ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 4, 2015

INTERNAL INVERSE LIMITS AND RETRACTIONS

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ABSTRACT. We establish equivalences between compacta that admit a sequence of retractions that converge uniformly to the identity map and compacta that are inverse limits on subcompacta with retractions for bonding maps. We give partial answers to questions of Charatonik and Prajs, and of Krasinkiewicz. Our results are related to and use results from another paper of the authors [**11**].

1. Introduction and definitions. Given a compactum X with retractions $r_n : X \to X_n$ converging uniformly to the identity map on X, under what conditions can X be represented as an inverse limit of copies of some (or all) of the X_n's with retractions as bonding maps? This is a central question to this investigation. More precisely, we want to find an *internal inverse limit structure* on X (definition follows), with some or all of the X_n's as factor spaces and retractions as bonding maps. Our results are presented in three cases.

Case 1, presented in Section 2, shows that, if the compacta X_n are nested and the retractions r_n commute, then all of the X_n's, with the corresponding restrictions of the r_n's, form the desired inverse limit structure.

Case 2, also presented in Section 2, shows that, if the compacta X_n are nested but the retractions r_n do not necessarily commute, then there is an internal inverse limit structure on X with a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ as factor spaces and the corresponding restrictions of r_{n_k}'s as the bonding maps.

Case 3, presented in Section 3, shows that, if the compacta X_n are not nested, then we have a positive result under the additional assumption that the X_n's are graphs of order at most three. Specifically, for such X_n's, we find a desired internal inverse limit structure on X made

2010 AMS *Mathematics subject classification.* Primary 54C15, 54F15, Secondary 54A20, 54F65.

Keywords and phrases. Internal inverse limit, r-maps, retractions.

Received by the editors on July 17, 2013.

DOI:10.1216/RMJ-2015-45-4-1209

of copies of a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ as factor spaces, and retractions, "nearly congruent" to the restrictions of the r_{n_k}'s, as bonding maps.

To better understand the motivation for this study, notice that many important examples in topology and continuum theory are, or can be represented as, inverse limits of some simple spaces with retractions as bonding maps. For instance, inverse limits of arcs with open bonding maps, also called *Knaster continua*, are inverse limits of arcs with open r-maps as bonding maps. From our results in Section 2 it follows that Knaster continua have internal inverse limit structures of arcs with open retractions for bonding maps. Moreover, replacing arcs with trees, this result can be generalized to a similar one for all inverse limits of trees with open bonding maps, which is a much larger class of spaces still having very regular properties. One of these properties is that all the members of this class are *absolute retracts for tree-like continua* [**4**]. Having retractions as bonding maps creates a special form of convergence of the factor spaces to the inverse limit space. This convergence was used in the construction of the following examples of Bellamy.

(i) A planar dendroid M with a connected set of endpoints E, where only endpoints are accessible from $\mathbb{R}^2 - M$, and no single point of E separates E but each pair of points of E does separate E [**1**].
(ii) A Hausdorff indecomposable continuum with exactly one composant and a Hausdorff indecomposable continuum with exactly two composants [**2**].

Our results are related to a recent study by Mańka [**9,** Theorem 1.1], in which he has shown that locally connected curves admit ϵ-retractions onto graphs. Additionally, Mańka states [**9,** page 650] that he has proved, in a paper presently in preparation, that these graphs can be chosen to form a nested increasing sequence. Assuming this last result, it follows from our Corollary 2.3 that each locally connected curve admits an internal inverse limit structure on graphs with retractions for bonding maps. This provides a partial answer to a question of Krasinkiewicz (see [**9,** pages 650, 651]).

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A *mapping* (or *map*) is a continuous function. If A is a closed subset of X, a surjective mapping $f\colon X \to A$ is a *retraction* if

$f(x) = x$ for each $x \in A$. A map $f\colon X \to Y$ of topological spaces is an *r-map* if there is a map $g\colon Y \to X$ such that $f \circ g\colon Y \to Y$ is the identity map on Y (see [**3,** page 7]). The class of r-maps coincides with the class of maps of the form $h \circ r$, where h is a homeomorphism and r is a retraction [**3,** 2.1, page 10]. For the composition of functions f after g, we will use both notations $f \circ g$ and fg.

