathbb{H}$, there exists $T \in O''$ such that $T(P)$ and x are on the same side of the plane. Experimental calculations suggest the dimension of that fractal is 1.415 ± 0.003. For a more detailed description of how the calculation is done, see [**2**].

5. Descent. One often uses a method of descent to navigate through a lattice. Setting up an appropriate height and algorithm for descent is sometimes difficult to accomplish and verify if viewed strictly algebraically. Geometrically, for a group $G = \langle R_{\mathbf{n}_1}, \dots, R_{\mathbf{n}_k} \rangle$ consisting of reflections, the set up is trivially accomplished. Pick any ample divisor D in the interior of the fundamental domain, and for each reflection $R_{\mathbf{n}_i}$, verify that $D \cdot \mathbf{n}_i > 0$, replacing $\mathbf{n}_i$ with its negative, if necessary. Define $h(P) = D \cdot P$. For any P not in the closure of the fundamental domain, there exists an $\mathbf{n}_i$ such that $\mathbf{n}_i \cdot P < 0$. We descend by replacing P with $R_{\mathbf{n}_i}(P)$, since clearly $h(R_{\mathbf{n}_i}(P)) < h(P)$. Descent ends when no such $\mathbf{n}_i$ exists, which means P is in the closure of the fundamental domain.

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