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SLIT UNIVALENT HARMONIC MAPPINGS

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ABSTRACT. In this paper we consider the class of complex-valued harmonic univalent functions that map the unit disc onto the complex plane, half-plane or a strip slit along finitely many horizontal half-lines.

1. Introduction. Denote by $\mathcal{H}(\Delta)$ the linear topological space of all complex harmonic mappings of the disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$ endowed with the topology of locally uniform convergence, and let $\mathcal{A}(\Delta)$ be the linear subspace of $\mathcal{H}(\Delta)$ of all analytic functions on Δ. If $f \in \mathcal{H}(\Delta)$ and functions $F, G \in \mathcal{A}(\Delta)$ are chosen so that $\operatorname{Re} F = \operatorname{Re} f$, $\operatorname{Re} G = \operatorname{Im} f$, then $f = h + \overline{g}$, where $h = (F + iG)/2$ and $g = (F - iG)/2$. Thus,

$$\mathcal{H}(\Delta) = \{h + \overline{g} : h,\, g \in \mathcal{A}(\Delta),\, g(0) = 0\}.$$

By a result of Lewy (see **[11]**), the function $f \in \mathcal{H}(\Delta)$ is locally univalent if and only if its Jacobian

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0 \quad \text{on } \Delta.$$

We will consider only orientation preserving harmonic mappings, i.e., those $f \in \mathcal{H}(\Delta)$ with $J_f(z) > 0$ on Δ.

For a given simply connected domain $D \subsetneq \mathbb{C}$ with $0 \in D$, denote by $\mathcal{S}_H(\Delta, D)$ the class of all functions $f \in \mathcal{H}(\Delta)$ which map Δ onto D univalently, $f(0) = 0 < f_z(0)$ and $f_z(0) > |f_{\overline{z}}(0)|$, and let

$$\mathcal{S}_H^0(\Delta, D) = \{f \in \mathcal{S}_H(\Delta, D) : f_{\overline{z}}(0) = 0\}.$$

The class $\mathcal{S}_H(\Delta, D)$ is very wide and, clearly, contains only one analytic function called the *conformal associate* of the class. It appears that the closure $\overline{\mathcal{S}_H^0(\Delta, D)}$ of the class $\mathcal{S}_H^0(\Delta, D)$ consists of only univalent

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functions, see [**9,** Theorem 3]. However, in some cases when D is a disc, a strip, a wedge, a half-plane and $\mathbb{C} \setminus (-\infty, a]$ with $a < 0$ or $\mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty))$ with $-\infty < a < 0 < b < +\infty$, collapsing takes place, i.e., $\overline{\mathcal{S}_H^0(\Delta, D)} \neq \mathcal{S}_H^0(\Delta, D)$ and $f(\Delta) \subsetneqq D$ for $f \in \overline{\mathcal{S}_H^0(\Delta, D)} \setminus \mathcal{S}_H^0(\Delta, D)$ although the radial limits $\widehat{f}(e^{i\theta})$ of f belong to ∂D for almost all real θ, see [**2, 5, 8, 10, 12, 13**]. Moreover, the set $\overline{\mathcal{S}_H(\Delta, D)} \setminus \mathcal{S}_H(\Delta, D)$ may contain some nonunivalent members, and this always happen when D is a bounded domain, see [**9,** Theorem 3].

This article contains results relating the classes $\mathcal{S}_H(\Delta, D)$ and $\mathcal{S}_H^0(\Delta, D)$, where the boundary ∂D of the domain D is the union of finitely many half-lines (slits). The case when D is the complex plane $\mathbb{C}$ slit along one half-line has been studied in [**12**], two opposite half-lines in [**8, 13**] (see also [**3**]), and the case of four horizontal slits symmetric with respect to the real axis recently has been investigated by Dorff, Nowak and Wołoszkiewicz [**3**]. In this paper, we consider a much more general case–the domain D will be the plane $\mathbb{C}$, a half-plane, or a strip, each slit along finitely many horizontal half-lines.

Let
$$c > c_1 > \cdots > c_k, \qquad d < d_1 < \ldots < d_s,$$

and for any $a_1, \ldots, a_k \in \mathbb{R}$, $b_1, \ldots, b_s \in \mathbb{R}$, consider the following domains with a finite number of horizontal slits directed to the left, to the right or to both the directions:

$$\Omega^l((a_j), (c_j)) = \Omega \Big\backslash \bigcup_{j=1}^{k} \{x + ic_j : x \leq a_j\}, \tag{1.1}$$

$$\Omega^r((b_m), (d_m)) = \Omega \Big\backslash \bigcup_{m=1}^{s} \{x + id_m : x \geq b_m\}, \tag{1.2}$$

and

$$\Omega^l((a_j), (c_j)) \cap \Omega^r((b_m), (d_m)), \tag{1.3}$$

whenever $a_u < b_v$ in any case when $c_u = d_v$, where Ω is one of the sets: the complex plane $\mathbb{C}$, the lower half-plane

$$L = L(c) = \{w \in \mathbb{C} : \operatorname{Im} w < c\} \quad \text{if } d_s < c, \tag{1.4}$$

the upper half-plane

$$U = U(d) = \{w \in \mathbb{C} : \operatorname{Im} w > d\} \quad \text{if } c_k > d, \tag{1.5}$$

the strip

$$S = S(c,d) = \{w \in \mathbb{C} : d < \operatorname{Im} w < c\} \quad \text{if } d_s < c,\ c_k > d. \tag{1.6}$$

We will assume that all the sets (1.1)–(1.3) with Ω described above contain the origin. For a short formulation, by (1.3) with $k = 0$ or $s = 0$, we will mean (1.2) or (1.1), respectively.

In this way, we may consider 12 families of horizontally slit domains that have their slits on the same levels and just the same direction. Since f is a harmonic normalized homeomorphism together with the functions

$$z \longmapsto \overline{f(\overline{z})} \quad \text{and} \quad z \longmapsto -f(-z),$$

we may consider only six kinds of the classes $\mathcal{S}_H(\Delta, D)$, namely, when D is a member of the set families

$$\mathsf{C}^l(c_1,\dots,c_k),\ \mathsf{L}^l(c_1,\dots,c_k),\ \mathsf{S}^l(c_1,\dots,c_k), \tag{1.7}$$

consisting of all sets of the type (1.1) with $\Omega = \mathbb{C}$, $\Omega = L = L(c)$ and $\Omega = S = S(c,d)$, respectively, and arbitrary $a_1,\dots,a_k \in \mathbb{R}$, or

$$\mathsf{C}((c_j),(d_m)),\ \mathsf{L}((c_j),(d_m)), \quad \text{and} \quad \mathsf{S}((c_j),(d_m)), \tag{1.8}$$

consisting of all sets of the type (1.3) with $\Omega = \mathbb{C}$, $\Omega = L = L(c)$ and $\Omega = S = S(c,d)$, respectively, and arbitrary $a_1,\dots,a_k \in \mathbb{R}$, $b_1,\dots,b_s \in \mathbb{R}$, where, like before, $a_u < b_v$ whenever $c_u = d_v$.

