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PICARD GROUPS AND TORSION-FREE CANCELLATION FOR ORDERS IN $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

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ABSTRACT. We examine the question of direct-sum cancellation of finitely generated, torsion-free modules over ring orders R contained in $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Using conditions involving the Picard group and the unit group of R, we give a nearly complete classification of those orders R for which torsion-free cancellation holds. There is exactly one 'exceptional' order to which our methods do not apply.

1. Introduction. In the paper [**4**], *Krull-Schmidt uniqueness fails dramatically over subrings of* $Z \oplus \cdots \oplus Z$, Levy gives, among other pathologies, examples of how direct-sum cancellation of finitely generated modules can fail for subrings R of $Z \oplus \cdots \oplus Z$. In that paper's notation, Z can be taken to be $\mathbb{Z}$, the ring of integers. The examples depend on rings Z 'involving non-liftability of units modulo maximal ideals to units of Z itself.'

In a later paper [**6**], Wiegand considers direct-sum cancellation of torsion-free modules over one-dimensional reduced rings R (having finite integral closure). In Section 4 of that paper, conditions involving liftable units and the Picard groups of R and its integral closure are used to obtain explicit results for torsion-free cancellation over orders R in real and imaginary quadratic number fields.

In this work, we combine aspects of the two cancellation problems mentioned above as we examine the question of torsion-free cancellation over orders in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Working exclusively in the context of Picard groups and unit groups we give a nearly complete determination of those orders R in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for which torsion-free cancellation holds (this is our main theorem, stated in the following section). There is

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exactly one 'exceptional' order to which our methods do not apply; see the next section and Section 8 for details.

2. Definitions and assumptions. *All rings are Noetherian and unital. All modules are finitely generated and unital.*

Throughout this paper, we set $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. We always regard Γ as a $\mathbb{Z}$-algebra with respect to the diagonal embedding $\mathbb{Z} \subseteq \Gamma$.

Let R be a subring of Γ containing (the diagonal copy of) $\mathbb{Z}$ such that R has rank three as a $\mathbb{Z}$-module. Then R is a $\mathbb{Z}$-*order* in Γ, or just an *order* for short. The order Γ itself is the *maximal order*. To avoid trivialities, we will often assume $R \neq \Gamma$.

We say that *torsion-free cancellation* holds for R if the implication

$$M \oplus C \cong N \oplus C \implies M \cong N$$

holds for all (finitely generated) torsion-free R-modules C, M and N. The main goal of this paper is to establish the following:

> *Let R be an order in $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ such that $R \neq \mathbb{Z} + (4, 6, 12)\,\Gamma$. Then torsion-free cancellation holds for R if and only if the Picard group of R is trivial.*

This is the content of our main theorem in Section 7. See Section 3 for definitions pertaining to the Picard group. We briefly point out the well-known fact that torsion-free cancellation holds for the maximal order Γ, since direct-sum cancellation holds for *arbitrary* finitely generated Γ-modules.

2.1. Notation and terminology. We shall use the notation $A^\times$ for the unit group of an arbitrary ring A. To avoid confusion, we use the term 'size' (instead of 'order') when we refer to the cardinality of a group.

3. Picard groups and conductors. We begin with some preliminaries regarding the Picard group of a commutative ring A. Recall that all modules are finitely generated. Let A be an arbitrary (Noetherian, unital) commutative ring. Let M be an A-module. Then M is *invertible* if there exists an A-module N such that $M \otimes_A N \cong A$. The set of isomorphism classes of invertible A-modules forms an abelian group, with tensor product over A as the group multiplication. This is called the *Picard group* of A, or $\operatorname{Pic} A$. The equation $\operatorname{Pic} A = 1$ will mean that $\operatorname{Pic} A$ is the trivial group.

For later use, we now mention a well-known fact concerning Picard groups and pairs of commutative rings A and B: Pic . If $A \subseteq B$, then there is a natural surjection $\operatorname{Pic} A \to \operatorname{Pic} B$.

