DING PROJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING BIMODULE

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ABSTRACT. Let R and S be rings and $_SC_R$ a faithfully semidualizing bimodule. A left S-module M is called *Ding C-projective* if there exists an exact sequence of C-projective left S-modules

$$X = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

such that $M \cong \operatorname{Coker}(C \otimes_R P_1 \to C \otimes_R P_0)$ and the complexes $\operatorname{Hom}_S(C \otimes_R P, X)$ and $\operatorname{Hom}_S(X, C \otimes_R F)$ are exact for any projective left R-module P and any flat left R-module F. The properties of Ding C-projective modules and dimensions are given. Among others, the Foxby equivalences between some subclasses of the Auslander class and the Bass class are also investigated.

Introduction. Throughout this work, R and S are fixed associated rings with identity and all R- and S-modules are understood to be unitary left R- and S-modules. Right R- and S-modules are identified with left modules over the opposite rings R^{op} and S^{op}. Also, $C = {}_SC_R$ is a fixed faithfully semidualizing bimodule, cf., Definition 1.1 below.

In basic homological algebra, projective, injective and flat modules play an important and fundamental role. Over a commutative Noetherian ring, Holm and Jørgensen [9] introduced the C-Gorenstein projective and C-Gorenstein injective modules using semidualizing modules and their associated projective and injective classes, and they connected the study of semidualizing modules to associated Auslander and Bass classes for semidualizing modules, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$, which are subcategories of the category of R-modules. White [17] further considered these modules when R is a commutative ring, and she called C-Gorenstein projective modules as G_C-projectives and C-Gorenstein

injective modules as G_C-injectives. In particular, many results about the Gorenstein projectivity and Gorenstein injectivity in [**3, 8, 10**] were generalized in [**17**]. Recently, Liu et al. [**11**] extended these modules to arbitrary associative rings; also, they have proved that many results for Auslander and Bass classes associated to a semidualizing module over a commutative Noetherian ring can be extended to arbitrary associative rings.

For a faithfully semidualizing bimodule $_SC_R$, denote by $\mathcal{GP}(R)$, $\mathcal{GI}(S)$, $\mathcal{GP}_C(S)$ and $\mathcal{GI}_C(R)$ the classes of Gorenstein projective, Gorenstein injective, G_C-projective and G_C-injective modules, respectively. From [**11,** Theorem 4.6], we have the following Foxby equivalence diagram:

$$
\begin{array}{ccc}
\mathcal{A}_C(R) \cap \mathcal{GP}(R) & \underset{\underset{\text{Hom}_S(C,-)}{\longleftarrow}}{\overset{C \otimes_R -}{\underset{\sim}{\longrightarrow}}} & \mathcal{B}_C(S) \cap \mathcal{GP}_C(S) \\
\big\uparrow & & \big\uparrow \\
\mathcal{A}_C(R) & \underset{\underset{\text{Hom}_S(C,-)}{\longleftarrow}}{\overset{C \otimes_R -}{\underset{\sim}{\longrightarrow}}} & \mathcal{B}_C(S) \\
\big\uparrow & & \big\uparrow \\
\mathcal{A}_C(R) \cap \mathcal{GI}_C(R) & \underset{\underset{\text{Hom}_S(C,-)}{\longleftarrow}}{\overset{C \otimes_R -}{\underset{\sim}{\longrightarrow}}} & \mathcal{B}_C(S) \cap \mathcal{GI}(S).
\end{array}
$$

On the other hand, Ding et al. [**2, 13**] considered two special cases of the Gorenstein projective and Gorenstein injective modules using projective, flat classes and injective, FP-injective classes, which they called strongly Gorenstein flat and Gorenstein FP-injective modules, respectively. The same modules were studied by Gillespie [**6**] with the different names: Ding projective and Ding injective modules, respectively. By $\mathcal{DP}(R)$ and $\mathcal{DI}(S)$, we denote the classes of Ding projective and Ding injective modules, respectively. Thus, for a faithfully semidualizing bimodule $_SC_R$, a natural question arises: What are the counterparts to the Ding projective and Ding injective modules under Foxby equivalence? So, the motivation of this article is the "?" and

"??" in the following Foxby equivalence diagram:

$$
\begin{array}{ccc}
\mathcal{A}_C(R) \cap \mathcal{DP}(R) & \underset{\underset{\mathrm{Hom}_S(C,-)}{\sim}}{\overset{C\otimes_R-}{\rightleftarrows}} & ? \\
\downarrow & & \uparrow \\
\mathcal{A}_C(R) \cap \mathcal{GP}(R) & \underset{\underset{\mathrm{Hom}_S(C,-)}{\sim}}{\overset{C\otimes_R-}{\rightleftarrows}} & \mathcal{B}_C(S) \cap \mathcal{GP}_C(S) \\
\uparrow & & \downarrow \\
\mathcal{A}_C(R) & \underset{\underset{\mathrm{Hom}_S(C,-)}{\sim}}{\overset{C\otimes_R-}{\rightleftarrows}} & \mathcal{B}_C(S) \\
\uparrow & & \uparrow \\
\mathcal{A}_C(R) \cap \mathcal{GI}_C(R) & \underset{\underset{\mathrm{Hom}_S(C,-)}{\sim}}{\overset{C\otimes_R-}{\rightleftarrows}} & \mathcal{B}_C(S) \cap \mathcal{GI}(S) \\
\uparrow & & \uparrow \\
?? & \underset{\underset{\mathrm{Hom}_S(C,-)}{\sim}}{\overset{C\otimes_R-}{\rightleftarrows}} & \mathcal{B}_C(S) \cap \mathcal{DI}(S).
\end{array}
$$

We shall introduce the notions of Ding C-projective and Ding C-injective modules which play the roles of "?" and "??".

In Section 2, we define and study Ding C-projective and Ding C-injective modules. An S-module M is called Ding C-projective if there exists an exact sequence of C-projective S-modules

$$X = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots$$

such that $M \cong \mathrm{Coker}\,(C \otimes_R P_1 \to C \otimes_R P_0)$ and the complexes $\mathrm{Hom}_S(C \otimes_R P, X)$ and $\mathrm{Hom}_S(X, C \otimes_R F)$ are exact for any projective R-module P and any flat R-module F. An R-module N is called *Ding C-injective* if there exists an exact sequence of C-injective R-modules

$$X = \cdots \longrightarrow \mathrm{Hom}_S(C, I_1) \longrightarrow \mathrm{Hom}_S(C, I_0)$$
$$\longrightarrow \mathrm{Hom}_S(C, I^0) \longrightarrow \mathrm{Hom}_S(C, I^1) \longrightarrow \cdots$$

such that $N \cong \mathrm{Coker}\,(\mathrm{Hom}_S(C, I_1) \to \mathrm{Hom}_S(C, I_0))$ and the complexes $\mathrm{Hom}_R(X,\, \mathrm{Hom}_S(C, I))$ and $\mathrm{Hom}_R(\mathrm{Hom}_S(C, E), X)$ are exact for any injective S-module I and any FP-injective S-module E. We prove that the class of Ding C-projective (Ding C-injective) modules is closed under extensions and direct summands.