Let us now consider the function

$$\Delta \ni z \longmapsto k(z,F,q) = \operatorname{Re} \int_0^z F'(t)q(t)\,dt + i \operatorname{Im} F(z) \tag{1.9}$$

for F, $q \in \mathcal{A}(\Delta)$, and let

$$\mathcal{P} = \{p \in \mathcal{A}(\Delta) : \operatorname{Re} p(z) > 0 \quad \text{on} \quad \Delta,\ p(0) = 1\}, \tag{1.10}$$

the known Carathéodory class. Clearly, if $f = h + \overline{g} \in \mathcal{H}(\Delta)$ is locally univalent, $\operatorname{Re} h(0) = 0 = g'(0) < |h'(0)|$, then we have a simple representation $f = k(\cdot, F, p)$ with a locally univalent $F = h - g$ and $p = (h' + g')/(h' - g') \in \mathcal{P}$. Conversely, for every $p \in \mathcal{P}$ and a locally univalent $F \in \mathcal{A}(\Delta)$ with $\operatorname{Im} F(0) = 0$, the functions $f = k(\cdot, F, p)$ and more general

$$f_{\lambda,\,\nu} = k(\cdot, F, \lambda p + i\,\nu)$$

with $\lambda > 0$ and $\nu \in \mathbb{R}$ are harmonic locally univalent on Δ:

$$|(f_{\lambda,\,\nu})_z|^2 - |(f_{\lambda,\,\nu})_{\overline{z}}|^2 = \lambda\,|F'(z)|^2 \operatorname{Re} p(z) > 0,$$

$$f_{\lambda,\,\nu}(0) = 0 \neq (1 + \lambda + i\,\nu)F'(0)/2 = (f_{\lambda,\,\nu})_z(0)$$

and

$$(f_{\lambda,\,\nu})_{\overline{z}}(0) = (\lambda + i\,\nu - 1)F'(0)/2.$$

Let $F \in \mathcal{A}(\Delta)$, $p \in \mathcal{P}$ and $\lambda > 0$, $\nu \in \mathbb{R}$. The following important fact, known as the "shear construction" method of producing univalent harmonic mappings, comes from **[2]**: *$k(\cdot, F, \lambda p + i\,\nu)$ is harmonic univalent convex in the direction of the real axis if and only if F is analytic univalent convex in the direction of the real axis.* If, for instance, the ranges of the above considered univalent functions are in any of the set families (1.7)–(1.8), then they are convex in the direction of the real axis.

In this paper, we prove two theorems.

Theorem 1.1. *Let A be one of the set families* (1.7)–(1.8)*, $p \in \mathcal{P}$ and $F \in \mathcal{A}(\Delta)$ univalent with $F(0) = 0 < F'(0)$. Denote by $p_r(z) = p(rz)$ for $z \in \Delta$ and $f_r = k(\cdot, F, p_r)$, where $r \in [0, 1]$. If $F(\Delta) \in \mathsf{A}$, then the range $f_r(\Delta) \in \mathsf{A}$ for all $r \in [0, 1)$.*

Under the assumptions and the notations of Theorem 1.1 in general $f_1(\Delta) \notin \mathsf{A}$ (see for example **[3, 8, 10, 12, 13]**), it may happen that $f_r(\Delta) \in \mathsf{A}$ for all $r \in [0, 1]$. In the opposite direction of Theorem 1.1, we prove

Theorem 1.2. *Suppose A is any of the set families* (1.7)–(1.8) *with*

$$\{c_1, \ldots, c_k\} \cap \{d_1 \ldots, d_s\} = \varnothing,$$

$D \in \mathsf{A}$ and $f \in \mathcal{S}_H^0(\Delta, D)$. Then there is a $p \in \mathcal{P}$ and a univalent $F \in \mathcal{A}(\Delta)$ such that $F(0) = 0 < F'(0)$, $F(\Delta) \in \mathsf{A}$ and

$$f(z) = k(z, F, p) \quad \textit{for } z \in \Delta. \tag{1.11}$$

The case $\{c_1, \dots, c_k\} \cap \{d_1 \dots, d_s\} \neq \varnothing$ of Theorem 1.2 remains open. One can prove that, for any family A of the form (1.7)–(1.8) and a univalent $F \in \mathcal{H}(\Delta)$ with $F(0) = 0 < F'(0)$ and $F(\Delta) \in \mathsf{A}$, the ranges $k(\Delta, F, \lambda p)$ belong to the same family A whenever $\lambda > 0$ and the function $p \in \mathcal{P}$ is chosen that, for every $\eta \in \partial\Delta$ with $F(\eta) = \infty$, the function p is analytic at η and $\operatorname{Re} p(\eta) > 0$, see the proof of Theorem 1.1.

2. Auxiliary lemmas and remarks. In the next lemmas and remarks we consider a slit domain D of the form (1.1) or (1.3) with Ω being $\mathbb{C}$, (1.4) or (1.6), under the assumption that $0 \in D$ and $a_u < b_v$ whenever $c_u = d_v$. The unique $F \in \mathcal{A}(\Delta)$ which maps univalently the disc Δ onto D with $F(0) = 0 < F'(0)$ is given by the Schwarz-Christoffel transformation (for the Schwarz-Christoffel mappings see, e.g., [**1, 4**]). We start with the following technical lemma.

Lemma 2.1. *Let $P, Q, R, S \in \mathbb{C}$. If there exists a finite limit*

$$\lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} \left[P \ln \frac{\sin \varepsilon/2}{\sin \delta/2} + Q\left(\cot \frac{\varepsilon}{2} + \cot \frac{\delta}{2} \right) + R\left(\csc^2 \frac{\varepsilon}{2} - \csc^2 \frac{\delta}{2} \right) \right] = S,$$

then $P = Q = R = S = 0$.

Proof. Taking $\varepsilon = \delta \to 0^+$, we get $Q = 0$ (because the limit is finite) and $S = 0$. Next if $\delta = 2\varepsilon \to 0^+$, then $R = 0$, and hence, finally $P = 0$. $\square$

In this paper, log means the principal branch of the logarithm.