Returning to the context of orders, let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and let R be an order in Γ. We assume throughout that $R \neq \Gamma$. Next, we let

$$\mathfrak{f} = (R : \Gamma) = \{x \in R : x\Gamma \subseteq R\}$$

denote the *conductor* of R. This is the largest common ideal of R and Γ.

From the ring inclusion $R \subseteq \Gamma$ we obtain $R/\mathfrak{f} \subseteq \Gamma/\mathfrak{f}$ and a corresponding conductor square, which yields a Mayer-Vietoris exact sequence (see [**5**])

$$\cdots \longrightarrow \Gamma^{\times} \times (R/\mathfrak{f})^{\times} \longrightarrow (\Gamma/\mathfrak{f})^{\times} \longrightarrow \operatorname{Pic} R \longrightarrow \operatorname{Pic} \Gamma \longrightarrow 1.$$

We note that $\Gamma^{\times}$ is generated by $(-1,1,1)$, $(1,-1,1)$ and $(1,1,-1)$ and thus is finite of order 8. Let Λ denote the image of $\Gamma^{\times}$ in $(\Gamma/\mathfrak{f})^{\times}$ under the natural projection. We call this image the group of *liftable units* of $\Gamma/\mathfrak{f}$.

Since $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a principal ideal ring, we have $\operatorname{Pic} \Gamma = 1$. From the above exact sequence it follows that

$$\frac{(\Gamma/\mathfrak{f})^{\times}}{\Lambda \cdot (R/\mathfrak{f})^{\times}} \cong \operatorname{Pic} R.$$

Intuitively, if $\mathfrak{f}$ is large, then both Λ and $(R/\mathfrak{f})^{\times}$ will be small compared to $(\Gamma/\mathfrak{f})^{\times}$. This will force $\operatorname{Pic} R$ to be nontrivial in 'most' cases, leading to a failure of cancellation for R (by Proposition 7.1 in Section 7).

4. Structure of the orders. Before continuing our investigation of orders R in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, let us note the following. It is an easy exercise to prove that every order S in $\mathbb{Z} \times \mathbb{Z}$ is equal to $\mathbb{Z} + n(\mathbb{Z} \times \mathbb{Z})$ for some positive integer n. Furthermore, every ideal of such an S is two-generated. The methods used in [**6,** Section 4] can be directly applied to this *much* easier situation. Details are left to the interested reader.

Let R be an order in $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ of conductor $\mathfrak{f}$. Then $\mathfrak{f} \cap \mathbb{Z} = n\mathbb{Z}$ for some positive integer n, with $n > 1$ if and only if $R \neq \Gamma$. Suppose

$n = p^m$ for some *prime* integer p and some positive integer m. In this paper, we will refer to such an order as a *p-order*. For such an R we have $\mathbb{Z} + p^m\Gamma \subseteq R$, and m is the smallest integer k for which the containment $\mathbb{Z} + p^k\Gamma \subseteq R$ holds.

Most of the results relevant to our main theorem follow from the structure of p-orders. The next theorem follows directly from the work of Drozd and Skuratovskii. It is essentially a restatement of Theorem 2.5 in [**1**], but specialized to our context.

Theorem 4.1. [**1**]. *Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Suppose R is a p-order in Γ, and let m be the smallest positive integer such that $\mathbb{Z} + p^m\Gamma \subseteq R$. Then, for some primitive idempotents $e \neq e'$ in Γ, the following hold*:

(i) *R is contained in an order of the form*

$$S = \mathbb{Z} + p^l(e + p^q ae')\,\mathbb{Z} + p^{2l+q}\Gamma,$$

where $a \in \mathbb{Z}$ is a unit modulo p and $l, q \geq 0$;

(ii) *If $q = 0$, then $a \not\equiv 1$ modulo p*;

(iii) *$R = \mathbb{Z} + p^k S$ for some $k \geq 0$ with $k + 2l + q = m$.*

Remark 4.2. In the above, if $2l + q = 0$ then $S = \Gamma$, necessarily $k \geq 1$ since $R \neq \Gamma$. Similarly, if $k = 0$, then necessarily $2l + q > 0$.