For the convenience of the reader in the remaining part of this section we state some definitions and results from [**11**].

We use *inverse sequences* and *inverse limits* throughout this paper. Definitions and general properties of these notions can be found in [**6,** page 7–14], [**8,** subsections 2.1–2.3], or [**12,** Section II, part One]. Two inverse sequences $\{X_n, f_n^{n+1}\}$ and $\{Y_n, g_n^{n+1}\}$ are said to be *topologically equivalent* if, for each $n \geq 1$, there exists a homeomorphism $h_n\colon X_n \to Y_n$ such that $g_n^{n+1} \circ h_{n+1} = h_n \circ f_n^{n+1}$.

If A_n is a sequence of non-empty subsets of X converging with respect to the Hausdorff distance, then $\operatorname{Lim} A_n$ denotes the Hausdorff limit of the A_n's.

In the spirit of the Anderson-Choquet embedding theorem (see [**12,** Theorem 2.10]), the following definition was introduced in [**13,** page 104].

Let $\{X_n\}$ be a sequence of closed sets in a metric space (X, d) and $\{f_n : X_{n+1} \to X_n\}$ a sequence of maps. We say the inverse sequence $\{X_n, f_n\}$ *converges* in X provided that:

(i) Each thread $(x_1, x_2, \ldots)$ of the inverse sequence is a convergent sequence in X,
(ii) the assignment f defined by $(x_1, x_2, \ldots) \mapsto \lim x_n$ is a continuous map from $\varprojlim\{X_n, f_n\}$ to X, and
(iii) the projections $\pi_n : \varprojlim\{X_n, f_n\} \to X_n$ converge uniformly to $f\colon \varprojlim\{X_n, f_n\} \to \operatorname{Lim} X_n$.

If, additionally, f is an embedding, we say that $\{X_n, f_n\}$ *converges exactly* in X.

If $\{X_n, f_n\}$ converges exactly in X and $f(\varprojlim\{X_n, f_n\}) = X$, that is, f is a homeomorphism onto the entire space X, we call $\{X_n, f_n\}$ an *internal inverse limit structure* on X. Identifying $\varprojlim\{X_n, f_n\}$ with X

by f, in this case, we have the projection maps in (iii) converging to the identity map on X.

We note that this terminology is slightly different from that used in [**13**]. What we call here an *exactly convergent inverse sequence* in [**13**] was referred to as a *convergent inverse sequence.*

Let X be a space and A a non-empty subspace of X. For any map $f : A \to X$, we write $\widetilde{d}(f) = \sup\{d(x, f(x)) \mid x \in A\}$.

Let $\mathcal{K}$ be a class of compacta. We say that X is *internally $\mathcal{K}$-like* if, for each $\epsilon > 0$, there is a $K \in \mathcal{K}$ with $K \subset X$, and a map $f\colon X \to K$ such that $\widetilde{d}(f) < \epsilon$. We say that X is *internally $\mathcal{K}$-representable* if X has an internal inverse limit structure with factor spaces in $\mathcal{K}$. We investigated these ideas in our previous paper [**11**]. If X is internally $\mathcal{K}$-like and, additionally, the mappings f are retractions, we say that X is *retractably $\mathcal{K}$-like.* If X is internally $\mathcal{K}$-representable and the bonding maps are retractions, we say that X is *retractably $\mathcal{K}$-representable.*

A collection $\mathcal{S}$ of convergent sequences in a space X is called *uniformly convergent* provided, for every $\epsilon > 0$, there is an N such that $d(s_m, \lim_{n\to\infty} s_n) < \epsilon$ for each $m > N$ and $\{s_n\} \in \mathcal{S}$. In a similar way, we define *uniformly Cauchy* collections of sequences. Clearly, in a complete metric space, a collection of sequences is uniformly convergent if and only if it is uniformly Cauchy.