Lemma 2.2. *Let $F \in \mathcal{A}(\Delta)$ be univalent, $F(0) = 0 < F'(0)$ and $F(\Delta) = \mathbb{C}^l((a_j), (c_j)) \cap \mathbb{C}^r((b_m), (d_m))$ (see (1.3)). Then there are the unique $2k + 2s$ points $\eta_j = e^{i\theta_j} \in \partial\Delta$, $j = 1, \dots, 2k + 2s$, where $\theta_1 < \theta_2 < \dots < \theta_{2k+2s-1} < \theta_{2k+2s} < \theta_1 + 2\pi$, such that*

$$F(z) \equiv \lambda \int_0^z \prod_{j=1}^{k+s} \frac{1 - \overline{\eta}_{2j-1} t}{1 - \overline{\eta}_{2j} t} \cdot \frac{dt}{(1 - \overline{\eta}_{2k} t)(1 - \overline{\eta}_{2k+2s} t)} \tag{2.1}$$

for some $\lambda > 0$,

(2.2)

$$F(z) \equiv \sum_{j=1}^{k+s} \frac{c_j - c_{j+1}}{\pi} \log(1 - \overline{\eta}_{2j} z) + \frac{i\alpha\overline{\eta}_{2k} z}{1 - \overline{\eta}_{2k} z} + \begin{cases} \frac{i\beta\overline{\eta}_{2k+2s} z}{1-\overline{\eta}_{2k+2s} z} & \textit{if } s \geq 1, \\ \frac{\beta\overline{\eta}_{2k} z}{(1-\overline{\eta}_{2k} z)^2} & \textit{if } s = 0, \end{cases}$$

where $c_{k+s+1} = c_1$ *and either* $s \geq 1$, $c_{k+1} = d_1, \ldots, c_{k+s} = d_s$, $\alpha < 0 < \beta$, *or* $s = 0$, $\alpha \in \mathbb{R}$, $\beta > 0$. *Moreover, the points* $F(\eta_{2j-1})$ *are the slit ends*: $a_j + ic_j$ *or* $b_{j-k} + id_{j-k}$ *depending on* $1 \leq j \leq k$ *or* $k + 1 \leq j \leq k + s$.

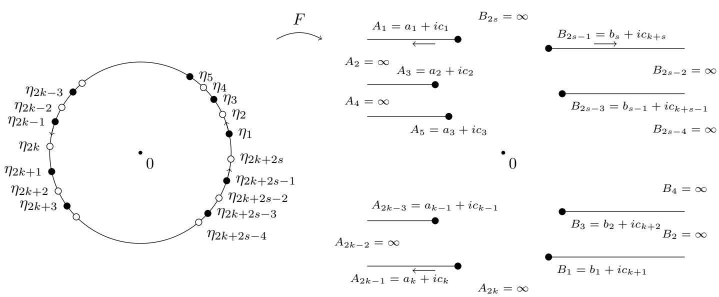

FIGURE 1. The function F.

Proof. Let $A_{2j} = B_{2m} = \infty$, $A_{2j-1} = a_j + ic_j$, $B_{2m-1} = b_m + id_m$, $1 \leq j \leq k$, $1 \leq m \leq s$. The univalent $F \in \mathcal{A}(\Delta)$ with $F(0) = 0 < F'(0)$ that maps Δ onto the unbounded polygon $F(\Delta) = A_1 A_2 \cdots A_{2k} B_1 B_2 \cdots B_{2s}$ has the form (2.1) by the Schwarz-Christoffel formula (see Figure 1). Partial fraction decomposition and integration of formula (2.1) gives

$$F(z) = \sum_{j=1}^{k+s} (-\lambda_j \, \eta_{2j}) \log(1 - \overline{\eta}_{2j} z) + \frac{\mu_1 z}{1 - \overline{\eta}_{2k} z} + \begin{cases} \frac{\mu_2 z}{1-\overline{\eta}_{2k+2s} z} & \text{for } s \geq 1, \\ \frac{\mu_2 z}{(1-\overline{\eta}_{2k} z)^2} & \text{for } s = 0, \end{cases} \tag{2.3}$$

where λ_j, $j = 1, \ldots, k + s$, and μ_1, μ_2 are some complex numbers which we determine will prove (2.2). By the Schwarz reflection principle, the function F can be analytically reflected about any open arc

$(\eta_j, \eta_{j+1}) = \{e^{i\theta}\colon \theta_j < \theta < \theta_{j+1}\}$, $j = 1, \dots, 2k+2s$. Actually, the Schwarz reflection principle was invented for the purpose of the Schwarz-Christoffel formula (see [**4**]). We denote by $\eta_{2k+2s+1} = \eta_1$ and $\theta_{2k+2s+1} = \theta_1$ for short notation. Because the slits are parallel we note two facts. First, $\operatorname{Im} F$ is constant on each arc (η_j, η_{j+1}), $j = 1, \dots, 2k+2s$. Second, for each $j = 1, \dots, 2k+2s$, the tangent vector of the curve $(\theta_j, \theta_{j+1}) \ni t \to F(e^{it})$, i.e., the vector $ie^{it}F'(e^{it})$, is parallel to the real axis at each point $t \in (\theta_j, \theta_{j+1})$. We will need the following obvious formula:

$$\log(1 - e^{i\theta}) = \ln\left(2\sin\frac{|\theta|}{2}\right) + \frac{i(|\theta| - \pi)\operatorname{sgn}\theta}{2} \quad \text{for } 0 < |\theta| < \pi. \tag{2.4}$$

Let us first compute λ_j for $j = 1, \dots, k+s-1$, $j \neq k$. Fix $j = 1, 2, \dots, k+s-1$, $j \neq k$. For each sufficiently small $\varepsilon > 0$ and $\delta > 0$ we have

$$c_j - c_{j+1} = \operatorname{Im} F(\eta_{2j}e^{-i\varepsilon}) - \operatorname{Im} F(\eta_{2j}e^{i\delta}).$$

Hence there exists the finite limit

$$\lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} (\operatorname{Im} F(\eta_{2j}e^{-i\varepsilon}) - \operatorname{Im} F(\eta_{2j}e^{i\delta})) = c_j - c_{j+1}.$$

On the other hand, by (2.3) and (2.4), we have

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} (\operatorname{Im} F(\eta_{2j}e^{-i\varepsilon}) - \operatorname{Im} F(\eta_{2j}e^{i\delta}))\\
&= \lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} \operatorname{Im}\left[(-\lambda_j\eta_{2j})(\log(1 - e^{-i\varepsilon}) - \log(1 - e^{i\delta}))\right]\\
&= \lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} \big[\operatorname{Re}(-\lambda_j\eta_{2j}) \operatorname{Im}(\log(1 - e^{-i\varepsilon}) - \log(1 - e^{i\delta}))\\
&\quad + \operatorname{Im}(-\lambda_j\eta_{2j}) \operatorname{Re}(\log(1 - e^{-i\varepsilon}) - \log(1 - e^{i\delta}))\big]\\
&= \operatorname{Re}(-\lambda_j\eta_{2j}) \cdot \pi + \lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} \left[\operatorname{Im}(-\lambda_j\eta_{2j}) \cdot \ln\frac{\sin\varepsilon/2}{\sin\delta/2}\right].
\end{aligned}$$

By Lemma 2.1, we get $\operatorname{Im}(-\lambda_j\eta_{2j}) = 0$, and so

$$c_j - c_{j+1} = \lim_{\varepsilon\to 0^+ \delta\to 0^+} (\operatorname{Im} F(\eta_{2j}e^{-i\varepsilon}) - \operatorname{Im} F(\eta_{2j}e^{i\delta})) = -\lambda_j\eta_{2j} \cdot \pi \in \mathbb{R}.$$

