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ON SINGLETONS AND ADJACENCIES OF SET PARTITIONS

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ABSTRACT. The number of singleton blocks in all partitions of a set $\{a_1, \ldots, a_n\}$ is known to be equal to the number adjacencies, that is, pairs of consecutively numbered elements (a_i, a_{i+1}) in a block. We give a generalization of this relation by introducing the d-adjacency which is a pair of elements (a_i, a_j) satisfying $j - i = d > 0$. It is proved that the number of d-adjacencies in all partitions is independent of d. Then we show that the number of d-adjacencies in noncrossing partitions is a function of d by means of an exact formula.

1. Adjacencies and singletons. A partition of a set of n distinguishable objects, $A_n = \{a_1, a_2, \ldots, a_n\}$, is a decomposition of A_n into nonempty subsets called blocks. The blocks are usually arranged in standard order, that is, in increasing order of least label-numbers.

The number of partitions of A_n into k blocks is the Stirling number of the second kind, $S(n, k)$, while the Bell numbers B_n are defined by $B_n = \sum_k S(n, k)$. These numbers may be computed using the formula (see for example [**3**]),

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

For any positive integer d, a (*circular*) *d-adjacency* is the occurrence of an ordered pair of elements (a_i, a_j) in a block such that $j - i \equiv d \pmod{n}$. We define a 0-adjacency to be a singleton, that is, a block containing one element.

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Callan [**2**] has proved that the number of singletons in all partitions of A_n equals the number of 1-adjacencies, by giving a bijection in terms of an algorithm that interchanges singletons and 1-adjacencies.

We remark that the number of singletons in all partitions of A_n is nB_{n-1}. This may be proved by fixing an index $j \in \{1, \dots, n\}$, and noting that the number of partitions containing the singleton $\{a_j\}$ is B_{n-1}, which is the number of ways of inserting the block $\{a_j\}$ into a partition of $A_n \setminus \{a_j\}$.

The purpose of this note is to prove the following general result and establish a formula for the number of d-adjacencies in noncrossing partitions (for the latter, see Section 2).

Theorem 1.1. *Let n, d be integers, $0 \leq d < n$. Then the number of d-adjacencies in all partitions of A_n is independent of d and equal to nB_{n-1}.*

The proof of Theorem 1.1 is a consequence of either of the following lemmas.

We will identify A_n with the label set $[n] = \{1, 2, \dots, n\}$. Clearly, (a_i, a_j) is a d-adjacency in a partition of A_n if and only if (i, j) is a d-adjacency in a partition of $[n]$.

Lemma 1.2. *The number of d-adjacencies in all partitions of $[n]$ is nB_{n-1} for all integers $0 \leq d \leq n-1$.*

Proof. Since the number of singletons, or 0-adjacencies, is known to be nB_{n-1}, we consider the case $d > 0$. There are precisely n distinct d-adjacencies for each $d \in [n-1]$, namely,

$$(a, a+d), \quad 1 \leq a \leq n-d \quad \text{with} \quad (n-d+c, c),\ 1 \leq c \leq d.$$

Fix a d-adjacency $(a, a+d) \pmod{n}$, $a \in [n]$. Note that the range of a implies that $a + d \not\equiv 0 \pmod{n}$. Then the number of partitions in which a and $a + d \pmod{n}$ belong to the same block is given by B_{n-1}, which is obtained as the number of ways of partitioning the set $[n] \setminus \{a + d \pmod{n}\}$, followed by putting $a + d \pmod{n}$ into the block containing a. Hence, the result. □

The proof of the second lemma contains a solution to the problem, raised in **[1, 4]**, of finding a bijection between 1-adjacencies and singletons.

Lemma 1.3. *The multi-set of d-adjacencies in all partitions of $[n]$ is in one-to-one correspondence with the set of singletons in all partitions of $[n]$, for all integers n, d, $1 \leq d < n$.*

Proof. The type of bijection described below was popularized by Richard Stanley (see **[5]**). Here, "adjacency" means "d-adjacency."

We associate a partition of $[n]$ containing m adjacencies with m different partitions of $[n]$ containing singletons so that the number of times a given partition π of $[n]$ appears is the same as the number of adjacencies in π.

Let π be a partition of $[n]$ containing $m > 0$ adjacencies. Write down π a total of m times, each corresponding to an adjacency. Then, for a fixed adjacency $x, x + d \pmod{n}$ the image of π is obtained by creating a new singleton block containing $x + d \pmod{n}$, and then rearranging the blocks in standard order. For example, consider the partition $\pi = 129/368/45/7$. When $d = 1$, π maps to 19/2/368/45/7, 129/368/4/5/7 and 1/29/368/45/7, corresponding to the adjacencies $(1, 2)$, $(4, 5)$ and $(9, 1)$, respectively; when $d = 2$, π maps to 129/36/45/7/8 and 19/2/368/45/7, corresponding to the adjacencies $(6, 8)$ and $(9, 2)$, and so forth.

Conversely, delete each singleton $\{x\}$, and put x into the block containing $x - d$ if $x > d$, or into the block containing $x + n - d$ if $x \leq d$. This gives the inverse image of a partition with respect to the singleton. For example, since it contains a singleton, the inverse image of $\pi = 129/368/45/7$ is 129/3678/45 when $d = 1$, and 129/368/457 when $d = 2$.

This gives the desired bijection.

