

## SKEWNESS, KURTOSIS AND NEWTON'S INEQUALITY

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ABSTRACT. We show that an inequality related to Newton's inequality provides one more relation between skewness and kurtosis. This also gives simple and alternative proofs of the bounds for skewness and kurtosis.

**1. Introduction.** The Pearson [4] inequality:

$$(1) \quad \alpha_4 \geq 1 + \alpha_3^2$$

gives a one-sided relation between skewness and kurtosis, respectively, defined as

$$(2) \quad \alpha_3 = \sqrt{\frac{m_3^2}{m_2^3}} \quad \text{and} \quad \alpha_4 = \frac{m_4}{m_2^2},$$

where

$$(3) \quad m_r = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^r, \quad r = 1, 2, \dots,$$

is the  $r$ th central moment and  $\bar{x}$  is the arithmetic mean of  $n$  real numbers  $x_i$  ( $i = 1, 2, \dots, n$ ). For two different proofs of (1), see [7, 8].

It is also in statistical interest to bound the sample statistics in terms of the sample size  $n$ . For example, the [5] inequality states that the standardized maximum deviate  $M - \bar{x}/s$  is bounded by  $\sqrt{n-1}$ , where  $M = \max_i x_i$  and  $s = \sqrt{m_2}$  is the standard deviation. Wilkins [8] uses the method of Lagrange multipliers to prove that

$$(4) \quad |\alpha_3| \leq \frac{n-2}{\sqrt{n-1}}.$$

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For an alternative proof of (4), see [6]. Dalen [1] proves an upper bound for the kurtosis,

$$(5) \quad \alpha_4 \leq \frac{n^2 - 3n + 3}{n - 1}.$$

For a brief history and motivation of these inequalities, see Nicholas [3]. We state here a two-sided relation between skewness and kurtosis and show that the inequalities (4) and (5) follow as a consequence. A refinement of the inequality (5) is given for symmetric distributions.

In an entirely different context, a result due to Newton [2] says that, if all the roots of the polynomial equation

$$(6) \quad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

are real, then

$$(7) \quad a_i^2 \geq \frac{n-i+1}{n-i} \frac{i+1}{i} a_{i+1}a_{i-1}, \quad i = 1, 2, \dots, n-1.$$

An inequality of our present concern is (7),  $i = 3$ ,

$$(8) \quad a_3^2 \geq \frac{4}{3} \frac{n-2}{n-3} a_2a_4.$$

We first prove a refinement of the inequality (8) for the case when the coefficient of  $x^{n-1}$  in (6) is zero.

**Lemma 1.1.** *If all the roots of the monic polynomial equation*

$$(9) \quad g(x) = x^n + b_2x^{n-2} + \cdots + b_n = 0$$

are real, then for  $n \geq 4$ ,

$$(10) \quad b_3^2 \geq \frac{16}{9} \frac{n-2}{n-3} b_2b_4.$$

*Proof.* The  $(n-4)$ th derivative of  $g(x)$  shows that the roots of polynomial equations,

$$(11) \quad n(n-1)(n-2)(n-3)x^4 + 12(n-2)(n-3)b_2x^2 + 24(n-3)b_3x + 24b_4 = 0$$

and

$$(12) \quad h(y) = 24b_4y^4 + 24(n-3)b_3y^3 + 12(n-2)(n-3)b_2y^2 + n(n-1)(n-2)(n-3) = 0,$$

are all real,  $b_4 \neq 0$ . Likewise, it follows on differentiating  $h(y)$  that the roots of the equation

$$(13) \quad 4b_4y^2 + 3(n-3)b_3y + (n-2)(n-3)b_2 = 0$$

are real. The inequality (10) now follows from the fact that the discriminant of the quadratic equation (13) is non-negative. For  $b_4 = 0$ , (10) is obvious.  $\square$

We now show that the inequality (10) yields an interesting relation between skewness and kurtosis.

**Theorem 1.2.** *Let  $\alpha_3$  and  $\alpha_4$  be, respectively, the skewness and kurtosis of  $n$  real numbers  $x_1, x_2, \dots, x_n$ . Then*

$$(14) \quad 1 + \alpha_3^2 \leq \alpha_4 \leq \frac{1}{2} \frac{n-3}{n-2} \alpha_3^2 + \frac{n}{2}, \quad n \geq 3.$$

*Proof.* On using the well-known Newton's identity,

$$\alpha_k + b_1\alpha_{k-1} + b_2\alpha_{k-2} + \dots + b_{k-1}\alpha_1 + kb_k = 0,$$

where

$$\alpha_k = \sum_{i=1}^n x_i^k, \quad k = 1, 2, \dots, n,$$

we have the following relations between the moments (3) and coefficients  $b_i$  in polynomial equation (9),

$$m_2 = -\frac{2}{n}b_2, \quad m_3 = -\frac{3}{n}b_3, \quad m_4 = \frac{2}{n}(b_2^2 - 2b_4).$$

Equivalently,

$$(15) \quad b_2 = -\frac{n}{2}m_2, \quad b_3 = -\frac{n}{3}m_3, \quad b_4 = \frac{n^2}{8}m_2^2 - \frac{n}{4}m_4.$$

Substituting the values of  $b_2, b_3$  and  $b_4$  from (15) in (10), a simple calculation leads to second inequality (14). If  $b_4 = 0, 2m_4 = nm_2^2$

and the second inequality (14) is obviously true. The first inequality is (1).  $\square$

The above theorem provides a complete relation between skewness and kurtosis. The first inequality (14) is Pearson's inequality and the second inequality (14) is its complementary. The inequalities (4) and (5) are subsumed in (14). From (14), we have

$$(16) \quad 1 + \alpha_3^2 \leq \frac{1}{2} \frac{n-3}{n-2} \alpha_3^2 + \frac{n}{2}.$$

Inequality (4) follows easily from (16). Further, the limits of kurtosis in (14) increase with absolute values of skewness. The upper bound for the kurtosis therefore corresponds to the maximum value of  $\alpha_3$ . Substitute  $\alpha_3 = n - 2/\sqrt{n-1}$  in the second inequality (14), a little calculation leads to (5).

If  $\alpha_3 = 0$ , as in the case of symmetric distribution, a refinement of inequality (5) follows from (14),

$$\alpha_4 \leq \frac{n}{2}.$$

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