Section 3 is devoted to investigating the Ding C-projective (Ding C-injective) dimensions of modules. We give functorial descriptions of

these dimensions in Theorem 3.3.

In Section 4, we prove that the subcategories of Ding projective R-modules in the Auslander class $\mathcal{A}_C(R)$ (Ding injective S-modules in the Bass class $\mathcal{B}_C(S)$) and Ding C-projective S-modules (Ding C-injective R-modules) are equivalent under Foxby equivalence.

1. Preliminaries. In this section, we shall recall some notions and definitions which we need in the later sections.

The study of semidualizing modules over commutative Noetherian rings was initiated independently (with different names) by Foxby [**4**], Golod [**7**], and Vasconcelos [**16**].

Definition 1.1. Let R and S be rings. Following [**10**], an (S, R)-bimodule $_SC_R$ is *semidualizing* if

(1) $_SC$ admits a degreewise finite S-projective resolution.
(2) C_R admits a degreewise finite R^{op}-projective resolution.
(3) The homothety map $_SS_S \overset{_S\gamma}{\rightarrow} \mathrm{Hom}_R(C, C)$ is an isomorphism.
(4) The homothety map $_RR_R \overset{\gamma_R}{\rightarrow} \mathrm{Hom}_S(C, C)$ is an isomorphism.
(5) $\mathrm{Ext}_S^{\geqslant 1}(C, C) = 0$.
(6) $\mathrm{Ext}_{R^{op}}^{\geqslant 1}(C, C) = 0$.

A semidualizing bimodule $_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules $_SN$ and M_R:

(1) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$.
(2) If $\mathrm{Hom}_{R^{op}}(C, M) = 0$, then $M = 0$.

Let $C = {}_SC_R$ be a semidualizing bimodule. We set $\mathcal{P}_C(S)$ to be the subcategory of modules $C \otimes_R P$ where P is R-projective, $\mathcal{F}_C(S)$ to be the subcategory of modules $C \otimes_R F$ where F is R-flat, $\mathcal{I}_C(R)$ to be the subcategory of modules $\mathrm{Hom}_S(C, I)$ where I is S-injective and $\mathcal{FI}_C(R)$ to be the subcategory of modules $\mathrm{Hom}_S(C, E)$ where E is S-FP-injective. Modules in $\mathcal{P}_C(S)$, $\mathcal{F}_C(S)$, $\mathcal{I}_C(R)$ and $\mathcal{FI}_C(R)$ are called *C-projective*, *C-flat*, *C-injective* and *C-FP-injective*, respectively. By setting $_SC_R = {}_RR_R$ in the definitions above we see that $\mathcal{P}_R(R)$, $\mathcal{F}_R(R)$, $\mathcal{I}_R(R)$ and $\mathcal{FI}_R(R)$ are the classes of ordinary *projective*, *flat*, *injective* and FP-*injective* R-modules, respectively, which we denote by $\mathcal{P}(R)$, $\mathcal{F}(R)$, $\mathcal{I}(R)$ and $\mathcal{FI}(R)$.

Note that all semidualizing modules are faithfully semidualizing over a commutative ring (see [**10**, Proposition 3.1]).

Definition 1.2. Let $_SC_R$ be a semidualizing bimodule.

The *Auslander class* $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M)$, and
(2) the natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of R-modules).

The *Bass class* $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying

(1) $\mathrm{Ext}^{\geqslant 1}_S(C, N) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C, N))$, and
(2) the natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C, N) \to N$ is an isomorphism (of S-modules).

The following notions were introduced by Holm and Jørgensen [**9**] over commutative Noetherian rings and by White [**17**] for commutative rings.

Definition 1.3. A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules satisfying the following conditions:

(1) X is exact and $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact; and
(2) X_i is projective for $i \geq 0$ and X_i is C-projective for $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Coker}(\partial^X_1)$, and in this case X is called a *complete $\mathcal{PP}_C$-resolution of M*. Set

$$\mathcal{GP}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-projective } R\text{-modules.}$$

In the case $C = R$ we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules such that:

(1) Y is exact and $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact; and
(2) Y_i is injective for $i \leq 0$ and Y_i is C-injective for $i > 0$.

An R-module N is G_C-*injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $N \cong \mathrm{Ker}(\partial_0^Y)$, and in this case Y is called a *complete $\mathcal{I}_C\mathcal{I}$-resolution of N*. Set

$$\mathcal{GI}_C(R) = \text{the subcategory of } G_C\text{-injective } R\text{-modules.}$$

In the case $C = R$, we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

Definition 1.4. An R-module M is called *Ding projective* if there exists an exact sequence of projective R-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M = \mathrm{Coker}(P_1 \to P_0)$ and which remains exact after applying $\mathrm{Hom}_R(-, \mathcal{F}(R))$. Set

$$\mathcal{DP}(R) = \text{the subcategory of Ding projective } R\text{-modules.}$$

An S-module N is called *Ding injective* if there exists an exact sequence of injective S-modules

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $N = \mathrm{Coker}(I_1 \to I_0)$ and which remains exact after applying $\mathrm{Hom}_S(\mathcal{FI}(S), -)$. Set

$$\mathcal{DI}(S) = \text{the subcategory of Ding injective } S\text{-modules.}$$

Note that every Ding projective (respectively, Ding injective) module is Gorenstein projective (respectively, Gorenstein injective), and if R is Gorenstein, then every Gorenstein projective (respectively, Gorenstein injective) module is Ding projective (respectively, Ding injective).

Definition 1.5. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An *$\mathcal{X}$-resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$, and the following exact sequence is the *augmented $\mathcal{X}$-resolution* of M associated to X:

$$X^+ = \cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0.$$

The $\mathcal{X}$-*projective dimension* of M is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, one has $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of the $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$. We set

$$\overline{\mathcal{X}} = \text{the subcategory of } R\text{-modules with } \mathcal{X}\text{-pd}_R(M) < \infty.$$

An $\mathcal{X}$-resolution X of M is *proper* if the augmented resolution X^+ is $\text{Hom}_R(\mathcal{X}, -)$-exact.

We define (*proper*) $\mathcal{X}$-*coresolution* and $\mathcal{X}$-*injective dimension* dually. And the $\mathcal{X}$-*injective dimension* of M is denoted $\mathcal{X}\text{-id}_R(M)$.