Now we compute λ_{2k} and λ_{2k+2s}, provided that $s \geq 1$. For each sufficiently small $\varepsilon > 0$ and $\delta > 0$, we again have

$$c_k - c_{k+1} = \operatorname{Im} F(\eta_{2k} e^{-i\varepsilon}) - \operatorname{Im} F(\eta_{2k} e^{i\delta}).$$

Hence, there exists the finite limit

$$\begin{aligned} c_k - c_{k+1} &= \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \operatorname{Im}(F(\eta_{2k} e^{-i\varepsilon}) - F(\eta_{2k} e^{i\delta})) \\ &= \pi \operatorname{Re}(-\lambda_{2k}\eta_{2k}) + \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \bigg[\operatorname{Im}(-\lambda_{2k}\eta_{2k}) \ln \frac{\sin \varepsilon/2}{\sin \delta/2} \\ &\qquad + \operatorname{Im}\left(\frac{\mu_1 \eta_{2k} e^{-i\varepsilon}}{1 - e^{-i\varepsilon}} - \frac{\mu_1 \eta_{2k} e^{i\delta}}{1 - e^{i\delta}} \right) \bigg] \\ &= \pi \operatorname{Re}(-\lambda_{2k}\eta_{2k}) + \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \bigg[\operatorname{Im}(-\lambda_{2k}\eta_{2k}) \ln \frac{\sin \varepsilon/2}{\sin \delta/2} \\ &\qquad - \frac{1}{2} \operatorname{Re}(\mu_1 \eta_{2k}) \left(\cot \frac{\varepsilon}{2} + \cot \frac{\delta}{2} \right) \bigg]. \end{aligned}$$

Again, by Lemma 2.1, we get

$$\operatorname{Im}(-\lambda_{2k}\eta_{2k}) = 0, \qquad \operatorname{Re}(\mu_1 \eta_{2k}) = 0 \tag{2.5}$$

and

$$c_k - c_{k+1} = -\lambda_{2k}\eta_{2k} \cdot \pi.$$

In a similar way, we show that

$$\operatorname{Im}(-\lambda_{2k+2s}\eta_{2k+2s}) = 0, \qquad \operatorname{Re}(\mu_2 \eta_{2k+2s}) = 0 \tag{2.6}$$

and

$$c_{k+s} - c_{k+s+1} = c_{k+s} - c_1 = -\lambda_{2k+2s}\eta_{2k+2s} \cdot \pi.$$

It remains to determine λ_{2k} for the case $s = 0$. As before, we consider the finite limit

$$\begin{aligned} c_k - c_{k+1} &= \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \operatorname{Im}(F(\eta_{2k} e^{-i\varepsilon}) - F(\eta_{2k} e^{i\delta})) \\ &= \pi \operatorname{Re}(-\lambda_{2k}\eta_{2k}) + \lim_{\substack{\varepsilon\to 0^+ \\ \delta\to 0^+}} \bigg[\operatorname{Im}(-\lambda_{2k}\eta_{2k}) \ln \frac{\sin \varepsilon/2}{\sin \delta/2} \end{aligned}$$

$$
\begin{aligned}
&\qquad -\frac{1}{2}\operatorname{Re}(\mu_1\eta_{2k})\left(\cot\frac{\varepsilon}{2}+\cot\frac{\delta}{2}\right)\\
&\qquad +\operatorname{Im}\left(\frac{\mu_2\eta_{2k}e^{-i\varepsilon}}{(1-e^{-i\varepsilon})^2}-\frac{\mu_2\eta_{2k}e^{i\delta}}{(1-e^{i\delta})^2}\right)\Bigg]\\
&=\pi\operatorname{Re}(-\lambda_{2k}\eta_{2k})+\lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}}\Bigg[\operatorname{Im}(-\lambda_{2k}\eta_{2k})\ln\frac{\sin\varepsilon/2}{\sin\delta/2}\\
&\qquad -\frac{1}{2}\operatorname{Re}(\mu_1\eta_{2k})\left(\cot\frac{\varepsilon}{2}+\cot\frac{\delta}{2}\right)\\
&\qquad\quad -\frac{1}{4}\operatorname{Im}(\mu_2\eta_{2k})\left(\csc^2\frac{\varepsilon}{2}-\csc^2\frac{\delta}{2}\right)\Bigg].
\end{aligned}
$$

By Lemma 2.1, in this case, we have

$$
\operatorname{Im}(-\lambda_{2k}\eta_{2k})=0,\qquad \operatorname{Re}(\mu_1\eta_{2k})=0,\qquad \operatorname{Im}(\mu_2\eta_{2k})=0 \tag{2.7}
$$

and

$$
c_k-c_{k+1}=-\lambda_{2k}\eta_{2k}\cdot\pi.
$$

Before we compute μ_1 and μ_2, note that

$$
\frac{|1-e^{i\theta}|}{1-e^{i\theta}}\longrightarrow\begin{cases} i & \text{if } \theta\to 0^+,\\ -i & \text{if } \theta\to 0^-.\end{cases} \tag{2.8}
$$

Let us start with the case $s\geq 1$. We have already proved that, in this case, $\operatorname{Re}(\mu_1\eta_{2k})=\operatorname{Re}(\mu_2\eta_{2k+2s})=0$ (see (2.5) and (2.6)). The tangent vector $ie^{it}F'(e^{it})<0$ for all $t\in(\theta_{2k-1},\theta_{2k})$ (see also Figure 1). Hence, by (2.3) and (2.8), we have

$$
\begin{aligned}
0&\geq \lim_{t\to\theta_{2k}^-}\left(|1-\overline{\eta}_{2k}e^{it}|^2\cdot ie^{it}F'(e^{it})\right)\\
&=\lim_{t\to\theta_{2k}^-}\left(|1-\overline{\eta}_{2k}e^{it}|^2\frac{\lambda_{2k}ie^{it}}{1-\overline{\eta}_{2k}e^{it}}+|1-\overline{\eta}_{2k}e^{it}|^2\frac{\mu_1 ie^{it}}{(1-\overline{\eta}_{2k}e^{it})^2}\right)\\
&=-i\mu_1\eta_{2k}.
\end{aligned}
$$

Denote this by $\alpha=-i\mu_1\eta_{2k}$. Hence, $\mu_1=i\alpha\overline{\eta}_{2k}$. The tangent vector $ie^{it}F'(e^{it})>0$ for all $t\in(\theta_{2k+2s-1},\theta_{2k+2s})$. In the same manner, we obtain

$$
0\leq\lim_{t\to\theta_{2k+2s}^-}\left(|1-\overline{\eta}_{2k+2s}e^{it}|^2\cdot ie^{it}F'(e^{it})\right)=-i\mu_2\eta_{2k+2s}=\beta.
$$

As $\mu_1 \cdot \mu_2 \neq 0$, we have $\alpha < 0 < \beta$.

