

ON THE ADDITION OF UNITS AND NON-UNITS IN FINITE COMMUTATIVE RINGS

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ABSTRACT. Let R be a finite commutative ring. In this paper, we find the number of representations of a fixed member of R to be the sum of k units in R , and the sum of k non-units, and as a sum of a unit and a non-unit. We prove that, if \mathbb{Z}_2 is not a quotient of R , then every $r \in R$ can be written as a sum of k units, for each integer $k > 1$.

1. Introduction and preliminaries. The study of algebraic structures using the properties of graphs has become an exciting research topic in the last thirty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see for example, [1, 2, 6, 7, 8].

Interest in the question of how the units of a ring might generate the ring additively goes back to the middle of the last century. In 1958, Skornyakov [11, Page 167, Problem 31] posed the problem of determining which regular rings are generated by their units. More precisely, he asked: *Is every element of a Von Neumann regular ring, which cannot have \mathbb{Z}_2 as a quotient, a sum of units?* In 1974, Raphael [9] launched a systematic study of rings generated by their units, which he called S -rings.

Finally, in 1976, Fisher and Snider [5] proved that, if R is a Von Neumann regular ring with artinian primitive factor rings in which 2 is a unit, then every element of R can be expressed as the sum of two units.

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Theorem 1.1. [9]. *Let R be a finite commutative ring with nonzero identity. Then R is generated by its units if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*

In [10], the authors found the number of representations of a fixed residue class mod m as the sum of two units in \mathbb{Z}_m , the sum of two non-units, and the sum of mixed pairs, respectively. We generalize these results into commutative rings, as the sum of k units and the sum of k non-units. In addition, we prove Theorem 1.1. Our methods are different and much simpler.

Throughout this paper, R is a finite commutative ring with identity. We denote the set of unit elements by R^\times . We say that R is local if R has exactly one maximal ideal. If R is a finite commutative ring, then $R \simeq R_1 \times \cdots \times R_t$, where each R_i is a finite commutative local ring with maximal ideal M_i , by [3, Theorem 8.7]. It is obvious that $R/J_R \simeq R_1/M_1 \times \cdots \times R_t/M_t$, where J_R is the Jacobson radical of R , since the Jacobson radical is the product of the maximal ideals of the local factors. This decomposition is unique up to permutation of factors. We denote by k_i the (finite) residue field R_i/M_i , $\pi_i : R_i \rightarrow k_i$ the quotient map, and $f_i = |k_i|$. We also assume (after an appropriate permutation of factors) that $f_1 \leq f_2 \leq \cdots \leq f_t$. Clearly, (u_1, \dots, u_t) is a unit of R if and only if each u_i is a unit element in R_i for $i = 1, \dots, t$.

The motivation of this paper is the determination of the number of solutions of

$$(1.1) \quad x_1 + x_2 + \cdots + x_k = r$$

where x_i are either all units or all non-units.

Let A be an additive group with identity 0. For $S \subseteq A$ such that $-S = \{-s; s \in S\} = S$ the *Cayley graph* $X = \text{Cay}(A, S)$ is the undirected graph having vertex set $V(X) = A$ and edge set $E(X) = \{\{a, b\}; a - b \in S\}$. Clearly, if $0 \notin S$, then there is no loop in X , and if $0 \in S$, then there is exactly one loop at each vertex. It is easy to check that S generates A if and only if X is connected. The unitary Cayley graph of a ring R , denoted by G_R , is the graph whose vertex set is R , and in which $\{x, y\}$ is an edge if and only if x and y are elements of R such that $x - y \in R^\times$.

Let G be a graph. Suppose v_0 and v_n are two vertices of G . A

(v_0, v_n) -walk in G is a sequence $W := v_0e_0v_1e_1v_2 \cdots e_{n-2}v_{n-1}e_{n-1}v_n$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that v_{i-1} and v_i are the endpoints of e_i for all i . We denote by $w_k(G, x, y)$ the number of $x - y$ walks of length k in the graph G . It is interesting to find the number of walks in $\text{Cay}(R, R^\times)$.

The *tensor product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, with edges specified by putting (u, v) adjacent to (u', v') if and only if u is adjacent to u' in G_1 and v is adjacent to v' in G_2 . So, we immediately see that G_R is the tensor product of the graphs G_{R_1}, \dots, G_{R_t} .

Let A_G be the adjacency matrix of the simple graph G . The next lemma can be proved easily by induction.