Fact 1.6. *Let $_SC_R$ be a semidualizing bimodule. The Auslander class $\mathcal{A}_C(R)$ contains every projective and C-injective R-module, and the Bass class $\mathcal{B}_C(S)$ contains every injective and C-projective S-module; see* [**10,** *Lemmas 4.1 and 5.1*]. *These classes also satisfy the two-of-three property by* [**10,** *Corollary 6.3*]. *Hence, $\mathcal{A}_C(R)$ contains the R-modules of finite projective dimension and those of finite $\mathcal{I}_C$-injective dimension, and $\mathcal{B}_C(S)$ contains the S-modules of finite injective dimension and those of finite $\mathcal{P}_C$-projective dimension. From* [**5,** *Lemma 3.9*] *we know that an S-module M is in $\mathcal{B}_C(S)$ if and only if $\text{Hom}_S(C, M)$ is in $\mathcal{A}_C(R)$, and that $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(S)$.*

Let $\mathcal{C}$ be a class of modules. We define, for $i \geqslant 1$,

$$^{\perp}\mathcal{C} = \bigcap_{i=1}^{\infty} {}^{\perp_i}\mathcal{C}, \text{ where } {}^{\perp_i}\mathcal{C} = \{X \mid \text{Ext}^i(X, C) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\mathcal{C}^{\perp} = \bigcap_{i=1}^{\infty} \mathcal{C}^{\perp_i}, \text{ where } \mathcal{C}^{\perp_i} = \{X \mid \text{Ext}^i(C, X) = 0 \text{ for all } C \in \mathcal{C}\}.$$

2. Ding C-projective modules. The Ding projective modules of interest in this paper are built from semidualizing modules and their associated C-projective and C-flat classes.

Definition 2.1. A *complete $\mathcal{D}(\mathcal{P}_C)$-resolution* is an exact sequence of C-projective S-modules

$$X = \cdots C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots$$

such that X is $\mathrm{Hom}_S(\mathcal{P}_C(S), -)$- and $\mathrm{Hom}_S(-, \mathcal{F}_C(S))$-exact.

An S-module M is called *Ding C-projective* if there exists a complete $\mathcal{D}(\mathcal{P}_C)$-resolution X with $M \cong \mathrm{Coker}\,(C \otimes_R P_1 \to C \otimes_R P_0)$.

A *complete $\mathcal{D}(\mathcal{I}_C)$-resolution* is an exact sequence of C-injective R-modules

$$Y = \cdots \longrightarrow \mathrm{Hom}_S(C, I_1) \longrightarrow \mathrm{Hom}_S(C, I_0)$$
$$\longrightarrow \mathrm{Hom}_S(C, I^0) \longrightarrow \mathrm{Hom}_S(C, I^1) \longrightarrow \cdots$$

such that Y is $\mathrm{Hom}_R(\mathcal{FI}_C(R), -)$- and $\mathrm{Hom}_R(-, \mathcal{I}_C(R))$-exact.

An R-module N is called *Ding C-injective* if there exists a complete $\mathcal{D}(\mathcal{I}_C)$-resolution Y with $N \cong \mathrm{Coker}\,(\mathrm{Hom}_S(C, I_1) \to \mathrm{Hom}_S(C, I_0))$. We set

$$\mathcal{D}(\mathcal{P}_C) = \mathcal{D}(\mathcal{P}_C(S))$$
$$= \text{the subcategory of Ding } C\text{-projective } S\text{-modules},$$
$$\mathcal{D}(\mathcal{I}_C) = \mathcal{D}(\mathcal{I}_C(R))$$
$$= \text{the subcategory of Ding } C\text{-injective } R\text{-modules}.$$

Note that, when $_S C_R = {_R}R_R$, the definitions above correspond to the definitions of Ding projective and Ding injective modules.

Remark 2.2. In the following, we shall only deal with the results concerning Ding C-projective modules. But it should be pointed out that all of the obtained results have Ding C-injective counterparts by using dual arguments.

Fact 2.3.

(1) *It is clear that every C-projective S-module is Ding C-projective.*
(2) *The class $\mathcal{D}(\mathcal{P}_C)$ is closed under direct sums by definition.*
(3) *If $X = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$
is a $Hom_S(\mathcal{P}_C(S), -)$- and $Hom_S(-, \mathcal{F}_C(S))$-exact sequence of C-*

projective S-modules, then, by symmetry, all the images, the kernels and the cokernels of X are Ding C-projective.
(4) By [**5,** Proposition 3.6], we have $\mathcal{D}(\mathcal{P}_C) \subseteq \mathcal{B}_C(S) \cap \mathcal{GP}_C(S)$.

Notation 2.4. Let $_SC_R$ be a semidualizing bimodule. We use the following abbreviations.

$$\mathrm{pd}_R(-) = \mathcal{P}(R)\text{-pd}(-) \qquad\qquad \mathrm{id}_S(-) = \mathcal{I}(S)\text{-id}(-)$$
$$\mathcal{P}_C\text{-pd}_S(-) = \mathcal{P}_C(S)\text{-pd}(-) \qquad \mathcal{I}_C\text{-id}_R(-) = \mathcal{I}_C(R)\text{-id}(-)$$
$$\mathcal{F}_C\text{-pd}_S(-) = \mathcal{F}_C(S)\text{-pd}(-) \qquad \mathcal{FI}_C\text{-id}_R(-) = \mathcal{FI}_C(R)\text{-id}(-)$$
$$\mathrm{Dpd}_R(-) = \mathcal{DP}(R)\text{-pd}(-) \qquad\qquad \mathrm{Did}_S(-) = \mathcal{DI}(S)\text{-id}(-)$$
$$\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(-) = \mathcal{D}(\mathcal{P}_C(S))\text{-pd}(-) \quad \mathcal{D}(\mathcal{I}_C)\text{-id}_R(-) = \mathcal{D}(\mathcal{I}_C(R))\text{-id}(-).$$

The following proposition is immediate by definition.

Proposition 2.5. An S-module M is Ding C-projective if and only if:

(1) $M \in {}^\perp\mathcal{F}_C \cap \mathcal{P}_C^\perp$,
(2) M has a proper $\mathcal{P}_C$-resolution,
(3) M has a $\mathrm{Hom}_S(-, \mathcal{F}_C)$-exact $\mathcal{P}_C$-coresolution.

Remark 2.6. If L is an S-module with $\mathcal{F}_C\text{-pd}_S(L) < \infty$, then $L \in \mathcal{D}(\mathcal{P}_C)^\perp$ by dimension shifting.

Theorem 2.7. The class $\mathcal{D}(\mathcal{P}_C)$ is closed under extensions.

Proof. Just use Proposition 2.5 and the "Horseshoe lemma" ([**3,** Lemma 8.2.1] and [**8,** (1.7)]). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.8. The class $\mathcal{D}(\mathcal{P}_C)$ is closed under direct summands.