Proposition 1.1. *For each $n \geq 1$, let X_n be a compact non-empty subset of a complete metric space X, and let $f_n : X_{n+1} \to X_n$ be a map. The inverse sequence $\{X_n, f_n\}$ converges in X if and only if the threads of $\{X_n, f_n\}$ form a uniformly convergent collection of sequences in X.*

Proposition 1.2. *For each $n \geq 1$, let X_n be a compact non-empty subset of a complete metric space X, and let $f_n : X_{n+1} \to X_n$ be a map. The inverse sequence $\{X_n, f_n\}$ converges exactly in X if and only if it converges in X and the function $(x_1, x_2, \dots) \mapsto \lim x_n$ from $\varprojlim\{X_n, f_n\}$ to X is one-to-one.*

Proposition 1.3. *Let $\{X_n, f_n\}$ be an inverse sequence converging in a compact space X. If $\operatorname{Lim} X_n = X$, then the map f in the definition of the convergence of inverse sequences is surjective.*

A set $\{f_\alpha\colon X_\alpha \to Y_\alpha \mid \alpha \in \Gamma\}$ of maps is *uniformly equicontinuous* if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f_\alpha(x), f_\alpha(y)) < \epsilon$ for all $\alpha \in \Gamma$ and all $x, y \in X_\alpha$ such that $d(x, y) < \delta$.

Given a compact space X and a collection of maps $\{f_\alpha\colon X_\alpha \to X \mid X_\alpha \in 2^X, \alpha \in \Gamma\}$, Theorem 1.4 below provides a general condition under which X admits an internal inverse limit structure. Proofs of this theorem and the subsequent two corollaries can be found in [**11**].

Theorem 1.4. *Let X be a compact space and $\{Y_n\}$ a sequence of closed subsets of X with $\operatorname{Lim} Y_n = X$, and let $\mathcal{F}$ be a collection of maps. Suppose for each $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that for each $n > N(\epsilon)$, there exists a uniformly equicontinuous sequence of maps $f_n^m : Y_m \to Y_n$ in $\mathcal{F}$, for $m > n$, with $\widetilde{d}(f_n^m) < \epsilon$. Then there are a subsequence $\{Y_{n_k}\}$ of $\{Y_n\}$ and maps $g_k : Y_{n_{k+1}} \to Y_{n_k}$ in $\mathcal{F}$ such that the inverse sequence $\{Y_{n_k}, g_k\}$ is an internal inverse limit structure on X.*

Corollary 1.5. *Let X be a compactum. Each sequence of maps $\{f_n : X \to X\}$ converging uniformly to the identity map on X has a subsequence $\{g_k = f_{n_k}\}$ such that $\{g_k(X), g_k|_{g_{k+1}(X)}\}$ is an internal inverse limit structure on X.*

Corollary 1.6. *Let $\mathcal{K}$ be any class of compacta and X any compactum. The compactum X is internally $\mathcal{K}$-like if and only if X is internally $\mathcal{K}$-representable.*

2. r-maps, retractions, and internal inverse limit structures. We begin with a theorem for which special versions have been noted by others, typically with comments about a proof. For example, Bellamy notes and uses part (i) of this theorem in the construction of his examples mentioned earlier. Also, note the implication 5.2.1 $\Longrightarrow$ 5.2.2 in [**5,** page 889]. For completeness, we supply details of a more general version of these related theorems. We also use this theorem later in the paper. As an easy application of Theorem 2.1 below, we note that the "tent map" on $[0,1]$, or any surjective open map on $[0,1]$, is an r-map. Hence, each Knaster continuum is retractably arc-representable.