Lemma 1.2. [13]. *If A_G is the adjacency matrix of a graph G in such a way that the vertices are labeled by v_1, v_2, \dots, v_n , then the (i, j) entry of A^k , is the number of $v_i - v_j$ walks of length k in G .*

We refer the reader to [13] for general definitions of graph theory.

The next theorem determines the diameter of unitary Cayley graphs.

Theorem 1.3. [2]. *If $R \cong R_1 \times \cdots \times R_t$ is a product of local rings, then*

$$\text{diam}(G_R) = \begin{cases} 1 & \text{if } t = 1 \text{ and } R \text{ is a field;} \\ 2 & \text{if } t = 1 \text{ and } R \text{ is not a field;} \\ 2 & \text{if } t > 1, f_1 > 2; \\ 3 & \text{if } t > 1, f_1 = 2, f_2 > 2; \\ \infty & \text{if } t > 1, f_1 = f_2 = 2. \end{cases}$$

The above result generalizes Theorem 1.1, since G_R is a disconnected graph if and only if $t > 1$ and $f_1 = f_2 = 2$. In the other cases, G_R is connected and R is generated (additively) by its units.

2. Representation by units. The purpose of this section is to find the number of representations of a member of ring R as the sum of k units of R .

Lemma 2.1. [2]. *Let R be a finite commutative ring.*

- (i) G_R is a regular graph of degree $|R^\times|$.
- (ii) If $R \cong R_1 \times \cdots \times R_t$ is a product of local rings, then $G_R = \otimes_{i=1}^t G_{R_i}$.
- (iii) If R is a commutative local ring with maximal ideal M , then G_R is a complete multipartite graph whose partite sets are the cosets of M .

We denote the number of solutions of equation (1.1) in R^\times and $R - R^\times$, by $\mathcal{S}_u(R, r, k)$ and $\mathcal{S}_{nu}(R, r, k)$, respectively.

Next are some basic consequences of this definition.

Lemma 2.2. *Let R be a ring, and r be an arbitrary member of R . Let k be a natural number. Then the following hold:*

- (a) $\mathcal{S}_u(R, r, k) = w_k(\text{Cay}(R, R^\times), r, 0)$.
- (b) $\mathcal{S}_{nu}(R, r, k) = w_k(\text{Cay}(R, R - R^\times), r, 0)$.
- (c) If r' is a unit of R , then $\mathcal{S}_u(R, r, k) = \mathcal{S}_u(R, r'r, k)$, and $\mathcal{S}_{nu}(R, r, k) = \mathcal{S}_{nu}(R, r'r, k)$.

Proof.

- (a) Let S and W be the set of solutions of equation (1.1) in R^\times and the set of walks of length k between 0 and r , respectively. Let $\varphi : S \rightarrow W$ be defined as

$$\varphi((x_1, x_2, \dots, x_k)) = (0, x_1, x_1 + x_2, \dots, x_1 + \cdots + x_k).$$

Obviously, φ is one to one and onto. So,

$$\mathcal{S}_u(R, r, k) = w_k(\text{Cay}(R, R^\times), r, 0).$$

- (b) The proof is similar to (a).
- (c) Let S_r and $S_{r'r}$ be the set of solutions of equation (1.1) in R^\times , for r and $r'r$, respectively. Let $\psi : S_r \rightarrow S_{r'r}$ be defined as

$$\psi((x_1, x_2, \dots, x_k)) = (r'x_1, r'x_2, \dots, r'x_k).$$

Obviously, ψ is one to one and onto. So, $\mathcal{S}_u(R, r, k) = \mathcal{S}_u(R, r'r, k)$. Applying the same argument as above we get

$$\mathcal{S}_{nu}(R, r, k) = \mathcal{S}_{nu}(R, r'r, k). \quad \square$$

In what follows, we generalize Theorems 1.1 and 1.3. Our proofs are based on walks on unitary Cayley graphs. We need the following result, whose proof is clear from the definition of adjacency in the tensor product.

Lemma 2.3. *Let G and H be simple graphs. Then*

$$w_k(G \otimes H, (x_1, y_1), (x_2, y_2)) = w_k(G, x_1, x_2)w_k(H, y_1, y_2).$$

Theorem 2.4. *Let R be a finite local ring with maximal ideal M of size m . Then we have the following:*

$$\mathcal{S}_u(R, r, k) = \begin{cases} \left(\frac{(t-1)^k - (-1)^k}{t} + (-1)^k\right)m^{k-1} & \text{if } r \in M \\ \frac{(t-1)^k - (-1)^k}{t}m^{k-1} & \text{if } r \in R - M, \end{cases}$$

where $t = |R|/m$.