Proof. Let $M \oplus N$ be a Ding C-projective S-module. Then there is a $\mathrm{Hom}_S(\mathcal{P}_C, -)$- and $\mathrm{Hom}_S(-, \mathcal{F}_C)$-exact sequence of C-projective S-modules

$$X = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

with $M \oplus N = \operatorname{Coker}(C \otimes_R P_1 \to C \otimes_R P_0)$. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & M \oplus N & \longrightarrow & N & \longrightarrow & 0 \\
& & \Big\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C \otimes_R P^0 & \longrightarrow & L & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & K & =\!=\!= & K & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0. & &
\end{array}
$$

Note that the exactness of $0 \to N \to L \to K \to 0$ gives the exact sequence $0 \to M \oplus N \to M \oplus L \to K \to 0$, since $M \oplus N$ and K are Ding C-projective, $M \oplus L$ is also Ding C-projective by Theorem 2.7. Thus, L is also a direct summand of a Ding C-projective module, and so $L \in {}^{\perp}\mathcal{F}_C$. By repeating the above process, we get a $\operatorname{Hom}_S(-, \mathcal{F}_C)$-exact $\mathcal{P}_C$-coresolution of M. Dually, M has a proper $\mathcal{P}_C$-resolution. Note that $M \oplus N \in \mathcal{P}_C^{\perp} \cap {}^{\perp}\mathcal{F}_C$. Then $M \in \mathcal{P}_C^{\perp} \cap {}^{\perp}\mathcal{F}_C$. Thus, M is Ding C-projective by Proposition 2.5. $\qquad\square$

3. Ding C-projective dimensions of modules. In this section, we investigate some properties of the Ding C-projective dimensions of modules.

Proposition 3.1. *Let M be an S-module. Then M has a $\mathcal{P}_C(S)$-resolution if and only if M has a $\mathcal{D}(\mathcal{P}_C)$-resolution.*

Proof. It is sufficient to show the "if" part. Let $0 \to N \to D_0 \to M \to 0$ be an exact sequence with $D_0 \in \mathcal{D}(\mathcal{P}_C)$ and N having a $\mathcal{D}(\mathcal{P}_C)$-resolution. Since D_0 is Ding C-projective, there exists a short exact sequence of S-modules $0 \to D' \to C \otimes_R P_0 \to D_0 \to 0$ with $P_0 \in \mathcal{P}(R)$

and $D' \in \mathcal{D}(\mathcal{P}_C)$. Consider the following pullback diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
D' & =\!\!=\!\!= & D' \\
\downarrow & & \downarrow \\
0 \longrightarrow H \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0 \\
\downarrow & & \downarrow & & \| \\
0 \longrightarrow N \longrightarrow D_0 \longrightarrow M \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

And, we have the following pullback diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K & =\!\!=\!\!= & K \\
\downarrow & & \downarrow \\
0 \longrightarrow D' \longrightarrow L \longrightarrow D_1 \longrightarrow 0 \\
\| & & \downarrow & & \downarrow \\
0 \longrightarrow D' \longrightarrow H \longrightarrow N \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
$$

where K has a $\mathcal{D}(\mathcal{P}_C)$-resolution and $D_1 \in \mathcal{D}(\mathcal{P}_C)$. So $L \in \mathcal{D}(\mathcal{P}_C)$ by Theorem 2.7, and then H has a $\mathcal{D}(\mathcal{P}_C)$-resolution. Note that $0 \rightarrow H \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0$ is exact from the first diagram. By repeating the preceding process, we have that M has a $\mathcal{P}_C(S)$-resolution. $\qquad\square$

Remark 3.2. Let M be an S-module with $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = n \geq 1$, and let $0 \rightarrow N \rightarrow D_0 \rightarrow M \rightarrow 0$ be an exact sequence with $D_0 \in \mathcal{D}(\mathcal{P}_C)$ and $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(N) = n - 1$. By the proof of the proposition above, we have an exact sequence $0 \rightarrow H \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0$ such that P_0 is projective and $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(H) = \mathcal{D}(\mathcal{P}_C)\text{-pd}_S(N)$.

The main result of this section gives both functorial descriptions of Ding C-projective dimensions of modules and some criteria for computing Ding C-projective dimensions of modules.

Theorem 3.3. *Let M be an S-module with finite Ding C-projective dimension and n a nonnegative integer. Then the following are equivalent*:

(1) $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) \leq n$.
(2) *There is an exact sequence* $0 \to D \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$ *with* $P_i \in \mathcal{P}(R)$ *for* $0 \leqslant i \leqslant n-1$ *and* $D \in \mathcal{D}(\mathcal{P}_C)$.
(3) *M has a proper $\mathcal{D}(\mathcal{P}_C)$-resolution of length n.*
(4) *There is an exact sequence* $0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to D \to M \to 0$ *with* $P_i \in \mathcal{P}(R)$ *for* $1 \leqslant i \leqslant n$ *and* $D \in \mathcal{D}(\mathcal{P}_C)$.
(5) *There is an exact sequence* $0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_{i+1} \to D \to C \otimes_R P_{i-1} \cdots \to C \otimes_R P_0 \to M \to 0$ *with* $P_j \in \mathcal{P}(R)$ *for* $1 \leqslant j \leqslant n$, $j \neq i$, $0 \leqslant i \leqslant n$ *and* $D \in \mathcal{D}(\mathcal{P}_C)$.
(6) $\text{Ext}^i_S(M, C \otimes_R F) = 0$ *for all* $i > n$ *and all flat R-modules F.*
(7) $\text{Ext}^i_S(M, L) = 0$ *for all* $i > n$ *and all S-modules L with* $\mathcal{F}_C\text{-pd}_S(L) < \infty$.
(8) $\text{Ext}^{n+1}_S(M, L) = 0$ *for all S-modules L with* $\mathcal{F}_C\text{-pd}_S(L) < \infty$.

Consequently, the Ding C-projective dimension of M is determined by the formulas:

$$\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = \sup\{i \in \mathbb{N}_0 \mid \exists L \in \overline{\mathcal{F}_C(S)} : \text{Ext}^i_S(M, L) \neq 0\}$$
$$= \sup\{i \in \mathbb{N}_0 \mid \exists Q \in \mathcal{F}_C(S) : \text{Ext}^i_S(M, Q) \neq 0\}.$$

Proof. We first prove the equivalences of (1)–(5). The case $n = 0$ is trivial. Now, we assume that $n \geq 1$.

(1) $\Rightarrow$ (2). By (1), there exists an exact sequence $0 \to N \to D_0 \to M \to 0$ with $D_0 \in \mathcal{D}(\mathcal{P}_C)$ and $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(N) \leq n-1$. By Remark 3.2, we have an exact sequence $0 \to H \to C \otimes_R P_0 \to M \to 0$ such that $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(H) = \mathcal{D}(\mathcal{P}_C)\text{-pd}_S(N)$. By repeating this process, we have an exact sequence $0 \to D_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$ with $P_i \in \mathcal{P}(R)$ for all $0 \leqslant i \leqslant n-1$ and $D_n \in \mathcal{D}(\mathcal{P}_C)$.

