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SOME CHARACTERIZATIONS OF THE EULER GAMMA FUNCTION

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ABSTRACT. Assume that $f : (0, \infty) \to (0, \infty)$ is bounded from above on a set of positive Lebesgue measures or on a set of the second category with the Baire property and satisfies the functional equation $f(x+1) = xf(x)$ for $x > 0$ and $f(1) = 1$. We prove that, if there is a positive sequence (p_n), $\lim_{n\to\infty} p_n = \infty$, such that for every $n \in \mathbb{N}$, the function $x \mapsto \log(x^{p_n})$ is Jensen convex in the interval $(1, \infty)$; or there are two positive sequences (p_n) and (q_n), $\lim_{n\to\infty} p_n = \infty$, $\lim_{n\to\infty} q_n = 0$ such that, for every $n \in \mathbb{N}$, the function $x \mapsto [f(x^{p_n})]^{q_n}$ is Jensen convex in the interval $(1, \infty)$, then f is the Euler gamma function.

1. Introduction. In 1922, Bohr and Mollerup [**3**] proved that if a function $f : (0, \infty) \to (0, \infty)$ satisfies the functional equation

$$f(x+1) = xf(x), \quad x > 0; \; f(1) = 1, \tag{1}$$

and $\log \circ f$ is convex, then f is the Euler gamma function Γ (cf., also Artin [**2**]).

Gronau and Matkowski [**4**] in 1993 gave an improvement of this result, showing, in particular, that (under weak regularity of f) it remains true if the convexity of $\log \circ f$ is replaced by the much weaker condition of the geometrical convexity of f in some interval (b, ∞), that is,

$$f(\sqrt{xy}) \le \sqrt{f(x) f(y)}, \quad x, y > b,$$

and equivalent to the Jensen convexity of the function $\log \circ f \circ \exp$.

In a recent paper, Alzer and Matkowski [**1**] have obtained a characterization of the Gamma function, making use of some properties of the composition of the power functions with the function $\Gamma \circ \exp$ which

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reads as follows. Assume that $f:(0,\infty)\to(0,\infty)$ satisfies (1). If f is bounded on a set of positive Lebesgue measure (or on a set of the second category with the Baire property) and there are $a>0$ and a sequence of positive numbers q_n with $\lim_{n\to\infty} q_n = 0$ such that, for every n the function $(f\circ\exp)^{q_n}$ is Jensen convex, then f is the gamma function.

The characterization of the gamma function presented in this note is also based on equation (1). The main result, Theorem 2 in Section 3, reads as follows. Assume that $f:(0,\infty)\to(0,\infty)$ is bounded from above on a set of positive Lebesgue measures or on a set of the second Baire category with the Baire property and satisfies the functional equation (1). If there is a positive sequence (p_n), $\lim_{n\to\infty} p_n = \infty$ such that, for every $n\in\mathbb{N}$, the function $x\mapsto \log f(x^{p_n})$ is Jensen convex in the interval $(1,\infty)$; or there are two positive sequences (p_n) and (q_n), $\lim_{n\to\infty} p_n=\infty$, $\lim_{n\to\infty} q_n = 0$ such that, for every $n\in\mathbb{N}$, the function $x\mapsto [f(x^{p_n})]^{q_n}$ is Jensen convex in the interval $(1,\infty)$, then f is the Euler gamma function. In Section 1 we present a simple argument assuming that f is a twice differentiable function (Theorem 1). In Section 2, we present the counterparts of these results under a little stronger assumption that can be regarded as a motivation of the main results.

As an immediate corollary, we obtain the following. If $f:(0,\infty)\to(0,\infty)$ satisfies (1) and, for every positive integer n, the function $x\mapsto[f(x^n)]^{1/n}$ is convex, then f is the Gamma function.

2. A characterization for regular functions. Let us note that the following is easy to verify

Remark 2.1. Let $I\subset(0,\infty)$ be an open interval, and let $f:I\to(0,\infty)$be twice differentiable. The following conditions are pairwise equivalent:

(i) the function f is geometrically convex, that is,

$$f(x^t y^{1-t}) \le f(x)^t f(y)^{1-t}, \quad x,y\in I,\ t\in(0,1);$$

(ii) the function $\log\circ f\circ\exp$ is convex in the interval $J:=\log(I)$;

(iii) the function $f:I\to(0,\infty)$ satisfies the inequality

$$f(x)\,f''(x)\,x + f(x)\,f'(x) \ge [f'(x)]^2\,x, \quad x\in I.$$

We prove the following:

Theorem 2.2. *Suppose that a function* $f : (0, \infty) \to (0, \infty)$ *is twice differentiable and satisfies equation* (1)*. If* f *satisfies one of the following two conditions*:

(i) *there is a sequence* (p_n), $p_n \to \infty$, *such that for every* $n \in \mathbb{N}$, *the function* $x \mapsto \log f(x^{p_n})$ *is convex in* $(1, \infty)$;
(ii) *there exist some sequences of positive numbers* (p_n), (q_n); $p_n \to \infty$, $q_n \to 0$, *such that, for every* $n \in \mathbb{N}$, *the function* $x \mapsto [f(x^{q_n})]^{p_x}$ *is convex in* $(1, \infty)$,

then f *is the Euler gamma function.*

Proof. To prove the first result, take $p > 0$. Since f is twice differentiable, the function $x \mapsto \log f(x^p)$ is convex in $(1, \infty)$ if, and only if,

$$(\log f(x^p))'' = \frac{p^2 x^{p-2}}{[f(x^p)]^2} \Big\{ \frac{p-1}{p} f(x^p) f'(x^p) + x^p f(x^p) f''(x^p) - x^p [f'(x^p)]^2 \Big\} \geq 0,$$

for all $x \in (1, \infty)$. Since, for $p > 0$, the function $x \mapsto x^p$ maps the interval $(1, \infty)$ onto itself, this inequality is satisfied if, and only if,

$$\frac{p-1}{p} f(x) f'(x) + x f(x) f''(x) - x [f'(x)]^2 \geq 0, \quad x \in (1, \infty).$$

Replacing here p by p_n such that $p_n \to \infty$, and then letting $n \to \infty$, we obtain

$$f(x) f'(x) + x f(x) f''(x) - x [f'(x)]^2 \geq 0, \quad x \in (1, \infty).$$

In view of Remark 2.1, the function f is geometrically convex in $(1, \infty)$. Since f satisfies (1), in view of the Gonau-Matkowski result **[4]**, the function f must be the Euler gamma function.

To prove the second result take arbitrary positive real numbers p and q. The function $x \mapsto [f(x^p)]^q$ is convex in the interval $(1, \infty)$ if, and only if,

$$\frac{\left([f(x^p)]^q\right)''}{p^2 q x^{p-2} [f(x^p)]^{q-2}}$$
$$= (q-1) x^p [f'(x^p)]^2 + x^p f(x^p) f''(x^p) + \frac{p-1}{p} f(x^p) f'(x^p) \geq 0,$$

for all $x \in (1, \infty)$. Since p and q are positive, and the function $x \mapsto x^p$ maps the interval $(1, \infty)$ onto itself, we see that this inequality is satisfied if, and only if,

$$(q-1) x [f'(x)]^2 + x f(x) f''(x) + \frac{p-1}{p} f(x) f'(x) \geq 0, \quad x \in (1, \infty).$$

Setting here $p = p_n$, $q = q_n$, $p_n \to \infty$ and $q_n \to 0$, and letting $n \to \infty$, we obtain

$$-x [f'(x)]^2 + x f(x) f''(x) + f(x) f'(x) \geq 0, \quad x \in (1, \infty),$$

whence, by Remark 2.1, the function f is geometrically convex. Now the result follows from the main result of [**4**]. $\square$

3. Main results. Let $D \subset \mathbb{R}^k$ be convex and open, and let $A \subset D$ be of positive Lebesgue measure. We need the following result of Ostrowski [**8**] (see also [**6,** page 210]).

If $f : D \to R$ is Jensen convex, that is,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}, \quad x, y \in D,$$

and bounded from above on A, then f is convex.

A subset A of a topological space X is said to have the Baire property if $A = (D \cup P) \backslash R$, where the set D is open, and the sets P, R are of the first category. (The family of sets having the Baire property is a σ-algebra.)

Let $D \subset \mathbb{R}^k$ be convex and open, and let $A \subset D$ be of the second category with the Baire property. We shall also need the following result, which is due to Mehdi [**7**] (see also [**6,** page 210]).

If $f : D \to R$ is Jensen convex and bounded from above on A, then f is convex.

The main result of the present paper reads as follows:

Theorem 3.1. *Assume that* $f:(0,\infty)\to(0,\infty)$ *is bounded from above on a set of positive Lebesgue measure or on a set of the second category with Baire property, and satisfies equation* (1):

$$f(x+1)=xf(x), \quad x>0;\ f(1)=1.$$

If one of the following two conditions is fulfilled,

(i) *there is a positive sequence* (p_n), $\lim_{n\to\infty} p_n=\infty$ *such that, for every* $n\in\mathbb{N}$, *the function*

$$x\longmapsto \log f(x^{p_n})$$

is Jensen convex in the interval $(1,\infty)$;