Note that Γ has exactly three primitive idempotents: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Some results that follow depend on the idempotents e and e', but only up to a permutation of the components of $\Gamma = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. We will use the phrase *up to permutation* in situations that involve such a result.

We shall denote by $R(p; k, l, m)$ a p-order R having the form described above in Theorem 4.1. Also, note that $q = m - k - 2l$ is determined by k, l, m. Let us now describe how to generate such an order R as a $\mathbb{Z}$-algebra.

Corollary 4.3. *Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and let $R \neq \Gamma$ be a p-order, so that R has the form $R(p; k, l, m)$ as described above. Then, up to permutation, R is generated as a $\mathbb{Z}$-algebra by these three elements of Γ:*

$$(1, 1, 1),\ (0, p^{k+l}, ap^{k+l+q}),\ (0, 0, p^m)$$

where $m = k+2l+q$ *and* $a \in \mathbb{Z}$ *satisfies the conditions in Theorem* 4.1.

Proof. By permuting the components of Γ, we may assume (using the notation from Theorem 4.1) that $e = (0,1,0)$ and $e' = (0,0,1)$. The corollary follows upon verifying that $(0, p^m, 0)$ and $(p^m, 0, 0)$ are already in the $\mathbb{Z}$-algebra of the three given generators. $\square$

Now, we will compute the conductor of a p-order R. Before we proceed, let us make a couple of observations. Suppose R is a p-order. Let $\mathfrak{f}$ be the conductor of R. By the definition of a p-order, $\mathfrak{f} \cap \mathbb{Z} = p^m\mathbb{Z}$ for some $m \geq 1$. Since $\mathfrak{f}$ is an ideal of Γ, we have $\mathfrak{f} = (p^x, p^y, p^z)\,\Gamma$ for some nonnegative integer exponents x, y, z, with at least one of these exponents being positive.

However, suppose we start with an arbitrary ideal $(p^x, p^y, p^z)\,\Gamma$, with x, y, z as just described. Such an ideal is not necessarily the conductor of a p-order. Furthermore, there may be several distinct p-orders that have this same ideal as a conductor. Having said that, we proceed to the next result, which helps clarify these issues.

Lemma 4.4. *Let* $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, *and let* R *be a* p*-order, so that* R *has the form* $R(p; k, l, m)$ *as described above. The conductor* $\mathfrak{f}$ *of* R *equals* $(p^m, p^{k+2l}, p^m)\,\Gamma$, *up to permutation.*

Proof. Using Corollary 4.3, we see right away that $(0, p^{k+2l}, 0)$ is in R. Additionally, the fact that $(p^m, 0, 0)$ is in R was required as part of the verification of Corollary 4.3. Thus, we have $(p^m, p^{k+2l}, p^m)\,\Gamma \subseteq R$.

It remains to show that $(p^m, p^{k+2l}, p^m)\,\Gamma$ is the *largest* ideal of Γ contained in R. We leave this to the reader but mention in passing that the condition 'if $q = 0$, then $a \not\equiv 1$ modulo p,' given in Theorem 4.1 must be invoked at some point. $\square$

5. Orders modulo their conductors. To analyze the Picard group of an order $R \neq \Gamma$ having conductor $\mathfrak{f}$, we will need some results concerning the unit groups of $R/\mathfrak{f}$ and $\Gamma/\mathfrak{f}$. As noted above, we can write $\mathfrak{f} \cap \mathbb{Z} = n\mathbb{Z}$ for some positive integer n. Let S be the finite set of primes p that divide n.

Since $\mathfrak{f}$ is an ideal of Γ, we can factor $\mathfrak{f}$ as follows:

$$\mathfrak{f} = \prod_{p \in S} \mathfrak{f}_p, \tag{5.1}$$

where each $\mathfrak{f}_p$ is an ideal of Γ such that the radical of $\mathfrak{f}_p \cap \mathbb{Z}$ equals $p\mathbb{Z}$. Using the fact that the ideals $\mathfrak{f}_p \subseteq \Gamma$ are coprime in pairs, we have

$$\Gamma/\mathfrak{f} \cong \prod_{p \in S} \Gamma/\mathfrak{f}_p.$$

By considering the contraction of each ideal $\mathfrak{f}_p$ to R, we are led to the following.