Theorem 2.1. *Suppose* $X = \varprojlim\{X_n, r_n^{n+1}\}$, *where each* X_n *is a compactum and each* $r_n^{n+1}\colon X_{n+1} \to X_n$ *is an* r*-map. Then there exists a monotone increasing sequence of compacta* $\{\widehat{X}_n\}$ *in* X *and retractions* $\widehat{r}_n^{n+1}\colon \widehat{X}_{n+1} \to \widehat{X}_n$ *such that*:

(i) *the inverse sequence* $\{\widehat{X}_n, \widehat{r}_n^{n+1}\}$ *is topologically equivalent to the inverse sequence* $\{X_n, r_n^{n+1}\}$,
(ii) $\{\widehat{X}_n, \widehat{r}_n^{n+1}\}$ *is an internal inverse limit structure on* X, *and*
(iii) *identifying* X *with its internal inverse limit structure, the projection mappings* $\widehat{r}_n\colon X \to \widehat{X}_n$ *are retractions satisfying* $\widehat{r}_n \circ \widehat{r}_m = \widehat{r}_n$ *for all* $n < m$.

Proof. For each r-map $r_n^{n+1}\colon X_{n+1} \to X_n$, let $g_n\colon X_n \to X_{n+1}$ be the embedding satisfying $r_n^{n+1} \circ g_n = \mathrm{id}\,|_{X_n}$. Define $g_n^m = g_{n-1} \circ \ldots \circ g_{m+1} \circ g_m$ for $m < n$, and note that each $g_n^m : X_m \to X_n$ is an embedding. Given a p in X_n, we let

$$h_n(p) = (r_1^n(p), r_2^n(p), \ldots, r_{n-1}^n(p), p, g_{n+1}^n(p), g_{n+2}^n(p), \ldots)$$

and note that $h_n(p)$ is a thread in X, and $h_n : X_n \to X$ is an embedding. Moreover, for $q = g_n(p)$, we note

$$\begin{aligned} h_n(p) &= h_{n+1}(q) \\ &= (r_1^{n+1}(q), r_2^{n+1}(q), \ldots, r_n^{n+1}(q), q, g_{n+2}^{n+1}(q), g_{n+3}^{n+1}(q), \ldots). \end{aligned}$$

So, $h_n(p) \in h_{n+1}(X_{n+1})$. Therefore, letting $\widehat{X}_n = h_n(X_n)$, we have $\widehat{X}_n \subset \widehat{X}_{n+1}$ for each $n \geq 1$.

Given $\mathbf{x} \in \widehat{X}_{n+1}$, there is a unique $q \in X_{n+1}$ such that $h_{n+1}(q) = \mathbf{x}$. The assignment $\mathbf{x} \mapsto h_n(r_n^{n+1}(q))$ defines a map $\widehat{r}_n^{n+1} : \widehat{X}_{n+1} \to \widehat{X}_n$. Moreover, for each $q \in X_{n+1}$, we have $h_n(r_n^{n+1}(q)) = \widehat{r}_n^{n+1}(h_{n+1}(q))$. By this last commutativity, the one-to-one maps h_n define an equivalence between the inverse systems $\{X_n, r_n^{n+1}\}$ and $\{\widehat{X}_n, \widehat{r}_n^{n+1}\}$.

If $\mathbf{x} \in \widehat{X}_n \subset \widehat{X}_{n+1}$, there is a unique $p \in X_n$ such that $h_n(p) = \mathbf{x}$. Letting $q = g_n(p)$, we have $h_n(p) = h_{n+1}(q)$ as above. Consequently $\mathbf{x} = h_{n+1}(q)$, and $\widehat{r}_n^{n+1}(\mathbf{x}) = h_n(r_n^{n+1}(q)) = h_n(r_n^{n+1}(g_n(p))) = h_n(p) = \mathbf{x}$. Thus, $\widehat{r}_n^{n+1}$ is a retraction for each $n \geq 1$.