Finally, we consider the case $s = 0$. We have already proved that, in this case, $\operatorname{Re}(\mu_1 \eta_{2k}) = \operatorname{Im}(\mu_2\, \eta_{2k}) = 0$ (see (2.7)). So $\alpha = -i\mu_1\eta_{2k} \in \mathbb{R}$. Moreover,

$$\begin{aligned}
0 &\geq \lim_{t \to \theta_{2k}^-} \left(|1 - \overline{\eta}_{2k} e^{it}|^3 \cdot ie^{it} F'(e^{it})\right) \\
&= \lim_{t \to \theta_{2k}^-} \left(|1 - \overline{\eta}_{2k} e^{it}|^3 \cdot ie^{it} \frac{\mu_2(1 + \overline{\eta}_{2k} e^{it})}{(1 - \overline{\eta}_{2k} e^{it})^3}\right) \\
&= -2\mu_2\eta_{2k} = -2\beta.
\end{aligned}$$

As $\mu_2 \neq 0$, we have $\beta = \mu_2\eta_{2k} > 0$. ☐

Lemma 2.3. *Let $c_1 > c_2 > \cdots > c_k$, $c_{k+s+1} = c_1$, and either $s \geq 1$, $c_{k+1} = d_1 < c_{k+2} = d_2 < \cdots < c_{k+s} = d_s$, $\alpha < 0 < \beta$ or $s = 0$, $\alpha \in \mathbb{R}$, $\beta > 0$. Choose arbitrary $\eta_{2j} = e^{i\theta_{2j}} \in \partial\Delta$, $j = 1, \dots, k + s$, with $\theta_2 < \theta_4 < \cdots < \theta_{2k+2s} < \theta_2 + 2\pi$, and consider F defined by (2.2). Then F is analytic on $\overline{\Delta} \setminus \{\eta_{2j} : j = 1, \dots, k + s\}$ and F' vanishes there only at points $\eta_{2j-1} = e^{i\theta_{2j-1}} \in \partial\Delta$, $j = 1, \dots, k + s$, such that $\theta_1 < \theta_2 < \theta_3 < \cdots < \theta_{2k+2s-1} < \theta_{2k+2s} < \theta_1 + 2\pi$. Moreover,*

- (i) *F is univalent on Δ if $\operatorname{Re} F(\eta_{2j-1}) < \operatorname{Re} F(\eta_{2(k+m)-1})$ each time as $c_j = d_m$.*
- (ii) *$F(\Delta) = \mathbb{C}^l((a_j), (c_j)) \cap \mathbb{C}^r((b_m), (d_m))$ if $\operatorname{Re} F(\eta_{2j-1}) = a_j$ for $j \leq k$, $\operatorname{Re} F(\eta_{2j-1}) = b_{j-k}$ for $k + 1 \leq j \leq k + s$ and $\operatorname{Im} F(\eta_1) = c_1$.*

Proof. The tangent vector

$$\xi \longmapsto i\xi F'(\xi), \quad \xi \in \partial\Delta \setminus \{\eta_{2j} \colon j = 1, \dots, k + s\}$$

is a continuous real function. Indeed, by (2.2), we have

$$\operatorname{Im}\left(i\xi F'(\xi)\right) = \sum_{j=1}^{k+s} \left(\frac{c_j - c_{j+1}}{\pi} \cdot \operatorname{Im} \frac{-i\overline{\eta}_{2j}\xi}{1 - \overline{\eta}_{2j}\xi}\right) = 0$$

for every $\xi \in \partial\Delta \setminus \{\eta_{2j} \colon j = 1, \dots, k + s\}$. Calculations of the one-sided limits at each point $\eta_{2j} = e^{i\theta_{2j}}$ give

$$\lim_{\varepsilon \to 0^{\pm}} \left[ie^{i(\theta_{2m}+\varepsilon)} \cdot F'(e^{i(\theta_{2m}+\varepsilon)})\right] = \begin{cases} \pm\infty & \text{for } j = 1, \dots, k-1, \\ \mp\infty & \text{for } j = k+1, \dots, k+s-1, \end{cases}$$

$$\lim_{\varepsilon\to 0}\left[ie^{i(\theta_{2k}+\varepsilon)}\cdot F'\big(e^{i(\theta_{2k}+\varepsilon)}\big)\right]=-\infty,$$
$$\lim_{\varepsilon\to 0}\left[ie^{i(\theta_{2(k+s)}+\varepsilon)}\cdot F'\big(e^{i(\theta_{2(k+s)}+\varepsilon)}\big)\right]=+\infty,$$

if $s\ge 1$, and

$$\lim_{\varepsilon\to 0^\pm}\left[ie^{i(\theta_{2j}+\varepsilon)}\cdot F'\big(e^{i(\theta_{2j}+\varepsilon)}\big)\right]=\pm\infty\quad\text{for } j=1,\dots,k,$$

if $s=0$. Hence, by the Darboux property for real continuous functions, the tangent vector $\partial\Delta\setminus\{\eta_{2j}\colon j=1,\dots,k+s\}\ni\xi\mapsto i\xi F'(\xi)$ is a real function vanishing at not less than $k+s$ distinct points on $\partial\Delta$. Since the $(k+s+1)$th coefficient of the polynomial

$$z\longmapsto F'(z)\prod_{j=1}^{k+s}(1-\overline{\eta}_{2j}\,z)(1-\overline{\eta}_{2k}\,z)(1-\overline{\eta}_{2k+2s}\,z)$$

vanishes, the function F' has no more zeros on $\overline{\Delta}$.

Note also that

$$\lim_{\varepsilon\to 0}\operatorname{Re}[F(e^{i(\theta_{2j}+\varepsilon)})]=\begin{cases}-\infty & \text{for } j=1,\dots,k-1,\\ +\infty & \text{for } j=k+1,\dots,k+s-1,\end{cases}$$
$$\lim_{\varepsilon\to 0^\pm}\operatorname{Re}[F(e^{i(\theta_{2k}+\varepsilon)})]=\pm\infty,$$
$$\lim_{\varepsilon\to 0^\pm}\operatorname{Re}[F(e^{i(\theta_{2(k+s)}+\varepsilon)})]=\mp\infty,$$

in the case $s\ge 1$, and

$$\lim_{\varepsilon\to 0}\operatorname{Re}[F(e^{i(\theta_{2j}+\varepsilon)})]=-\infty\quad\text{for } j=1,\dots,k,$$

in the case $s=0$. Hence, the image of each arc $(\eta_{2j},\eta_{2(j+1)})$ under the mapping F is a horizontal half-line with the tip $F(\eta_{2j+1})$.

(i) Since $\operatorname{Re}F(\eta_{2j-1})<\operatorname{Re}F(\eta_{2k+2m-1})$ whenever $c_j=d_m$, the finite values of $F|_{\partial\Delta}$, except slit ends, cover every slit exactly twice, i.e., F is univalent.