Proposition 5.1. *With notation and definitions as above, we have a ring isomorphism*

$$R/\mathfrak{f} \cong \prod_{p \in S} (R + \mathfrak{f}_p)/\mathfrak{f}_p,$$

where each order $R + \mathfrak{f}_p$ is a p-order whose conductor equals $\mathfrak{f}_p$.

Proof. For each $p \in S$ we have a ring isomorphism $(R + \mathfrak{f}_p)/\mathfrak{f}_p \cong R/(\mathfrak{f}_p \cap R)$. Since $\mathfrak{f} = \bigcap(\mathfrak{f}_p \cap R)$, it suffices to show that the ideals $\mathfrak{f}_p \cap R$ are coprime in pairs. Take primes $p \neq q$ in S. Since $\mathfrak{f}_p + \mathfrak{f}_q = \Gamma$, there exist $x \in \mathfrak{f}_p$ and $y \in \mathfrak{f}_q$ such that $x + y = 1$.

Next, notice that $\mathfrak{f}_p \cap \mathbb{Z} = p^a\mathbb{Z}$ for some a, and likewise $\mathfrak{f}_q \cap \mathbb{Z} = q^b\mathbb{Z}$ for some b. It is easy to see that there exist x' and y' in Γ such that $x'x = p^a$ and $y'y = q^b$. Now we have

$$p^a = x'x \in \mathfrak{f}_p \cap \mathbb{Z} \subseteq \mathfrak{f}_p \cap R$$

and

$$q^b = y'y \in \mathfrak{f}_q \cap \mathbb{Z} \subseteq \mathfrak{f}_q \cap R.$$

Since p^a and q^b are relatively prime in $\mathbb{Z} \subseteq R$, it follows that $\mathfrak{f}_p \cap R + \mathfrak{f}_q \cap R = R$, as required.

Now, since $\mathfrak{f}$ is the conductor of R, the ring $\Gamma/\mathfrak{f}$ contains no nonzero $R/\mathfrak{f}$-ideals. It follows that $\Gamma/\mathfrak{f}_p$ contains no nonzero $(R + \mathfrak{f}_p)/\mathfrak{f}_p$-ideals. Hence, $\mathfrak{f}_p$ is the conductor of $R + \mathfrak{f}_p$. ☐

We now work with the p-orders. We assume R has the form $R(p;k,l,m)$ for some $k,l,m \geq 0$ and some fixed prime p, and $\mathfrak{f}$ is the conductor of R. We also assume (from Lemma 4.4 above, and up to permutation) that

$$\mathfrak{f} = (p^m, p^{k+2l}, p^m)\,\Gamma.$$

In what follows, we have $m > 0$, but $k+2l = 0$ is allowed.

Clearly, the order of the unit group of $\Gamma/\mathfrak{f}$ equals

$$\varphi(p^m)\varphi(p^{k+2l})\varphi(p^m) = (p-1)^2 p^{2(m-1)}\varphi(p^{k+2l}) \tag{5.2}$$

Additionally, we have the following result that gives the size of the unit group of $R/\mathfrak{f}$, assuming $R = R(p;k,l,m)$ as above.

Proposition 5.2. *With notation as above, let $R = R(p;k,l,m)$ with conductor $\mathfrak{f}$. Then the size of the unit group of $R/\mathfrak{f}$ equals $\varphi(p^m)p^l = (p-1)p^{m-1+l}$.*

Proof. By Corollary 4.3, every element ρ of R has the form

$$r(1,1,1) + s(0, p^{k+l}, ap^{k+l+q}) + t(0,0,p^m),$$

where $r,s,t \in \mathbb{Z}$.