(2) $\Rightarrow$ (3).   Since D is Ding C-projective by (2), there is a $\operatorname{Hom}_S(-, \mathcal{F}_C(S))$-exact sequence $0 \to D \to C \otimes_R P^0 \to \cdots \to C \otimes_R P^{n-1} \to D' \to 0$ with each $P^i \in \mathcal{P}(R)$ and $D' \in \mathcal{D}(\mathcal{P}_C)$. So the diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & D & \longrightarrow & C \otimes_R P^0 & \longrightarrow & \cdots & \longrightarrow & C \otimes_R P^{n-1} & \longrightarrow & D' & \longrightarrow 0 \\
& & \| & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & D & \longrightarrow & C \otimes_R P_{n-1} & \longrightarrow & \cdots & \longrightarrow & C \otimes_R P_0 & \longrightarrow & M & \longrightarrow 0
\end{array}
$$

can be completed to a commutative diagram by the comparison lemma [**8**, Proposition 1.8].   Then the mapping cone $0 \to D \to D \oplus (C \otimes_R P^0) \to C \otimes_R (P_{n-1} \oplus P^1) \to \cdots \to C \otimes_R (P_1 \oplus P^{n-1}) \to (C \otimes_R P_0) \oplus D' \to M \to 0$ gives an exact sequence $X_\bullet = 0 \to C \otimes_R P^0 \to C \otimes_R (P_{n-1} \oplus P^1) \to \cdots \to C \otimes_R (P_1 \oplus P^{n-1}) \to (C \otimes_R P_0) \oplus D' \to M \to 0$. Note that each cokernel of $X_\bullet$ except M has a finite $\mathcal{P}_C(S)$-resolution. So $X_\bullet$ is $\operatorname{Hom}_S(\mathcal{D}(\mathcal{P}_C), -)$-exact by Remark 2.6. It follows that $X_\bullet$ is just a proper $\mathcal{D}(\mathcal{P}_C)$-resolution of M of length n.

(3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are trivial.

(2) $\Rightarrow$ (4). Note that $X_\bullet$ in the proof of (2) $\Rightarrow$ (3) is just the desired exact sequence.

(1) $\Rightarrow$ (5) is immediate by Remark 3.2, and the equivalence of (1) and (4).

Next we show the equivalences of (1), (6), (7) and (8).

(1) $\Rightarrow$ (6). By hypothesis, there exists an exact sequence $0 \to D_n \to \cdots \to D_1 \to D_0 \to M \to 0$ with $D_i \in \mathcal{D}(\mathcal{P}_C)$ for $0 \leqslant i \leqslant n$. So $\operatorname{Ext}_S^{n+j}(M, C \otimes_R F) \cong \operatorname{Ext}_S^j(D_n, C \otimes_R F) = 0$ for all $j > 0$ and all flat R-modules F by Proposition 2.5.

(6) $\Rightarrow$ (7) by the usual dimension shifting argument.

(7) $\Rightarrow$ (8) is clear.

(8) $\Rightarrow$ (1). By assumption, let $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = s < \infty$. If $s \leqslant n$, there is nothing to prove. So we assume $s > n$. Then there is an exact sequence $0 \to D_s \to \cdots \to D_1 \to D \to M \to 0$ with $D_i \in \mathcal{P}_C(S)$ for $1 \leqslant i \leqslant s$ and $D \in \mathcal{D}(\mathcal{P}_C)$ by the equivalence of (1) and (4). Let $K_i = \operatorname{Coker}(D_{i+1} \to D_i)$ for $1 \leqslant i \leqslant s-1$.

If $n = 0$, then $\operatorname{Ext}_S^{n+1}(M, K_1) = 0$ by (8) since $\mathcal{F}_C\text{-pd}_S(K_1) < \infty$. Thus, the exact sequence $0 \to K_1 \to D \to M \to 0$ splits, and so

$M \in \mathcal{D}(\mathcal{P}_C)$ by Theorem 2.8, as desired.

Let $n \geqslant 1$. Since $\mathcal{F}_C\text{-pd}_S(K_{n+1}) < \infty$, we have that $\text{Ext}_S^1(K_n, K_{n+1})$ $\cong \text{Ext}_S^{n+1}(M, K_{n+1}) = 0$ by (8). So the exact sequence $0 \to K_{n+1} \to D_n \to K_n \to 0$ splits, and then $K_n \in \mathcal{P}_C(S)$. Hence, (1) follows.

The last formulas in the theorem for determination of $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M)$ are a direct consequence of the equivalences between (1), (6) and (7).   □

Let $\mathcal{X}$ be any class of R-modules and M an R-module. An $\mathcal{X}$-*precover* of M is an R-homomorphism $\varphi : X \to M$, where $X \in \mathcal{X}$ and such that the sequence $\text{Hom}_R(X', X) \xrightarrow{\text{Hom}_R(X', \varphi)} \text{Hom}_R(X', M) \to 0$ is exact for every $X' \in \mathcal{X}$.

From the equivalence of (1) and (3) in Theorem 3.3, we immediately have the following:

Corollary 3.4. *Let M be an S-module with $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = n < \infty$. Then there is a surjective $\mathcal{D}(\mathcal{P}_C)$-precover $\varphi : N \to M$ such that $K = \text{Ker}\,\varphi$ satisfies $\mathcal{P}_C\text{-pd}_S(K) = n - 1$ (if $n = 0$, this should be interpreted as $K = 0$).*

The following result is the Ding C-projective version of [1, Lemma 2.17].

Corollary 3.5. *Let M be an S-module with $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = n < \infty$. Then there exists an exact sequence $0 \to M \to H \to D \to 0$, where D is Ding C-projective and $\mathcal{P}_C\text{-pd}_S(H) = n$.*

Proof. If M is Ding C-projective, we take $0 \to M \to H \to D \to 0$ to be the first short exact sequence in the "right half" of a complete $\mathcal{D}(\mathcal{P}_C)$-resolution of M.

Now assume that $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M) = n > 0$. By Corollary 3.4, there is an exact sequence $0 \to K \to D' \to M \to 0$, where $D' \in \mathcal{D}(\mathcal{P}_C)$ and $\mathcal{P}_C\text{-pd}_S(K) = n - 1$. Since D' is Ding C-projective, there exists an exact sequence $0 \to D' \to C \otimes_R P \to D \to 0$ such that $P \in \mathcal{P}(R)$ and

$D \in \mathcal{D}(\mathcal{P}_C)$. Consider the pushout diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K & = & K \\
\downarrow & & \downarrow \\
0 \longrightarrow D' \longrightarrow C \otimes_R P \longrightarrow D \longrightarrow 0 \\
\downarrow \quad\quad \downarrow \quad\quad \| \\
0 \longrightarrow M \longrightarrow H \longrightarrow D \longrightarrow 0 \\
\downarrow \quad\quad \downarrow \\
0 \quad\quad 0.
\end{array}
$$

The bottom row of this diagram is the desired sequence. To see this, we must argue that $\mathcal{P}_C\text{-pd}_S(H) = n$. Clearly, $\mathcal{P}_C\text{-pd}_S(H) \nleq n-1$. If $n = 1$, then we must have $\mathcal{P}_C\text{-pd}_S(H) = 1$; otherwise, $M \in \mathcal{D}(\mathcal{P}_C)$ by Corollary 3.4. If $n > 1$, from the short exact sequence $0 \to K \to C \otimes_R P \to H \to 0$, we have $\mathcal{P}_C\text{-pd}_S(H) = \mathcal{P}_C\text{-pd}_S(K) + 1 = n$ since $\mathcal{P}_C\text{-pd}_S(K) \neq \mathcal{P}_C\text{-pd}_S(C \otimes_R P)$. Then, in conclusion, we have $\mathcal{P}_C\text{-pd}_S(H) = n$. $\qquad\square$