(ii) By (2.2) and (2.4), we have

$$\begin{aligned}\operatorname{Im}F(\eta_{2j-1})-\operatorname{Im}F(\eta_{2j+1}&=\lim_{\varepsilon\to 0^+}\big[\operatorname{Im}F(\eta_{2j}e^{-i\varepsilon})-\operatorname{Im}F(\eta_{2j}e^{i\varepsilon})\big]\\&=c_j-c_{j+1}\end{aligned}$$

for all $j = 1, \dots, k+s$. Hence, if $\operatorname{Im} F(\eta_1) = c_1$, then all the remaining slits will lie at the prechosen heights of $c_2, c_3, \dots, c_{k+s}$. □

Remark 2.4. Generally, it is very hard to determine all the $k+s+2$ parameters $\eta_2, \eta_4, \dots, \eta_{2k+2s}$, α, β for the Riemann mapping (2.2) whose range is $\mathbb{C}^l((a_j),(c_j)) \cap \mathbb{C}^r((b_m),(d_m))$. From Lemma 2.3, it follows that the points $\eta_1, \eta_3, \dots, \eta_{2(k+s)-1}$ are uniquely determined by the unknown points $\eta_2, \eta_4, \dots, \eta_{2k+2s}$, α, β, so we have for them exactly $k+s+2$ equations: to those of Lemma 2.3 (ii), add $F'(0) > 0$. Let us consider the following examples of univalent functions $F \in \mathcal{A}(\Delta)$.

(i)

$$F(z) \equiv \log \frac{1+\overline{\eta}\, z}{1-\overline{\eta}\, z} + \frac{2i(\tan\gamma)\overline{\eta}\, z}{1-\overline{\eta}^2\, z^2}$$

with $\eta = e^{i\gamma}$, $\gamma \in (0, \pi/2)$. The function has the form (2.2) ($k = s = 1$, $\eta_2 = -\eta$, $\eta_4 = \eta$ and $\beta = -\alpha = \tan\gamma$). By Lemma 2.3, there exist exactly two points η_1 and η_2 such that $F'(\eta_1) = 0$ and $F'(\eta_3) = 0$. Computing F', we see that $F'(-\eta^2) = F'(\eta^2) = 0$. Hence, $\eta_1 = \eta^2$ and $\eta_3 = -\eta^2$. Moreover, $F(0) = 0 < F'(0)$, $F(\eta^2) = a(\gamma) + i\pi/2$ and $F(-\eta^2) = b(\gamma) - i\pi/2$, where $a(\gamma) = \ln(\cot(\gamma/2)) - \sec\gamma$, $b(\gamma) = -a(\gamma)$. By Lemma 2.3, the function F is univalent and

$$F(\Delta) = \mathbb{C} \setminus (\{F(\eta^2) - x : x \geq 0\} \cup \{F(-\eta^2) + x : x \geq 0\}).$$

Observe that, in the limit case $\gamma \to 0^+$, we get the strip mapping

$$\Delta \ni z \longmapsto \log \frac{1+z}{1-z},$$

which maps Δ onto the strip $\{w \in \mathbb{C} \colon |\operatorname{Im} z| < \pi/2\}$.

(ii)

$$F(z) \equiv \log \frac{1+z}{1-z} + 2\left(\tan^2 \frac{\gamma}{2}\right) \frac{z}{(1-z)^2} \quad \text{with } 0 < \gamma < \pi.$$

In this case, $k = 2$, $s = 0$, $\eta_2 = -1$, $\eta_4 = 1$ and $\beta = 2\tan^2(\gamma/2)$ in (2.2). Clearly, $F(0) = 0 < F'(0)$, $F'(e^{\pm i\gamma}) = 0$ (so $\eta_1 = e^{i\gamma}$, $\eta_3 = e^{-i\gamma}$), $F(e^{\pm i\gamma}) = a(\gamma) \pm i\pi/2$, where $a(\gamma) = \ln(\cot(\gamma/2)) -$

$\sec^2(\gamma/2)/2$ and

$$F(\Delta) = \mathbb{C} \setminus \{x \pm i\pi/2 : x \leq a(\gamma)\}.$$

In the limit case $\gamma \to 0^+$, we again get the strip mapping

$$\Delta \ni z \longmapsto \log \frac{1+z}{1-z}.$$

(iii)

$$F(z) \equiv \log\left(1 + \frac{2z}{(1-z)^2}\right) + 2\beta \frac{z}{1+z^2} \quad \text{with } \beta > 0.$$

In this case, $k = 1$, $s = 2$, $\eta_2 = -i$, $\eta_4 = 1$, $\eta_6 = i$ and $\alpha = -\beta$ in (2.2). Clearly, $F(0) = 0 < F'(0)$ and $F'(\xi) = F'(-1) = F'(\overline{\xi}) = 0$, where

$$\xi = \frac{\beta + i\sqrt{1+2\beta}}{1+\beta}$$

(so $\eta_1 = -1$, $\eta_3 = \overline{\xi}$, $\eta_5 = \xi$), and

$$\begin{aligned} F(\xi) &= 1 + \beta + \ln\beta + \pi i, \\ F(-1) &= -\ln 2 - \beta, \\ F(\overline{\xi}) &= \overline{F(\xi)}. \end{aligned}$$

Hence,

$$\begin{aligned} \partial F(\Delta) = \{&x \pm \pi i : x \geq 1 + \beta + \ln\beta\} \\ &\cup (-\infty, -\ln 2 - \beta] \cup \{\infty\}. \end{aligned}$$

In the limit case, as $\beta \to 0^+$, we get the univalent function

$$\Delta \ni z \longmapsto \log\left(1 + \frac{2z}{(1-z)^2}\right)$$

which maps Δ onto the slit strip $\{w \in \mathbb{C} \colon |\operatorname{Im} w| < \pi\} \setminus (-\infty, -\ln 2]$.

Remark 2.5. Since every slit half-plane and slit strip of the set families (1.7)–(1.8) can be obtained by passing in the kernel convergence sense (with respect to zero) to the limit with a suitable sequence of above considered slit planes, some similar results for other domains hold (for the kernel convergence and the Carathéodory theorem, see e.g., [**6,** pages 76–78], [**14,** pages 13–15]).

For instance, if $F \in \mathcal{H}(\Delta)$ is univalent, $F(0) = 0 < F'(0)$ and

$$F(\Delta) = L^l((a_j),(c_j)) \cap L^r((b_m),(d_m)),$$

then there are the unique $2k + 2s + 1$ points $\eta_j = e^{i\theta_j} \in \partial\Delta$, $j = 0, \dots, 2k + 2s$, $\theta_0 < \theta_1 < \cdots < \theta_{2k+2s-1} < \theta_{2k+2s} < \theta_0 + 2\pi$, such that

$$F(z) \equiv \lambda \int_0^z \prod_{j=1}^{k+s} \frac{1-\overline{\eta}_{2j-1}t}{1-\overline{\eta}_{2j}t} \cdot \frac{dt}{(1-\overline{\eta}_{2k}t)(1-\overline{\eta}_0 t)} \quad \text{for some } \lambda > 0, \tag{2.1$'$}$$

$$F(z) \equiv \sum_{j=0}^{k+s} \frac{c_j - c_{j+1}}{\pi} \log(1-\overline{\eta}_{2j}z) + \frac{i\alpha\overline{\eta}_{2k}z}{1-\overline{\eta}_{2k}z} \tag{2.2$'$}$$

where $\alpha < 0$, $c_0 = c = c_{k+s+1}$ for $s \geq 1$, and like before, $c_{k+1} = d_1, \dots, c_{k+s} = d_s$. Clearly, the points $F(\eta_{2j-1})$ are the slit ends: $a_j + ic_j$ if $1 \leq j \leq k$ and $b_{j-k} + id_{j-k}$ if $k+1 \leq j \leq k+s$. To obtain (2.1′), consider the sequence of the domains

$$D_n = F(\Delta) \cup \{x + ic_0 \colon x \in (n, +\infty)\} \cup \{z \in \mathbb{C} \colon \operatorname{Im} z > c_0\}.$$

Observe that $D_n \xrightarrow[n\to\infty]{} F(\Delta)$ in the kernel convergence sense with respect to zero.