Suppose ρ is a unit of R. Recall from Lemma 4.4 that the conductor of R is equal to (p^m, p^{k+2l}, p^m). We see that r uniquely determines a *unit* residue modulo p^m, and s uniquely determines a residue modulo p^l. Conversely, any triple of integers r,s,t such that p does not divide r gives a unit of R. $\square$

6. The sizes of the Picard groups. In this section, we begin by allowing R to be an arbitrary but proper order in $\Gamma = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ (i.e., R is not necessarily a p-order). Recall that Λ is the image of $\Gamma^\times$ in $(\Gamma/\mathfrak{f})^\times$ and has size at most 8. Since the image of $(-1,-1,-1)$ is already a unit in $R/\mathfrak{f}$, we see that, as subgroups of $(\Gamma/\mathfrak{f})^\times$,

$$|\Lambda \cdot (R/\mathfrak{f})^\times| \leq 4\,|(R/\mathfrak{f})^\times|.$$

Employing the isomorphism

$$\frac{(\Gamma/\mathfrak{f})^\times}{\Lambda\cdot(R/\mathfrak{f})^\times} \cong \operatorname{Pic} R,$$

we see that if the strict inequality

$$4\,|(R/\mathfrak{f})^\times| < |(\Gamma/\mathfrak{f})^\times| \tag{6.1}$$

holds, then $|\operatorname{Pic} R| > 1$, and hence $\operatorname{Pic} R$ is nontrivial.

Some special cases must be mentioned at this point. If $\mathfrak{f}\, e_i = 1$ or $\mathfrak{f}\, e_i = 2$ for one of the primitive idempotents e_i of Γ, then the size of Λ is now at most 4, but again $(-1, -1, -1)$ is a unit in $R/\mathfrak{f}$. Hence, we can conclude

$$|\Lambda \cdot (R/\mathfrak{f})^\times| \le 2\,|(R/\mathfrak{f})^\times|$$

so that the strict inequality

$$2\,|(R/\mathfrak{f})^\times| < |(\Gamma/\mathfrak{f})^\times| \tag{6.2}$$

implies the nontriviality of $\operatorname{Pic} R$ in these two special cases.

At this point, we specialize to p-orders R in Γ. Equation (5.2) and Proposition 5.2 above allow us to rewrite the inequality in (6.1) as

$$4 < (p-1)^2 p^{m-1} p^{k+l-1} \quad \text{if } k + 2l > 0, \tag{6.3}$$

and rewrite the inequality in (6.2) as

$$2 < (p-1) p^{m-1} \quad \text{if } k + 2l = 0 \tag{6.4}$$

(since $k+2l = 0$ implies $\mathfrak{f}\, e_i = 1$ for some e_i). In the special case where $\mathfrak{f}\, e_i = 2$ for some i, we can use the following:

$$2 < 2^{m-1} 2^{k+l-1} \quad \text{if } k + 2l > 0, \tag{6.5}$$

or

$$2 < 2^{m-1} \quad \text{if } k + 2l = 0. \tag{6.6}$$

We immediately put these inequalities to work.

Lemma 6.1. *Let $R \neq \Gamma$ be an order in $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with conductor $\mathfrak{f}$. Write*

$$\mathfrak{f} = \prod_{p \in S} \mathfrak{f}_p$$

as in (5.1) *above. Then* $\operatorname{Pic} R$ *is nontrivial whenever one of the following holds*:

(i) *S contains any prime $p \ge 5$;*

(ii) *S contains $p = 3$ and $R + \mathfrak{f}_3 = R(3; k, l, m)$ with $m \geq 2$;*
(iii) *S contains $p = 2$ and $R + \mathfrak{f}_2 = R(2; k, l, m)$ with $m \geq 3$.*

Proof. For every p in S we have $R \subseteq R + \mathfrak{f}_p$. Since $\operatorname{Pic} R$ maps surjectively onto $\operatorname{Pic}(R + \mathfrak{f}_p)$, it suffices to show that the latter is nontrivial.

(i) If $p \geq 5$, it follows from either (6.3) or (6.4) that $\operatorname{Pic}(R + \mathfrak{f}_p)$ is nontrivial.
(ii) If $p = 3$ and $m \geq 2$, we use (6.3) or (6.4) again to conclude that $\operatorname{Pic}(R + \mathfrak{f}_p)$ is nontrivial.
(iii) If $p = 2$ and $m \geq 3$, we use (6.5) or (6.6) to conclude that $\operatorname{Pic}(R + \mathfrak{f}_p)$ is nontrivial. $\square$

7. Proof of the main theorem. Before we state the main theorem, we recall two important facts concerning Picard groups, unit groups and torsion-free cancellation. The following is a direct restatement of Corollary 2.4 in [**6**].