Let $\mathbf{x} = (p_1, p_2, \ldots) \in X$. Thus $(p_1, p_2, \ldots)$ is a thread in $\{X_n, r_n^{n+1}\}$, that uniquely defines a thread $h(\mathbf{x}) = (\mathbf{x}_1, \mathbf{x}_2, \ldots) =$

$(h_1(p_1), h_2(p_2), \ldots)$ in $\{\widehat{X}_n, \widehat{r}_n^{n+1}\}$, where h is the equivalence established above. For each $n \geq 1$, by definition,

$$\begin{aligned}\mathbf{x}_n &= h_n(p_n)\\ &= (r_1^n(p_n), \ldots, r_{n-1}^n(p_n), p_n, g_{n+1}^n(p_n), g_{n+2}^n(p_n), \ldots)\\ &= (p_1, \ldots, p_{n-1}, p_n, g_{n+1}^n(p_n), g_{n+2}^n(p_n), \ldots).\end{aligned}$$

Thus, identifying X with $\varprojlim\{\widehat{X}_n, \widehat{r}_n^{n+1}\}$ by the equivalence homeomorphism h, the projection on the nth-coordinate, which we denote by $\widehat{r}_n$, leaves the first n coordinates invariant. Therefore, these projections converge uniformly to the identity. It is also clear that $\lim \mathbf{x}_n = \mathbf{x}$. If $\mathbf{x} \in \widehat{X}_n$, there is a $p \in X_n$ such that

$$\mathbf{x} = h_n(p) = (r_1^n(p), r_2^n(p), \ldots, r_{n-1}^n(p), p, g_{n+1}^n(p), g_{n+2}^n(p), \ldots).$$

Since the nth-coordinate of $\mathbf{x}$ is p, for $h(\mathbf{x}) = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$, the nth-coordinate, $\mathbf{x}_n$, is $h_n(p) = \mathbf{x}$. Hence, the projections $\widehat{r}_n$ are retractions, and it is clear that $\widehat{r}_n \circ \widehat{r}_m = \widehat{r}_n$ for all $n < m$. The proof is complete. $\square$

Theorem 2.2. *Let X be a compactum, and let $\{f_n : X \to X\}$ be a sequence of retractions converging uniformly to the identity map on X. Suppose also that*:

(i) *for each $n \geq 1$, $f_n(X) \subset f_{n+1}(X)$, and*
(ii) *for all pairs $m, n \in \mathbb{N}$ with $m > n$, $f_n \circ f_m = f_n = f_m \circ f_n$.*

Then the inverse sequence $\{f_n(X), f_n|_{f_{n+1}(X)}\}$ is an internal inverse limit structure on X.

Proof. Note that the right-side equality in (ii) is immediate for a nested sequence $\{f_n(X)\}$. So, the left-side equality is the essential assumption. We begin with an observation that is straightforward to verify. Namely,

$$f_n \circ f_m = f_n \quad \text{for all } m > n$$

if and only if

$$f_n \circ f_{n+1} \circ \cdots \circ f_m = f_n \quad \text{for all } m > n.$$

For $n \geq 1$, let $X_n = f_n(X)$, and let $f_n^{n+1} = f_n|_{X_{n+1}}$. Since $\{X_n\}$ is nested, each f_n^{n+1} is a retraction onto X_n. Let $f_n^m \colon X_m \to X_n$ be defined by $f_n^m = f_n^{n+1} \circ f_{n+1}^{n+2} \circ \cdots \circ f_{m-1}^m$. So, by assumption (ii) and the observation above,

$$f_n^m = f_n|_{X_{n+1}} \circ f_{n+1}|_{X_{n+2}} \circ \cdots \circ f_{m-1}|_{X_m} = f_n|_{X_m}.$$

Consider $M = \varprojlim\{X_n, f_n^{n+1}\}$. We show that the conditions of Propositions 1.1 and 1.2 are satisfied for M interpreted as a collection of sequences in X.

Let $(x_1, x_2, \ldots) \in M$. From above, we have that, for $n < m$,

$$x_n = f_n^m(x_m) = f_n(x_m).$$

Since $\{f_n : X \to X\}$ converges uniformly to the identity map on X, given $\epsilon > 0$, for large N and $m > n > N$, we have that

$$d(x_n, x_m) = d(f_n^m(x_m), x_m) = d(f_n(x_m), x_m) < \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence and, hence, $\lim x_n$ exists. Let $x = \lim x_n$. For fixed n, it follows from the displayed equation above that the sequence $\{f_n(x_m)\}_{m>n}$ is the constant sequence $\{x_n\}$. So, $\lim f_n(x_m) = f_n(x)$, that is, $f_n(x) = x_n$.