The full correspondence is illustrated for $n = 4$ when $d = 2$ in Table 1. As a verification of the inverse mapping observe that the number of occurrences of a partition in the third column is equal to the number of singletons the partition contains. □

TABLE 1. Bijection between 2-adjacencies and singletons for $n = 4$.

partition	2-adjacency	image
1234	13	124/3
1234	24	123/4
1234	31	1/234
1234	42	134/2
123/4	13	12/3/4
123/4	31	1/23/4
124/3	24	12/3/4
124/3	42	14/2/3
134/2	13	14/2/3
134/2	31	1/2/34
13/24	13	1/24/3
13/24	24	13/2/4
13/24	31	1/24/3
13/24	42	13/2/4
1/234	24	1/23/4
1/234	42	1/2/34
1/24/3	24	1/2/3/4
1/24/3	42	1/2/3/4
13/2/4	13	1/2/3/4
13/2/4	31	1/2/3/4

2. Noncrossing partitions. A noncrossing partition of $[n]$ forbids the occurrence of four elements $w < x < y < z$ such that w, y belong to one block and x, z belong to another. Equivalently, a noncrossing partition is a partition of the vertices of a regular n-gon (labeled by $[n]$ and arranged clockwise on a circle) such that the convex hulls of its blocks are pairwise disjoint.

Denote the set of noncrossing partitions of $[n]$ by $NC(n)$. It is well known that

$$|NC(n)| = C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{2.1}$$

where C_n is the nth Catalan number.

We consider $NC(n)$ in the light of the correspondence established in Lemma 1.3.

It is not hard to see that 1-adjacencies and singletons are equidistributed in $NC(n)$, as already observed in **[2]**. This follows from the simple fact that connecting or disconnecting a pair of consecutive points on a circle cannot create a crossing.

However, the situation is different for d-adjacencies when $d > 1$: the (re-) connection of the member of a singleton $\{a\}$ to (the block containing) $a - d$, may create a crossing. For example, consider the inverse image of the noncrossing partition $\pi = 129/368/45/7$, when $d = 2$. The resulting partition is $129/368/457$ which is not noncrossing because of the integers $5, 6, 7, 8$ with $5, 7$ in the third block and $6, 8$ in the second.

Denote by $y_n(d)$ the number of d-adjacencies in all noncrossing partitions of $[n]$. The following rotational symmetry relation obviously holds

$$y_n(d) = y_n(n - d). \tag{2.2}$$

It is also easy to show, as with unrestricted partitions, that

$$y_n(0) = y_n(1) = nC_{n-1} = \binom{2n-2}{n-1}.$$

The full formula is stated below.

Theorem 2.1.

$$y_n(d) = nC_dC_{n-d}, \quad 1 \le d \le n - 1.$$

Proof. We first count how many noncrossing partitions π contain a certain d-adjacency $(a, a + d)$, for any $a \in [n]$, $d \in [n - 1]$.

The restriction of π to $\{a + 1, \dots, a + d\}$ (mod n) is a noncrossing partition for which there are C_d possibilities. Similarly, the restriction to $\{a + d + 1, \dots, n, 1, \dots, a\}$ (mod n) is a noncrossing partition with C_{n-d} possibilities.

This procedure can be reversed uniquely. Given a noncrossing partition of the set $\{a + 1, a + 2, \dots, a + d\}$ and another noncrossing partition of the set $\{a + d + 1, \dots, a\}$, combine them by merging the blocks containing a and $a + d$. By construction this merging process cannot create a crossing.

Hence, there are precisely C_dC_{n-d} noncrossing partitions containing the d-adjacency $(a, a+d)$. Since there are n possible choices for a, this gives a total of nC_dC_{n-d} d-adjacencies. □

It follows from the Catalan-number recurrence

$$C_0 = 1, \quad C_{n+1} = \sum_{j=0}^{n} C_jC_{n-j}, \tag{2.3}$$

that

$$\sum_{d=1}^{n-1} y_n(d) = nC_{n+1} - 2nC_n = 2\binom{2n}{n-2}.$$

We remark that Theorem 2.1 gives a seemingly new interpretation of the jth summand in (2.3) as the number of j-adjacencies $(a, a+j)$ in all noncrossing partitions of $[n]$, for each $a \in [n]$.

Lastly, we recall Stirling's asymptotic approximation of the factorial function:

$$n! \sim \sqrt{2\pi n}e^{-n}n^n, \tag{2.4}$$

where the standard notation $\sim$ is defined by $u \sim v$ if and only if $\lim_{n\to\infty} u/v = 1$. Using (2.4) and the Catalan-number formula (2.1) one can show that

$$C_n \sim \frac{4^n}{\sqrt{\pi n^3}}. \tag{2.5}$$

Consequently,

$$y_n(d) \sim \frac{4^n}{\pi\sqrt{d^3 n}}. \tag{2.6}$$

We can now state:

Theorem 2.2. *Given positive integers n and d, $0 < d < n$, the average number of d-adjacencies in a random noncrossing partition of $[n]$ is given by*

$$\frac{nC_dC_{n-d}}{C_n} = \frac{n(n+1)}{(d+1)(n-d+1)}\binom{2d}{d}\binom{2(n-d)}{n-d}\binom{2n}{n}^{-1} \sim \frac{n}{\sqrt{\pi d^3}}.$$

Proof. The first equality follows from (2.1). The asymptotic part may be obtained by using the exact formula in the theorem together with (2.4):

$$\frac{nC_dC_{n-d}}{C_n} \sim \frac{n4^d}{\sqrt{\pi d^3}} \frac{4^{n-d}}{\sqrt{\pi(n-d)^3}} \frac{\sqrt{\pi n^3}}{4^n} = \frac{n\sqrt{\pi n^3}}{\sqrt{\pi^2 d^3 (n-d)^3}},$$

which, for large n, is the same as

$$\frac{n\sqrt{\pi n^3}}{\sqrt{\pi^2 d^3 n^3}}. \qquad \square$$

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