Proposition 7.1. [**6**]. *Let R be a reduced, one-dimensional ring with integral closure $\mathcal{O}$. Assume $\mathcal{O}$ is finitely generated as an R-module. Let $\mathfrak{f} = (R : \mathcal{O})$ be the conductor of R.*

(i) *Torsion-free cancellation fails for R if the map $\operatorname{Pic} R \to \operatorname{Pic} \mathcal{O}$ is not injective.*
(ii) *Torsion-free cancellation holds for R if the map $\mathcal{O}^\times \to (\mathcal{O}/\mathfrak{f})^\times$ is surjective.*

In our context, $\mathcal{O} = \Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, so the conditions of Proposition 7.1 are clearly satisfied.

Theorem 7.2 (Main theorem). *Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and let R be an order in Γ. Assume that $R \neq \mathbb{Z} + (4, 6, 12)\,\Gamma$. Then R has torsion-free cancellation if and only if $\operatorname{Pic} R$ is trivial.*

Proof. The theorem is well known to be true if $R = \Gamma$, so we suppose throughout the proof that $R \neq \Gamma$.

First, suppose $\operatorname{Pic} R$ is nontrivial. By Proposition 7.1, torsion-free cancellation fails for R since $\operatorname{Pic}\Gamma = 1$. To establish the converse, we assume that $\operatorname{Pic} R$ is trivial for the remainder of the proof.

Let
$$\mathfrak{f} = (R : \Gamma) = \{x \in R : x\Gamma \subseteq R\}$$
be the conductor of R. As in (5.1), we factor $\mathfrak{f}$ in Γ as follows:
$$\mathfrak{f} = \prod_{p \in S} \mathfrak{f}_p.$$
By Lemma 6.1, $S \subseteq \{2, 3\}$. We now consider three cases.

(i) Suppose $S = \{3\}$ so that $\mathfrak{f} = \mathfrak{f}_3$. By Lemma 6.1, we must have $\mathfrak{f}_3 = (3, 1, 3)\,\Gamma$ or $\mathfrak{f}_3 = (3, 3, 3)\,\Gamma$, up to permutation. But then all units of $\Gamma/\mathfrak{f}$ lift to units of Γ. By Proposition 7.1, torsion-free cancellation holds for R and hence holds whenever $S = \{3\}$ and $\operatorname{Pic} R = 1$.

(ii) Suppose $S = \{2\}$ so that $\mathfrak{f} = \mathfrak{f}_2$. By Lemma 6.1, $\mathfrak{f}_2 = r\,\Gamma$, where r is equal to one of the following, up to permutation:

$$(2, 1, 2), (2, 2, 2), (4, 1, 4), (4, 2, 4), (4, 4, 4) \tag{7.1}$$

For each r above, all units of $\Gamma/\mathfrak{f}$ lift to units of Γ, so torsion-free cancellation holds (Proposition 7.1). Thus, torsion-free cancellation holds whenever $S = \{2\}$ and $\operatorname{Pic} R = 1$.

(iii) Finally, suppose $S = \{2, 3\}$. Then we have $\mathfrak{f} = \mathfrak{f}_2\,\mathfrak{f}_3$. In this case, by Proposition 5.1,
$$R/\mathfrak{f} \cong (R + \mathfrak{f}_2)/\mathfrak{f}_2 \times (R + \mathfrak{f}_3)/\mathfrak{f}_3.$$
Write $\mathfrak{f}_2 = r\,\Gamma$ and $\mathfrak{f}_3 = s\,\Gamma$ for some r and s in Γ. From Lemma 6.1, the only possibilities for s, up to permutation, are $(3, 1, 3)$ and $(3, 3, 3)$. Likewise, the only possibilities for r, up to permutation, are $(2, 1, 2)$, $(2, 2, 2)$, $(4, 1, 4)$, $(4, 2, 4)$ and $(4, 4, 4)$.