Corollary 3.6. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of S-modules with $M \in \mathcal{D}(\mathcal{P}_C)$.*

(1) *If $M' \in \mathcal{D}(\mathcal{P}_C)$, then $M'' \in \mathcal{D}(\mathcal{P}_C)$ if and only if $M'' \in {}^{\perp_1}\mathcal{F}_C$.*
(2) *If $M'' \in \mathcal{D}(\mathcal{P}_C)$, then $M' \in \mathcal{D}(\mathcal{P}_C)$ if and only if $M' \in \mathcal{P}_C^{\perp_1}$.*

Proof. (1) The necessity follows from Proposition 2.5. We now prove the sufficiency. Since $\mathcal{D}(\mathcal{P}_C)\text{-pd}_S(M'') \leq 1$, by Corollary 3.4, there is an exact sequence $0 \to C \otimes_R P \to D \to M'' \to 0$, where $P \in \mathcal{P}(R)$ and $D \in \mathcal{D}(\mathcal{P}_C)$. By assumption $\text{Ext}^1_S(M'', C \otimes_R P) = 0$, this sequence splits, and hence M'' is Ding C-projective by Theorem 2.8.

(2) The necessity follows from Proposition 2.5. Since $M'' \in \mathcal{D}(\mathcal{P}_C)$, there is an exact sequence $0 \to K \to C \otimes_R P_0 \to M'' \to 0$ with

$P_0 \in \mathcal{P}(R)$ and $K \in \mathcal{D}(\mathcal{P}_C)$. Consider the following pullback diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K & = & K \\
\downarrow & & \downarrow \\
0 \longrightarrow M' \longrightarrow L \longrightarrow C \otimes_R P_0 \longrightarrow 0 \\
\| & & \downarrow \qquad\qquad \downarrow \\
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \\
\downarrow \qquad \downarrow \\
0 \qquad\quad 0.
\end{array}
$$

From the middle column, we get that L is Ding C-projective by Theorem 2.7. Note that the middle row splits since $\mathrm{Ext}^1_S(C \otimes_R P_0, M') = 0$ by hypothesis, and hence, M' is Ding C-projective by Theorem 2.8. $\qquad\square$

Recall that a class of modules is called *projectively resolving* if it is closed under extensions, kernels of surjections and it contains all projective modules. By Corollary 3.6, we get that the class of Ding projective modules is projectively resolving, which was proved in [**18,** Theorem 2.6] and [**12,** Theorem 2.1].

When $_SC_R = {}_RR_R$, it was proved in [**2,** Lemma 2.4 (2)] that, if M is a Ding projective R-module of finite flat dimension, then M is projective. Naturally, it makes sense to give the relation between $\mathcal{P}_C$-projective dimensions and $\mathcal{D}(\mathcal{P}_C)$-projective dimensions.

Proposition 3.7. *If M is an S-module of finite $\mathcal{F}_C$-projective dimension, then*

$$\mathcal{D}(\mathcal{P}_C)\text{-}\mathrm{pd}_S(M) = \mathcal{P}_C\text{-}\mathrm{pd}_S(M).$$

In particular, there is an equality of classes $\mathcal{D}(\mathcal{P}_C) \cap \overline{\mathcal{F}_C(S)} = \mathcal{P}_C(S)$.

Proof. Using Theorem 3.3, it suffices to show that if M is Ding C-projective with $\mathcal{F}_C\text{-}\mathrm{pd}_S(M) < \infty$, then M is C-projective. To this end, consider an exact sequence of the form

$$0 \longrightarrow K \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0,$$

where $P \in \mathcal{P}(R)$ and $\mathcal{F}_C\text{-pd}_S(K) < \infty$. By Theorem 3.3, $\text{Ext}_S^{\geq 1}(M, K) = 0$, so the above sequence splits, forcing M to be a summand of $C \otimes_R P$. Since the class of C-projectives is closed under summands by [**10**, Proposition 5.5], M is C-projective, as desired. $\qquad\square$

4. The Foxby equivalence. In this section, we will study the Foxby equivalence between some subclasses of the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(S)$.

Lemma 4.1. *If $M \in \mathcal{B}_C(S)$, then M is Ding C-projective if and only if $\text{Hom}_S(C, M)$ is Ding projective.*

Proof. $\Rightarrow$. By Proposition 2.5, there exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots. \qquad (*)$$

Furthermore, the complex $\text{Hom}_S((*), C \otimes_R F)$ is exact for all flat R-modules F.

Since M and each $C \otimes_R P^i$ are in $\mathcal{B}_C(S)$, by [**10**, Theorem 1], the definition of $\mathcal{B}_C(S)$ gives that both $\text{Ext}_S^{\geq 1}(C, M) = 0$ and $\text{Ext}_S^{\geq 1}(C, C \otimes_R P^i) = 0$ for all $i \geq 0$. Hence, the sequence $(*)$ stays exact if we apply to it the functor $\text{Hom}_S(C, -)$, and doing so we obtain the exact sequence of R-modules

$$0 \longrightarrow \text{Hom}_S(C, M) \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots. \qquad (**)$$

By similar arguments, we see that if we apply $C \otimes_R -$ to the sequence $(**)$, then we have $(*)$ back. Let F be any flat R-module. Then $F \cong \text{Hom}_S(C, C \otimes_R F)$ since $F \in \mathcal{A}_C(R)$. So we have:

$$\begin{aligned}
\text{Hom}_R((**), F) &\cong \text{Hom}_R((**), \text{Hom}_S(C, C \otimes_R F)) \\
&\cong \text{Hom}_S(C \otimes_R (**), C \otimes_R F) \\
&\cong \text{Hom}_S((*), C \otimes_R F)
\end{aligned}$$

is exact. Thus, $(**)$ is a $\text{Hom}_R(-, \mathcal{F}(R))$-exact projective-coresolution of $\text{Hom}_S(C, M)$.