So if, for points $(x_1, x_2, \ldots) \in M$ and $(y_1, y_2, \ldots) \in M$, we have that $x = \lim x_n = \lim y_n = y$, it follows that, for $n \geq 1$, $x_n = f_n(x) = f_n(y) = y_n$, that is, the assignment $(x_1, x_2, \ldots) \mapsto \lim x_n$ is one-to-one.

For $\epsilon > 0$, let N be large enough so that, for $n > N$, $d(f_n, \mathrm{id}) < \epsilon$. Then, for $(x_1, x_2, \ldots) \in M$ and $x = \lim x_i$, we have that, for $n > N$, $d(x_n, \lim x_i) = d(f_n(x), x) < \epsilon$, that is, the threads of M form a uniformly convergent collection of sequences in X. It follows from Propositions 1.1, 1.2 and 1.3 that $\{X_n, f_n^{n+1}\}$ is an internal inverse limit structure on X. $\square$

We end this section with the result discussed in Case 2 of the introduction, the case in which the images under the retractions are nested but the retractions do not have to commute.

Theorem 2.3. *Let X be a compactum, and let $\{f_n : X \to X\}$ be a sequence of retractions converging uniformly to the identity map on X. Suppose also that, for each $n \geq 1$, $f_n(X) \subset f_{n+1}(X)$. Then there are a subsequence $\{Y_{n_k}\}$ of $\{f_n(X)\}$ and retractions $g_k : Y_{n_{k+1}} \to Y_{n_k}$ such* *that the inverse sequence* $\{Y_{n_k}, g_k\}$ *is an internal inverse limit structure on* X.

Proof. This result is a consequence of Corollary 1.5. Indeed, let $X_n = f_n(X)$ and $g_k = f_{n_k}$ be the subsequence guaranteed by Corollary 1.5. We note that, since the sequence $\{X_n\}$ is nested, the restrictions $g_k|X_{n_{k+1}}$ are retractions. Thus, the inverse system $\{g_k(X), g_k \mid X_{n_{k+1}}\}$ guaranteed by Corollary 1.5 is a desired internal inverse limit structure on X. $\square$

3. Retractably $\mathcal{K}$-like and retractably $\mathcal{K}$-representable compacta. In this section, we investigate compacta with near-identity retractions to subspaces belonging to some restricted classes of spaces. We begin with the following result, which is a corollary to Theorem 2.3 from the previous section.

Corollary 3.1. *Let* X *be a compactum and* $\mathcal{K}$ *a class of compacta. The following are equivalent*:

(i) X *is retractably* $\mathcal{K}$*-like onto a nested sequence of compacta.*
(ii) X *is retractably* $\mathcal{K}$*-representable.*
(iii) $X \overset{T}{\approx} \varprojlim\{X_n, r_n^{n+1}\}$, *where each* X_n *is in* $\mathcal{K}$ *and each* r_n^{n+1} *is an* r*-map.*

Corollary 3.1 provides a partial answer to Question 5.3 in [**5**]. Namely, if the images $r_k(X)$, for a decreasing sequence $\{\epsilon_k\} \to 0$, can be chosen to be nested, then 5.2.2 implies 5.2.1.

In the remainder of the paper, we show that the nested assumption in Corollary 3.1 can be removed for the class of graphs that have order at most three. This is Case 3 as discussed in the introduction. It is known that each graph-like continuum can be represented as an inverse limit on graphs with order at most three.

In the proof of the next result we use the notation ab for an arc from a to b. It may be helpful to draw pictures while reading the proof.