However, the two permutations (in the components of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$) invoked above need not be equal. We shall fix the permutation for $\mathfrak{f}_2$, so that all the possible choices for r are given in the list (7.1) above. Then, for each choice of r, s must equal one of

$$(1, 3, 3), (3, 1, 3), (3, 3, 1), (3, 3, 3). \tag{7.2}$$

If $r = (2,1,2)$ or $r = (2,2,2)$, then every possible choice for s implies that all units of $\Gamma/\mathfrak{f}$ lift to units of Γ. Hence, we may assume that $r = (4,1,4)$, $r = (4,2,4)$ or $r = (4,4,4)$. If $r = (4,4,4)$, then one can show that $\operatorname{Pic} R$ is nontrivial for every possible choice of s from the list (7.2).

Likewise, if either $r = (4,2,4)$ or $r = (4,1,4)$, but $s = (3,3,3)$, one can show that $\operatorname{Pic} R$ is nontrivial. Therefore, we are left with four possibilities:
(a) $r = (4,1,4)$ or $r = (4,2,4)$;
(b) $s = (1,3,3)$ or $s = (3,3,1)$.

By swapping e_1 with e_3, if necessary, we may now assume that either (a) $r = (4,1,4)$ and $s = (1,3,3)$ or (b) $r = (4,2,4)$ and $s = (1,3,3)$. Using Theorem 4.1, Proposition 5.1, and some analysis involving the generators listed in Corollary 4.3, we are left with the consideration of these two orders:

$$\mathbb{Z} + (4,6,12)\,\Gamma \subseteq \mathbb{Z} + (4,3,12)\,\Gamma.$$

However, it is easy to check that $\mathbb{Z} + (4,3,12)\,\Gamma$ has the property that every order between itself and Γ is Gorenstein. By [**6,** Proposition 2.6 and Theorem 2.7], this order has torsion-free cancellation.

In conclusion, we have shown in all three cases above that $\operatorname{Pic} R = 1$ implies torsion-free cancellation holds for R, provided $R \neq \mathbb{Z} + (4,6,12)\,\Gamma$. This establishes the converse, as required, and completes the proof of the Main Theorem. □

8. The exceptional order. The order 'in limbo' at the end of Theorem 7.2,

$$\mathbb{Z} + (4,6,12)\,\Gamma$$

is exceptional because it is one of the only two orders in $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ such that each possesses a trivial Picard group *and* each has units of $\Gamma/\mathfrak{f}$ that do *not* lift to units of Γ (where $\mathfrak{f}$ is the conductor of the order). The only other such order, as we saw (in the proof of the main theorem) above, is $\mathbb{Z} + (4,3,12)\,\Gamma$. For each of these two orders, Proposition 7.1 yields no conclusion. Hence, we were a bit 'lucky' in dispensing with the order $\mathbb{Z} + (4,3,12)\,\Gamma$.

It turns out that the remaining exceptional order has *finite representation type*. The author of this paper conjectures that this order does indeed have torsion-free cancellation, so that *every* order R in Γ has torsion-free cancellation if and only if $\operatorname{Pic} R$ is trivial.

If true, this situation stands in contrast with a cubic example found by the author in [**3**]: There is an order S in a cubic *field* extension of the rationals $\mathbb{Q}$ such that $\operatorname{Pic} S = 1$ but S does *not* have torsion-free cancellation. Of course, the total quotient ring of $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a cubic *algebra* over $\mathbb{Q}$, but not a field.

The calculations and techniques that support the following conjecture go beyond the scope of the present paper. However, we do offer a sketch of a strategy that might yield a proof of this conjecture.