Next, we claim that $\text{Ext}_R^{\geq 1}(\text{Hom}_S(C, M), F) = 0$ for all flat R-modules F. To this end, let $\mathbb{P}. \to M$ be a proper $\mathcal{P}_C$-projective

resolution of M. Then $\operatorname{Hom}_S(C, \mathbb{P}.)$ is a proper projective resolution of $\operatorname{Hom}_S(C, M)$ by [**15**, Lemma 2.1]. For every $i > 0$, we have:

$$
\begin{aligned}
\operatorname{Ext}_R^i(\operatorname{Hom}_S(C, M), F) &\overset{(a)}{=} \mathrm{H}^i(\operatorname{Hom}_R(\operatorname{Hom}_S(C, \mathbb{P}.), \operatorname{Hom}_S(C, C \otimes_R F))) \\
&\overset{(b)}{\cong} \mathrm{H}^i(\operatorname{Hom}_S(C \otimes_R \operatorname{Hom}_S(C, \mathbb{P}.), C \otimes_R F)) \\
&\overset{(c)}{=} \mathrm{H}^i(\operatorname{Hom}_S(\mathbb{P}., C \otimes_R F)) \\
&\overset{(d)}{=} \operatorname{Ext}_{\mathcal{P}_C}^i(M, C \otimes_R F) \\
&\overset{(e)}{\cong} \operatorname{Ext}_S^i(M, C \otimes_R F) \\
&= 0.
\end{aligned}
$$

Here (a) and (c) follow from $F \in \mathcal{A}_C(R)$ and $\mathbb{P}. \in \mathcal{B}_C(S)$ (see [**10**, Lemma 4.1; Theorem 1]); (b) is by adjointness, (d) is by the definition of relative cohomology modules [**15**, Lemma (1.5)] and (e) follows from [**15**, Corollary 4.2]. Therefore, we have proved that $\operatorname{Hom}_S(C, M)$ is a Ding projective R-module by [**12**, Proposition 2.1].

$\Leftarrow$. Assume that $\operatorname{Hom}_S(C, M)$ is a Ding projective R-module. It follows from [**12**, Proposition 2.1] that there exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$
0 \longrightarrow \operatorname{Hom}_S(C, M) \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots . \tag{+}
$$

Furthermore, the above exact sequence is $\operatorname{Hom}_R(-, \mathcal{F}(R))$-exact.

Since $P^0, P^1, \ldots$ and $\operatorname{Hom}_S(C, M)$ are in $\mathcal{A}_C(R)$, it follows from [**10**, Corollary 6.3] that every cokernel in $(+)$ is also in $\mathcal{A}_C(R)$. If $N \in \mathcal{A}_C(R)$, then $\operatorname{Tor}_{\geq 1}^R(C, N) = 0$, and consequently, the sequence $(+)$ stays exact when we apply to it the functor $C \otimes_R -$, and we have the following exact sequence:

$$
0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots . \tag{♯}
$$

Let F be any flat R-module. Then $F \cong \operatorname{Hom}_S(C, C \otimes_R F)$ since $F \in \mathcal{A}_C(R)$. Now by adjointness, we see that

$$
\begin{aligned}
\operatorname{Hom}_S((♯), C \otimes_R F) &\cong \operatorname{Hom}_S(C \otimes_R (+), C \otimes_R F) \\
&\cong \operatorname{Hom}_R((+), \operatorname{Hom}_S(C, C \otimes_R F)) \\
&\cong \operatorname{Hom}_R((+), F)
\end{aligned}
$$

is exact. Furthermore, from the first part of this proof, for any flat R-module F, we have

$$\operatorname{Ext}_{\overline{S}}^{\geq 1}(M, C \otimes_R F) \cong \operatorname{Ext}_{\overline{R}}^{\geq 1}(\operatorname{Hom}_S(C, M), F) = 0.$$

Dually, we can prove that M has a proper $\mathcal{P}_C$-resolution and $M \in \mathcal{P}_C^{\perp}$. Hence, M is Ding C-projective by Proposition 2.5. $\qquad\square$

Proposition 4.2. *There are equivalences of categories*:

$$\mathcal{A}_C(R) \cap \mathcal{DP}(R) \underset{\operatorname{Hom}_S(C,-)}{\overset{C \otimes_R -}{\rightleftarrows}} \mathcal{D}(\mathcal{P}_C(S)) \ .$$

Proof. By Fact 2.3 (4), Fact 1.6 and Lemma 4.1, it is obvious that the functor $\operatorname{Hom}_S(C, -)$ maps $\mathcal{D}(\mathcal{P}_C(S))$ to $\mathcal{A}_C(R) \cap \mathcal{DP}(R)$. So it suffices to show that the functor $C \otimes_R -$ maps $\mathcal{A}_C(R) \cap \mathcal{DP}(R)$ to $\mathcal{D}(\mathcal{P}_C(S))$. To this end, let $M \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)$. Then there exists a $\operatorname{Hom}_R(-, \mathcal{F}(R))$-exact sequence of R-modules

$$\mathbb{P} = \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} P^0 \xrightarrow{\partial^0} P^1 \xrightarrow{\partial^1} \cdots$$

with all P_i, $P^i \in \mathcal{P}(R)$ and $M \cong \operatorname{Im}\partial_0$. By [**17,** Lemma 4.1, Corollary 6.3] and [**2,** Remark 2.2(3)], every kernel and cokernel in $\mathbb{P}$ are in $\mathcal{A}_C(R) \cap \mathcal{DP}(R)$. Then, applying the functor $C \otimes_R -$ to it, we have the following exact sequence of S-modules

$$C \otimes_R \mathbb{P} = \cdots \xrightarrow{1_C \otimes_R \partial_2} C \otimes_R P_1 \xrightarrow{1_C \otimes_R \partial_1} C \otimes_R P_0$$

$$\xrightarrow{1_C \otimes_R \partial_0} C \otimes_R P^0 \xrightarrow{1_C \otimes_R \partial^0} \cdots .$$

Clearly, $C \otimes_R M \cong \operatorname{Im}(1_C \otimes_R \partial_0)$. Now by [**10,** Theorem 6.4], for any flat R-module F, we see that

$$\operatorname{Hom}_S(C \otimes_R \mathbb{P}, C \otimes_R F) \cong \operatorname{Hom}_R(\operatorname{Hom}_S(C, C \otimes_R \mathbb{P}),$$
$$\operatorname{Hom}_S(C, C \otimes_R F))$$
$$\cong \operatorname{Hom}_R(\mathbb{P}, F)$$

is exact. And, for any projective R-module P, we have

$$\operatorname{Hom}_S(C \otimes_R Q, C \otimes_R \mathbb{P}) \cong \operatorname{Hom}_R(\operatorname{Hom}_S(C, C \otimes_R Q),$$
$$\operatorname{Hom}_S(C, C \otimes_R \mathbb{P}))$$
$$\cong \operatorname{Hom}_R(Q, \mathbb{P})$$

is exact. So $C \otimes_R \mathbb{P}$ is a $\operatorname{Hom}_S(\mathcal{P}_C, -)$- and $\operatorname{Hom}_S(-, \mathcal{F}_C)$-exact sequence, and hence $C \otimes_R M \in \mathcal{D}(\mathcal{P}_C(S))$.

Finally, note that if $M \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)$ and $N \in \mathcal{D}(\mathcal{P}_C(S))$, then there exist natural isomorphisms $M \cong \operatorname{Hom}_S(C, C \otimes_R M)$ and $N \cong C \otimes_R \operatorname{Hom}_S(C, N)$. Then the desired equivalences of categories follow. □

Let n be a nonnegative integer. In the following, we denote by $\mathcal{DP}(R)_{\leq n}$ $(\mathcal{D}(\mathcal{P}_C(S))_{\leq n})$ the class of modules with Ding projective (Ding C-projective) dimension at most n.