By the Riemann mapping theorem, to each domain D_n there corresponds a unique conformal mapping F_n of the form (2.1) with the points η_j, say $\eta_j^{(n)}$, $j = -1, 0, 1, \dots, 2k+2s$ ($F_n(\eta_{-1}^{(n)}) = n$). If one adds the slit $\{x + ic_0 \colon x \in (-\infty, n]\}$ in the right side of the Figure 1, then it will demonstrate the function F_n.

By the Carathéodory theorem, $F_n \xrightarrow[n\to\infty]{} F$ locally uniformly. During this limiting process, the points $\eta_{-1}^{(n)}$ and $\eta_{2k+2s}^{(n)}$ tend to the single point, say η_{2k+2s}, and we obtain (2.1′) from (2.1) by passing $n \to \infty$. Of course, there are many other ways of approximating the domain $F(\Delta)$. For example, we could construct D_n by removing from the line

$\{x + ic_0 \colon x \in \mathbb{R}\}$, the segment $\{x + ic_0 \colon x \in (-1/n, 1/n)\}$, and adding

$$F(\Delta) \cup \{z \in \mathbb{C}\colon \operatorname{Im} z > c_0\}, \quad n \in \mathbb{N}.$$

Then $D_n \xrightarrow[n\to\infty]{} F(\Delta)$ in the kernel convergence sense with respect to zero and appropriate considerations lead to the formulas (2.1′) and (2.2′).

Similarly, if $F \in \mathcal{H}(\Delta)$ is univalent, $F(0) = 0 < F'(0)$ and $F(\Delta) = S^l((a_j), (c_j)) \cap S^r((b_m), (d_m))$, then there are the unique $2k + 2s + 2$ points $\xi_j = e^{i\tau_j}$, $\eta_m = e^{i\theta_m} \in \partial\Delta$, $j = 0, \dots, 2k$, $m = 0, \dots, 2s$, $\tau_0 < \tau_1 < \dots < \tau_{2k} < \theta_0 < \theta_1 < \dots < \theta_{2s} < \tau_0 + 2\pi$, such that

(2.1″)

$$F(z) \equiv \lambda \int_0^z \prod_{j=1}^{k} \frac{1 - \overline{\xi}_{2j-1} t}{1 - \overline{\xi}_{2j} t} \prod_{m=1}^{s} \frac{1 - \overline{\eta}_{2j-1} t}{1 - \overline{\eta}_{2j} t} \cdot \frac{dt}{(1 - \overline{\xi}_0 t)(1 - \overline{\eta}_0 t)}$$

for some $\lambda > 0$ (without the second product if $s = 0$),

(2.2″)

$$F(z) \equiv \sum_{j=0}^{k} \frac{c_j - c_{j+1}}{\pi} \log(1 - \overline{\xi}_{2j} z) + \sum_{m=0}^{s} \frac{d_m - d_{m+1}}{\pi} \log(1 - \overline{\eta}_{2m} z),$$

where $c_0 = d_{s+1} = c$ and $d_0 = c_{k+1} = d$. Obviously, $F(\xi_{2j-1}) = a_j + ic_j$ for $1 \le j \le k$ and $F(\eta_{2m-1}) = b_m + id_m$ for $1 \le m \le s$. By the way, the formulas (2.1′), (2.1″), (2.2′) and (2.2″) can be directly proved in much the same manner as formulas (2.1) and (2.2). Also, the analog of Lemma 2.3 for functions given by (2.2′) and (2.2″) holds.

3. Proofs of the main theorems. In this section, we use the Landau symbols O and o for complex functions of complex variables. In particular, $O(1)$ as $z \to \zeta$ means that $O(1)$ is a bounded function in a punctured neighborhood of ζ, while $o(1)$ as $z \to \zeta$ means that $\lim_{z\to\zeta} o(1) = 0$.

Proof of Theorem 1.1. It is sufficient to consider the case

$$F(\Delta) = \mathbb{C}^l((a_j), (c_j)) \cap \mathbb{C}^r((b_m), (d_m)),$$

so F has the form described in Lemma 2.2. Observe that, for every $0 \le r < 1$, the function p_r is analytic on $\overline{\Delta}$, $f_0 = F$ and, for $\eta \in \partial\Delta$, there exists the unrestricted limit

$$\lim_{\Delta \ni z \to \eta} (1 - \overline{\eta}\, z) \log(1 - \overline{\eta}\, z) = 0. \tag{3.1}$$

Denote $G(z) \equiv \int_0^z F'(t)p(t)\,dt$, and suppose that the function $p \in \mathcal{P}$ is analytic at the point η_{2u} for some $u \in \{1, 2 \ldots, k+s\}$. We would like to investigate the behavior of G near the point η_{2u} using the power series expansion of the function p around that point (this is possible as we assumed p is analytic at η_{2u}). Consider the following cases.

Case I. $u \neq k$ and $u \neq k+s$. By (2.1) from Lemma 2.2, we have

$$\begin{aligned} G(z) &= \int_0^z F'(t)\left(p(\eta_{2u}) + \sum_{n=1}^{\infty} \frac{p^{(n)}(\eta_{2u})}{n!}(t-\eta_{2u})^n\right) dt \\ &= F(z)p(\eta_{2u}) + \int_0^z F'(t) \sum_{n=1}^{\infty} \frac{p^{(n)}(\eta_{2u})}{n!}(t-\eta_{2u})^n dt \\ &= F(z)p(\eta_{2u}) + O(1) \quad \text{as } \Delta \ni z \to \eta_{2u}. \end{aligned} \tag{3.2}$$