Conjecture 8.1. *Let $R = \mathbb{Z} + (4, 6, 12)\,\Gamma$, where $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then torsion-free cancellation holds for R.*

Note that $\mathfrak{f} = (4, 6, 12)\,\Gamma$ is the conductor of R. We let $\mathfrak{f}_2 = (4, 2, 4)\,\Gamma$ and $\mathfrak{f}_3 = (1, 3, 3)\,\Gamma$. Thus, by Proposition 5.1,

$$R/\mathfrak{f} \cong (R + \mathfrak{f}_2)/\mathfrak{f}_2 \times (R + \mathfrak{f}_3)/\mathfrak{f}_3.$$

Furthermore, it is easy to see that $R + \mathfrak{f}_2 = \mathbb{Z} + \mathfrak{f}_2$ and $R + \mathfrak{f}_3 = \mathbb{Z} + \mathfrak{f}_3$. For ease of notation, let

$$R_2 = \mathbb{Z} + \mathfrak{f}_2 = \mathbb{Z} + (4, 2, 4)\,\Gamma$$

and

$$R_3 = \mathbb{Z} + \mathfrak{f}_3 = \mathbb{Z} + (1, 3, 3)\,\Gamma.$$

The inclusion $R/\mathfrak{f} \subseteq \Gamma/\mathfrak{f}$ is called an *Artinian pair*. The idea is to use results from [**8, 7, 2**] that reduce the question of torsion-free cancellation for R to certain *bimodules* defined over this pair. At this stage, we believe that there is a routine, but tedious, verification of the following steps that will lead to a proof.

1. Let M be an arbitrary torsion-free R-module, where $R = \mathbb{Z} + (4, 6, 12)\,\Gamma$. The conductor $\mathfrak{f}$ equals $(4, 6, 12)\,\Gamma$. By [**8**], it suffices to consider the bimodule $M/\mathfrak{f}M \subseteq \Gamma M/\mathfrak{f}M$ over the Artinian pair $R/\mathfrak{f} \subseteq \Gamma/\mathfrak{f}$. Call this bimodule $\mathcal{M}$.

2. By [**8**], it suffices to show that the *delta group* of $\mathcal{M}$, denoted Δ, which is a subgroup of $(\Gamma/\mathfrak{f})^{\times}$, has the property that $\Delta \cdot \Lambda = (\Gamma/\mathfrak{f})^{\times}$. (Recall that Λ is the group of units of $\Gamma/\mathfrak{f}$ that lift to units of Γ.)
3. Show that
$$\Delta \cong \Delta_2 \cdot \Delta_3 \leq \Gamma/\mathfrak{f}_2 \times \Gamma/\mathfrak{f}_3,$$
where Δ_2 and Δ_3 are two delta groups that correspond, respectively, to two bimodules, say $\mathcal{M}_2$ and $\mathcal{M}_3$, obtained from $\mathcal{M}$, one over the Artinian pair $R_2/\mathfrak{f}_2 \subseteq \Gamma/\mathfrak{f}_2$ and the other over the Artinian pair $R_3/\mathfrak{f}_3 \subseteq \Gamma/\mathfrak{f}_3$.
4. By [**8**], it suffices to assume that the bimodules $\mathcal{M}_2$ and $\mathcal{M}_3$, over their respective Artinian pairs, are indecomposable. (The main fact being that the delta group of a direct sum of bimodules equals the product of the delta groups.)
5. From [**2**], verify that each of the two Artinian pairs in step (3) above has *finite representation type*.
6. From [**2**], obtain a finite list of all (isomorphism classes of) indecomposable bimodules over the two Artinian pairs and compute all the delta groups involved.
7. From step (6), for each delta group Δ' of an indecomposable bimodule over the Artinian pair $R_2/\mathfrak{f}_2 \subseteq \Gamma/\mathfrak{f}_2$, verify that $\Delta' \cdot \Lambda = (\Gamma/\mathfrak{f}_2)^{\times}$. Do the same for each Δ'' obtained from an indecomposable bimodule over the Artinian pair $R_3/\mathfrak{f}_3 \subseteq \Gamma/\mathfrak{f}_2$ from step (6).
8. It should follow that $\Delta \cdot \Lambda = (\Gamma/\mathfrak{f})^{\times}$ for the delta group in Step (2) of the arbitrary bimodule $\mathcal{M}$ above from step (1), as was required.

This concludes a sketch of how to attack the conjecture regarding the exceptional order $\mathbb{Z} + (4, 6, 12)\,\Gamma$.

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