Theorem 4.3. *There are equivalences of categories*:

$$\mathcal{A}_C(R) \cap \mathcal{DP}(R)_{\leq n} \xrightarrow[\operatorname{Hom}_S(C,-)]{C \otimes_R -} \mathcal{D}(\mathcal{P}_C(S))_{\leq n} .$$

Proof. Let $M \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)_{\leq n}$. If $n = 0$, then $C \otimes_R M \in \mathcal{D}(\mathcal{P}_C(S))$ by Proposition 4.2. Now assume $n \geq 1$. Then by [**12,** Theorem 2.4], there exists an exact sequence of R-modules

$$0 \to D_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with $D_n \in \mathcal{DP}(R)$ and $P_i \in \mathcal{P}(R)$ for $0 \leq i \leq n-1$. By [**10,** Theorem 1, Corollary 6.3], every term and kernel in the above sequence are in $\mathcal{A}_C(R)$. So we have the following exact sequence of S-modules

$$0 \to C \otimes_R D_n \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R M \to 0$$

with $C \otimes_R D_n$ and all $C \otimes_R P_i$ in $\mathcal{D}(\mathcal{P}_C(S))$ by Proposition 4.2, which means that $C \otimes_R M \in \mathcal{D}(\mathcal{P}_C(S))_{\leq n}$.

Conversely, let $N \in \mathcal{D}(\mathcal{P}_C(S))_{\leq n}$. If $n = 0$, then $\operatorname{Hom}_S(C, N) \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)$ by Proposition 4.2. Now assume $n \geq 1$. Then by Theorem 3.3, there exists an exact sequence of S-modules

$$0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to D_0 \to N \to 0$$

with $D_0 \in \mathcal{D}(\mathcal{P}_C(S))$ and $P_i \in \mathcal{P}(R)$ for $1 \leq i \leq n$. By [**10,** Corollaries 6.1, 6.3], every term and cokernel in the above sequence are in $\mathcal{B}_C(S)$. So, applying the functor $\operatorname{Hom}_S(C, -)$ to it, we have the following exact sequence of R-modules

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to \operatorname{Hom}_S(C, D_0) \to \operatorname{Hom}_S(C, N) \to 0$$

with $\operatorname{Hom}_S(C, D_0) \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)$ by Proposition 4.2. Hence, $\operatorname{Hom}_S(C, N) \in \mathcal{A}_C(R) \cap \mathcal{DP}(R)_{\leq n}$.

The rest of the proof is similar to that of Proposition 4.2. $\qquad\square$

By assembling the information above and [**11,** Theorem 4.6], we have the following extension of the Foxby equivalence from the introduction.

Theorem 4.4. (*Foxby equivalence*). *Let n be a nonnegative integer, and let $\mathcal{P}(R)_{\leq n}$, $\mathcal{P}_C(S)_{\leq n}$, $\mathcal{I}(S)_{\leq n}$ and $\mathcal{I}_C(R)_{\leq n}$ be the classes of modules with projective, C-projective, injective and C-injective dimension at most n, respectively. Then there are equivalences of categories*:

$$
\begin{array}{ccc}
\mathcal{P}(R)_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{P}_C(S)_{\leq n} \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{A}_C(R) \cap \mathcal{DP}(R)_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{D}(\mathcal{P}_C(S))_{\leq n} \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{A}_C(R) \cap \mathcal{GP}(R)_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{B}_C(S) \cap \mathcal{GP}_C(S)_{\leq n} \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{A}_C(R) & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{B}_C(S) \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{A}_C(R) \cap \mathcal{GI}_C(R)_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{B}_C(S) \cap \mathcal{GI}(S)_{\leq n} \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{D}(\mathcal{I}_C(R))_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{B}_C(S) \cap \mathcal{DI}(S)_{\leq n} \\
\big\updownarrow & & \big\updownarrow \\
\mathcal{I}_C(R)_{\leq n} & \underset{\underset{\operatorname{Hom}_S(C,-)}{\overset{\sim}{\longleftarrow}}}{\overset{C \otimes_R -}{\longrightarrow}} & \mathcal{I}(S)_{\leq n}.
\end{array}
$$

Remark 4.5. By the definition of G_C-projective modules, one can define D_C-projective modules over a commutative ring S as follows:

An S-module M is called D_C-*projective* if there exists an exact sequence of S-modules

$$X = \cdots \to P_1 \to P_0 \to C \otimes_S P^0 \to C \otimes_S P^1 \to \cdots$$

with P_i, P^i projective for $i \geq 0$, such that $M \cong \mathrm{Coker}(P_1 \to P_0)$ and X is $\mathrm{Hom}_S(-, \mathcal{F}_C(S))$-exact. Set

$$\mathcal{DP}_C(S) = \text{the subcategory of } D_C\text{-projective } S\text{-modules.}$$

By Proposition 2.5, we have that $\mathcal{D}(\mathcal{P}_C(S)) = \mathcal{B}_C(S) \cap \mathcal{DP}_C(S)$.

REFERENCES

1. L.W. Christensen, A. Frankild and H. Holm, *On Gorenstein projective, injective and flat dimensions−a functorial description with applications*, J. Alg. **302** (2006), 231–279.

2. N. Ding, Y. Li and L. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), 323–338.

3. E.E. Enochs and O.M.G. Jenda, *Relative homological algebra*, de Gruyter Expo. Math. **30**, Walter de Gruyter Co., Berlin, 2000.

4. H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1973), 267–284.

5. Y. Geng and N. Ding, *W-Gorenstein modules*, J. Alg. **325** (2011), 132–146.

6. J. Gillespie, *Model structures on modules over Ding-Chen rings*, Homol. Homot. Appl. **12** (2010), 61–73.

7. E.S. Golod, *G-dimension and generalized perfect ideals*, Alg. Geom. Appl. **165** (1984), 62–66.

8. H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Alg. **189** (2004), 167–193.

9. H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Alg. **205** (2006), 423–445.

10. H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

11. Z. Liu, Z. Huang and A. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Alg. **41** (2013), 1–18.

12. N. Mahdou and M. Tamekkante, *Strongly Gorenstein flat modules and dimensions*, Chin. Ann. Math. **32** (2011), 533–548.

13. L. Mao and N. Ding, *Gorenstein FP-injective and Gorenstein flat modules*, J. Alg. Appl. **7** (2008), 491–506.

14. J.J. Rotman, *An introductions to homological algebra*, Academic Press, New York, 1979.

15. R. Takahashi and D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), 5–22.

16. W.V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974.

17. D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Comm. Alg. **2** (2010), 111–137.

18. G. Yang, Z. Liu and L. Liang, *Ding projective and Ding injective modules*, Alg. Colloq. **20** (2013), 601–612.

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