Proof. By Lemma 2.1, the graph G_R is a complete multipartite graph, so the adjacency matrix of G_R is equal to $A_G = (J_t - I_t) \otimes J_m$, where J_m is the $m \times m$ all 1-matrix. Thus, by the fact that, for arbitrary matrices A, B, C and D , we have $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, so:

$$(2.1) \quad A_G^k = \left(\sum_{i=1}^k ((-1)^{k-i} \binom{k}{i} t^{i-1} J_t) + (-1)^k I_t\right) \otimes (m^{k-1} J_m).$$

We can easily see that

$$\sum_{i=1}^k ((-1)^{k-i} \binom{k}{i} t^{i-1}) = \frac{(t-1)^k - (-1)^k}{t}.$$

Therefore, if $r \in M$, then

$$\mathcal{S}_u(R, r, k) = \left(\frac{(t-1)^k - (-1)^k}{t} + (-1)^k\right)m^{k-1};$$

otherwise,

$$\mathcal{S}_u(R, r, k) = \frac{(t-1)^k - (-1)^k}{t}m^{k-1}.$$

The next theorem, which can be easily obtained by Lemma 2.3 and Theorem 2.4, gives the number of solutions of equation (1.1) for a fixed member of R . □

Theorem 2.5. *Let R be a ring, where $R = R_1 \times R_2 \times \dots \times R_t$ and R_i is a local ring with maximal ideal M_i of size m_i , for all $i \in \{1, 2, \dots, t\}$. Let $r = (r_1, \dots, r_t)$ be a member of R . Then*

$$(2.2) \quad \mathcal{S}_u(R, r, k) = \prod_{i=1}^t \left(\left(\frac{(t_i - 1)^k - (-1)^k}{t_i} + (-1)^k \right) m_i^{k-1} \right)^{a_i} \\ \times \left(\frac{((t_i - 1)^k - (-1)^k) m_i^{k-1}}{t_i} \right)^{1-a_i},$$

where $t_i = |R_i|/m_i$ and $a_i = 1$ if and only if $r_i \in M_i$, otherwise $a_i = 0$.

Corollary 2.6. *A ring R is generated by units if and only if R does not have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. It is enough to check that, for $r \in R$, $w_k(G_R, r, 0)$ is not zero.*

Remark 2.7. It should be mentioned that Theorem 1.3 can be derived from Theorem 2.5. Theorem 2.5 is a generalization of [10, Theorem 1.1] and [4, Theorem 4.4].

In the following, we study the good number of elements of a ring.

Definition 2.8. [12]. An element $r \in R$ is called k -good if $r = u_1 + u_2 + \dots + u_k$ with $u_1, u_2, \dots, u_k \in R^\times$, and the ring R is called k -good if every element of R is k -good. The ring R is called ω -ring if it is not k -good for any k , but every element of R is k -good for some k (that is, when at least R^\times generates R additively); otherwise, R is called an ∞ -ring.

Clearly, a unit element is 1-good.

The good number of a ring has been studied in [12]. In the sequel, we prove that, if R is a ring where \mathbb{Z}_2 is not a quotient of R , then R is a k -good ring, for each $k > 1$. Otherwise, R is not 2-good.

Theorem 2.9. *Let R be a ring and $k > 1$ a natural number. Then R is k -good if and only if \mathbb{Z}_2 is not a quotient of R .*

Proof. Let R be a ring such that \mathbb{Z}_2 is not a quotient of R . By definition, it is enough to prove that each $r \in R$ is k -good. By

Theorem 2.5, it is enough to show that

$$(2.3) \quad \prod_{i=1}^t \left(\left(\frac{(t_i - 1)^k - (-1)^k}{t_i} + (-1)^k \right) m_i^{k-1} \right)^{a_i} \left(\frac{((t_i - 1)^k - (-1)^k) m_i^{k-1}}{t_i} \right)^{1-a_i} \neq 0.$$

To show this, we prove that:

$$(2.4) \quad \left(\frac{(t_i - 1)^k - (-1)^k}{t_i} + (-1)^k \right) m_i^{k-1} \neq 0$$

and

$$\frac{((t_i - 1)^k - (-1)^k) m_i^{k-1}}{t_i} \neq 0.$$

These inequalities are equivalent to:

$$(2.5) \quad (t_i - 1)^k - (-1)^k + t_i(-1)^k \neq 0$$

and

$$(t_i - 1)^k - (-1)^k \neq 0,$$

if and only if

$$(2.6) \quad (t_i - 1)^{k-1} \neq (-1)^k \quad \text{and} \quad (t_i - 1)^k \neq (-1)^k,$$

if and only if $t_i \neq 0, 2$ for all i . □

The proof of the next result is similar.