Case II. $s \geq 1$ and $u = k$ or $u = k+s$. We start with the case $u = k$, and we use (2.2) instead of (2.1). We have

$$\begin{aligned} G(z) &= F(z)p(\eta_{2k}) \\ &\quad + \int_0^z \Bigg[\Bigg(\sum_{j=1}^{k+s} \frac{c_j - c_{j+1}}{\pi}\, \frac{-\overline{\eta}_{2j}}{1-\overline{\eta}_{2j}t} + \frac{i\alpha\overline{\eta}_{2k}}{(1-\overline{\eta}_{2k}t)^2} + \frac{i\beta\overline{\eta}_{2k+2s}}{(1-\overline{\eta}_{2k+2s}t)^2}\Bigg) \\ &\qquad\qquad \times \sum_{n=1}^{\infty} \frac{p^{(n)}(\eta_{2k})}{n!}(t-\eta_{2k})^n\Bigg] dt \\ &= F(z)p(\eta_{2k}) + i\alpha\eta_{2k}p'(\eta_{2k})\log(1-\overline{\eta}_{2k}z) \\ &\quad + O(1) \text{ as } \Delta \ni z \to \eta_{2k}. \end{aligned} \tag{3.3}$$

In a similar way, we obtain

$$\begin{aligned} G(z) &= F(z)p(\eta_{2k+2s}) \\ &\qquad + i\beta\eta_{2k+2s}p'(\eta_{2k+2s})\log(1-\overline{\eta}_{2k+2s}z) + O(1) \end{aligned} \tag{3.4}$$

as $\Delta \ni z \to \eta_{2k+2s}$.

Case III. $s = 0$ and $u = k$. Proceeding as before, by Lemma 2.2 (the case $s = 0$), we get

$$\begin{aligned} G(z) &= F(z)p(\eta_{2k}) - 2\beta\, p'(\eta_{2k})z/(1-\overline{\eta}_{2k}z) \\ &\quad + [(i\alpha - \beta)p'(\eta_{2k}) - \beta\,\eta_{2k}p''(\eta_{2k})]\eta_{2k} \\ &\quad \times \log(1-\overline{\eta}_{2k}z) + O(1), \end{aligned} \tag{3.5}$$

as $\Delta \ni z \to \eta_{2k}$.

In any case, formulas (3.1)–(3.5) always lead to

$$G(z) = F(z)[p(\eta_{2u}) + o(1)] + O(1) \quad \text{as } \Delta \ni z \to \eta_{2u}. \tag{3.6}$$

Obviously, we can write (3.6) for the function p_r instead of p (p_r is analytic on the whole unit circle).

For $\gamma \in \mathbb{R}$, denote by $\ell_\gamma = \{z \in \Delta\colon \operatorname{Im} F(z) = \gamma\}$, i.e., ℓ_γ is the preimage of the horizontal line $\{w \in \mathbb{C}\colon \operatorname{Im} w = \gamma\}$ under the univalent function F. Observe that $\ell_{\gamma_1} \cap \ell_{\gamma_2} = \varnothing$ for $\gamma_1 \neq \gamma_2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and

$$\bigcup_{\gamma\in\mathbb{R}} \ell_\gamma = \Delta.$$

Fix $\gamma \in \mathbb{R}$. Note that a half-line of the horizontal line $\{w \in \mathbb{C}\colon \operatorname{Im} w = \gamma\}$ is included in $F(\Delta)$ if and only if there exists $u \in \{1, 2, \ldots, k+s\}$ such that $\eta_{2u} \in \overline{\ell_\gamma}$. Suppose $\eta_{2u} \in \overline{\ell_\gamma}$. Then, for $z \in \ell_\gamma$, by (3.6), we have

$$\begin{aligned} \operatorname{Re} f_r(z) &= \operatorname{Re} \int_0^z F'(t)p_r(t)\,dt \\ &= \operatorname{Re}(F(z)[p_r(\eta_{2u}) + o(1)]) + O(1) \\ &= \operatorname{Re} F(z)\operatorname{Re}(p_r(\eta_{2u}) + o(1)) \\ &\quad - \operatorname{Im} F(z)\operatorname{Im}(p_r(\eta_{2u}) + o(1)) + O(1) \\ &= \operatorname{Re} F(z)\operatorname{Re} p_r(\eta_{2u}) + O(1) \quad \text{as } \ell_\gamma \ni z \to \eta_{2u}, \end{aligned}$$

i.e., $f_r(\Delta) \in \mathsf{A}$. □

In fact, we have proved a little stronger

Theorem 1.1´. *Suppose* A *is any of the set families* (1.7)–(1.8), $F \in \mathcal{A}(\Delta)$ *is univalent with* $F(0) = 0 < F'(0)$ *and* $F(\Delta) \in \mathsf{A}$. *If*

$p \in \mathcal{P}$ is analytic at every point $\eta \in \partial\Delta$ such that $\operatorname{Re} p(\eta) > 0$ *and $F(\eta) = \infty$, then the range of the function $k(\cdot, F, p)$ is also in* A.

Proof of Theorem 1.2. For a fixed set family A choose $D \in \mathsf{A}$, and let $f \in \mathcal{S}_H^0(\Delta, D)$. Then there exists a univalent $F \in \mathcal{A}(\Delta)$ convex in the direction of the real axis such that $F(0) = 0 < F'(0)$ and $f = k(\cdot, F, p)$. For almost all real θ the angular limits $\widehat{f}(e^{i\theta})$, $\widehat{F}(e^{i\theta})$, $\widehat{p}(e^{i\theta})$ exist and are finite. Since $\operatorname{Im} \widehat{F} = \operatorname{Im} \widehat{f}$ almost everywhere there exists a set family $\widetilde{\mathsf{A}}$ consisting of such slit domains whose slits are on the same heights as domains from A and $F(\Delta) \in \widetilde{\mathsf{A}}$ (see also [**9,** Remark 3]). Perhaps $\widetilde{\mathsf{A}} \neq \mathsf{A}$, so on a suitable horizontal line $\{w \in \mathbb{C} : \operatorname{Im} w = \gamma\}$

(i) $F(\Delta)$ has two slits,
(ii) $f(\Delta)$ and $F(\Delta)$ have only one slit, but each of them extends in opposite directions.

Let $p_r(z) \equiv p(rz)$, $f_r = k(\cdot, F, p_r)$ and $0 \le r < 1$. By Theorem 1.1, $f_r(\Delta) \in \widetilde{\mathsf{A}}$, and by Lemma 2.2, w_r is an end of the slit $\{w \in \mathbb{C} : \operatorname{Im} w = \gamma\} \setminus f_r(\Delta)$ if and only if there exists the zero $\xi_j \in \partial\Delta$ of F' such that $f_r(\xi_j) = w_r$. Denote by Ω the kernel of the sequence $(f_{n/(n+1)}(\Delta))$ with respect to 0. Since $f_{n/(n+1)} \to f$ locally uniformly on Δ, then $f(\Delta) \subseteqq \Omega$, see [**2,** Theorem 3.5]. However,

$$|\operatorname{Re} w_r| \le \int_0^1 |F'(t\xi_j) p_r(t\xi_j)|\, dt \le \int_0^1 \left| \frac{F'(t\xi_j)}{1 - rt} \right| (1 + rt)\, dt < C \int_0^1 (1 + t)\, dt,$$

for some positive constant C. Hence, the possibilities (i)–(ii) fail, i.e., $\widetilde{\mathsf{A}} = \mathsf{A}$. $\square$

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