(ii) *there are two positive sequences* (p_n) *and* (q_n), $\lim_{n\to\infty} p_n=\infty$, $\lim_{n\to\infty} q_n=0$ *such that, for every* $n\in\mathbb{N}$, *the function*

$$x\longmapsto [f(x^{p_n})]^{q_n}$$

is Jensen convex in the interval $(1,\infty)$,

then f *is the Euler gamma function.*

Proof. If condition (i) is satisfied, then

$$\log f\left(\left(\frac{x+y}{2}\right)^{p_n}\right)\le\frac{\log f(x^{p_n})+\log f(y^{p_n})}{2}, \quad x,y>1;\ n\in\mathbb{N},$$

whence, replacing x and y, respectively, by x^{1/p_n} and y^{1/p_n}, we obtain

$$f\left(\left(\frac{x^{1/p_n}+y^{1/p_n}}{2}\right)^{p_n}\right)\le\sqrt{f(x)f(y)}, \quad x,y>1;\ n\in\mathbb{N}.$$

Since

$$\lim_{r\to\infty}\left(\frac{u^{1/r}+v^{1/r}}{2}\right)^{r}=\sqrt{uv}, \quad u,v>0, \tag{3.1}$$

letting $n\to\infty$, we hence get

$$f(\sqrt{xy})\le\sqrt{f(x)f(y)}, \quad x,y>1,$$

that is, f is Jensen geometrically convex in $(1,\infty)$ (or, equivalently, the function $\log\circ f\circ\exp$ is Jensen convex in the interval $(0,\infty)$).

If condition (ii) is satisfied, then

$$\left[f\left(\left(\frac{x+y}{2}\right)^{p_n}\right)\right]^{q_n} \le \frac{\left[f\left(x^{p_n}\right)\right]^{q_n}+\left[f\left(y^{p_n}\right)\right]^{q_n}}{2}, \quad x,y>1;\ n\in\mathbb{N}.$$

Replacing here x and y by x^{1/p_n} and y^{1/p_n}, respectively, we get

$$f\left(\left(\frac{x^{1/p_n}+y^{1/p_n}}{2}\right)^{p_n}\right) \le \left(\frac{\left[f\left(x\right)\right]^{q_n}+\left[f\left(y\right)\right]^{q_n}}{2}\right)^{1/q_n}, \quad x,y>1;\ n\in\mathbb{N}.$$

Since $\lim_{n\to\infty} p_n = \infty$, $\lim_{n\to\infty} q_n = 0$, letting here $n\to\infty$ and applying (3.1), we obtain

$$f\left(\sqrt{xy}\right) \le \sqrt{f\left(x\right) f\left(y\right)}, \quad x,y>1,$$

that is, f is Jensen geometrically convex in $(1,\infty)$.

By the assumption there are a set $A\subset(0,\infty)$ of positive Lebesgue measure or of the second Baire category and $M>0$ such that

$$f\left(x\right)\le M, \quad x\in A.$$

Since there is $n\in\mathbb{N}$ such that $A\cap(0,\infty)$ is also of positive Lebesgue measure or of the second Baire category, we may assume that A is bounded, that is, $m:=\sup A<\infty$. For sufficiently large k, we have $k+A\subset(a,\infty)$ and $k+A$ is of positive Lebesgue measure or of the second Baire category. From (1), by induction, we have

$$f\left(x+k\right)=x\left(x+1\right)\cdot\ldots\cdot\left(x+k-1\right)f\left(x\right), \quad x>0.$$

Hence,

$$f\left(x+k\right)\le m\left(m+1\right)\cdot\ldots\cdot\left(m+k-1\right)M, \quad x\in\left(k+A\right),$$

that is, f is bounded from above on the set $k+A\subset(a,\infty)$. By the theorem of Ostrowski and the theorem of Mehdi (cf., Kuczma [**6,** page 210], the function $\log\circ f\circ\exp$ is convex in the interval $(\log a,\infty)$, that is,

$$\log\circ f\circ\exp\left(tu+\left(1-t\right)v\right)\le t\log\circ f\circ\exp\left(u\right)+\left(1-t\right)\log\circ f\circ\exp\left(v\right),$$

for all $t\in(0,1)$ and $u,v\in(\log a,\infty)$, or equivalently,

$$f\left(x^t y^{1-t}\right)\le\left[f\left(x\right)\right]^t\left[f\left(y\right)\right]^{1-t}, \quad t\in\left(0,1\right),\ x,y\in\left(a,\infty\right).$$

Thus, f is geometrically convex, and the result follows from [**4**]. $\square$

Remark 3.2. The function f satisfies the assumed regularity conditions in Theorem 3.1 if it is Lebesgue measurable or continuous at a point (cf., Kuczma [**6**]).

Remark 3.3. The assumption of the convexity. From Theorem 3.1, we immediately obtain the following:

Corollary 3.4. *Assume that* $f : (0, \infty) \to (0, \infty)$ *is bounded from above on a set of positive Lebesgue measure or on a set of the second Baire category and satisfies* (1)*. If, for every positive integer* n*, the function* $x \mapsto [f(x^n)]^{1/n}$ *is Jensen convex in the interval* $(1, \infty)$*, then* f *is the Euler gamma function.*

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