Theorem 2.10. *Let R be a ring. Then R is an ω -ring if and only if $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a quotient of R , but \mathbb{Z}_2 is a quotient of R .*

Remark 2.11. Let R be a ring. Then R is an ∞ -ring if and only if $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a quotient of R .

3. Representation by non-units. In this section, we study those rings which are generated by their non-unit members. In what follows,

we find the number of solutions of the following equation in non-unit members of a ring R :

$$(3.1) \quad x_1 + x_2 + \dots + x_k = r.$$

Obviously, if R is a local ring, then there is no solution to the above equation for unit members.

The next theorem shows the relation between $\mathcal{S}_{nu}(R, r, k)$ and $\mathcal{S}_u(R, r, k)$:

Theorem 3.1. *Let $R = R_1 \times R_2 \times \dots \times R_s$ be a finite ring where, for $i \in \{1, 2, \dots, s\}$, each R_i is local of order n_i with maximal ideal M_i of order m_i . Let $n = \prod_{i=1}^s n_i$. Then the number of solutions of (3.1) in which all x_i are non-units is given by:*

$$(3.2) \quad \mathcal{S}_{nu}(R, r, k) = (-1)^k \mathcal{S}_u(R, r, k) + \sum_{t=1}^k \left((-1)^{k-t} \binom{k}{t} n^{t-1} \left(\prod_{i=1}^s (n_i - m_i)^{k-t} \right) \right).$$

Proof. We assign labels to the vertices of G_R such that $v_1 = 0$ and $v_2 = r$. We know that $J_n - I_n - A_{G_R}$ is the adjacency matrix of the complement graph G_R^c . The adjacency matrix of $\text{Cay}(R, R - R^\times)$ is $(J_n - I_n - A_G) + I_n$, since G_R^c is isomorphic to the $\text{Cay}(R, R - (R^\times \cup \{0\}))$. By Lemma 2.2, $\mathcal{S}_{nu}(R, r, k)$ is the $(1, 2)$ entry of matrix $(J_n - I_n - A_G + I_n)^k = (J_n - A_G)^k$. So,

$$(3.3) \quad \begin{aligned} \mathcal{S}_{nu}(R, r, k) &= ((J_n - A_G)^k)_{1,2} \\ &= \left(\sum_{t=0}^k (-1)^{k-t} \binom{k}{t} (J_n)^t A_G^{k-t} \right)_{1,2}. \end{aligned}$$

Since G_R is an $\prod_{i=1}^s (n_i - m_i)$ -regular graph, it follows that

$$J_n A_G^{k-t} = \prod_{i=1}^s (n_i - m_i)^{k-t} J_n$$

and

$$(J_n)^t = n^{t-1} J_n.$$

Therefore, we have:

$$(3.4) \quad \mathcal{S}_{nu}(R, r, k) = \left((-1)^k A_G^k + \sum_{t=1}^k (-1)^{k-t} \binom{k}{t} n^{t-1} \left(\prod_{i=1}^s (n_i - m_i) \right)^{k-t} J_n \right)_{1,2} .$$

Thus,

$$(3.5) \quad \mathcal{S}_{nu}(R, r, k) = (-1)^k \mathcal{S}_u(R, r, k) + \sum_{t=1}^k \left((-1)^{k-t} \binom{k}{t} n^{t-1} \left(\prod_{i=1}^s (n_i - m_i)^{k-t} \right) \right) . \quad \square$$

Remark 3.2. Let R be a finite local ring with maximal ideal M of size m . Then, for the number of solutions of equation (3.1) in non-unit members of R , we have the following:

- (a) If $r \in R - M$, then there is no solution.
- (b) If $r \in M$, then this number is m^{k-1} .

The following corollary is obtained from Theorems 2.4 and 3.1.

Corollary 3.3. *Let R be a finite ring. Then the number of solutions of the equation*

$$(3.6) \quad x_1 + x_2 = r,$$

where $x_1 \in R^\times$ and $x_2 \in R - R^\times$ is

$$(3.7) \quad \frac{|R| - \mathcal{S}_u(R, r, 2) - \mathcal{S}_{nu}(R, r, 2)}{2} .$$

It would be interesting to calculate the number of solutions of equation (1.1) for mixed pairs, and we leave it as an open problem.

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