Theorem 3.2. *Let* G *be a graph of order at most three contained in a space* X. *For every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that, if* $f : G \to X$ *is* *a map with* $\widetilde{d}(f) < \delta$*, then there is an embedding* $h : G \to f(G)$ *with* $\widetilde{d}(h) < \epsilon$*.*

Proof. Let G be a graph of order at most three in a space X, and let $\mathcal{E}$ be a finite collection of arcs in G such that $\bigcup \mathcal{E} = G$, and each two different members of $\mathcal{E}$ either are disjoint, or have a common end point as their intersection. Thus, $\mathcal{E}$ is the collection of edges of a simplicial complex on G.

Given $\epsilon > 0$, let n be an even positive integer such that each $E \in \mathcal{E}$ can be divided by n non-end points into $n+1$ arcs each having diameter less than $\epsilon/6$. Fix such a set of division points for each $E \in \mathcal{E}$. We group the resulting division arcs into two categories. Namely, suppose a and b are the end points of an $E \in \mathcal{E}$, and $p_1, \dots, p_n$ are the division points of E listed in the ordering from a to b. We call the arcs $ap_1, p_2p_3, \dots, p_{n-2}p_{n-1}$, p_nb category $\mathcal{A}$ division arcs, and the arcs $p_1p_2, p_3p_4, \dots, p_{n-1}p_n$, category $\mathcal{B}$ division arcs. Thus division arcs of category $\mathcal{A}$ and category $\mathcal{B}$ alternate on E, and at the end points of E we only have category $\mathcal{A}$ arcs.

Let $\mathcal{A}$ and $\mathcal{B}$ be the collections of all category $\mathcal{A}$ and category $\mathcal{B}$ arcs, respectively, for all $E \in \mathcal{E}$. Thus, each two different members of $\mathcal{B}$ are disjoint. Two different members of $\mathcal{A}$ are either disjoint, or they may intersect at a common end point, if they are end division arcs of adjacent edges in $\mathcal{E}$. At most three different members of $\mathcal{A}$ may meet at a point because G has order three, and each of them is disjoint from any other member of $\mathcal{A}$. If two or three members of $\mathcal{A}$ meet, we call them a *cluster* of $\mathcal{A}$. If $A \in \mathcal{A}$ contains an end point of G, we call it an *end member* of $\mathcal{A}$, and, if $A \in \mathcal{A}$ neither contains an end point of G nor is a member of a cluster, we call it a *regular member* of $\mathcal{A}$.

Let $\alpha = \min\{d(x, y) \mid x \text{ and } y \text{ are in disjoint members of } \mathcal{A} \cup \mathcal{B}\}$. Define $\delta = \min\{\alpha/3, \epsilon/6\}$. Let $f : G \to X$ be a map with $\widetilde{d}(f) < \delta$. By the choice of α and δ, and by the triangle inequality, if x and y are in two non-intersecting members of $\mathcal{A} \cup \mathcal{B}$, then $f(x) \neq f(y)$. Thus, two members of $\mathcal{A} \cup \mathcal{B}$ intersect if and only if their images under f intersect.

Let $B \in \mathcal{B}$. There are exactly two different members, A_1 and A_2, of $\mathcal{A}$ intersecting B at its endpoints. The set $f(B)$ is a locally connected continuum intersecting both $f(A_1)$ and $f(A_2)$, and $f(A_1) \cap f(A_2) = \emptyset$ because A_1 and A_2 are disjoint. Thus, there is an irreducible arc $\widehat{B}$

in $f(B)$ connecting $f(A_1)$ and $f(A_2)$. Fix a point $p_B \in B \setminus (A_1 \cup A_2)$ such that $f(p_B) \in \widehat{B} \setminus f(A_1 \cup A_2)$. Choose such $\widehat{B}$ and p_B for each $B \in \mathcal{B}$. Let $H = \bigcup\{f(A) \mid A \in \mathcal{A}\} \cup \bigcup\{\widehat{B} \mid B \in \mathcal{B}\}$ and note that $H \subset f(G)$. We write $h(p_B) = f(p_B)$ for each $B \in \mathcal{B}$. We will extend this last notation to define the desired embedding h.

If A is an end member of $\mathcal{A}$, with an end point $q \in A$ of G, there is exactly one member B of $\mathcal{B}$ intersecting A. The arc $\widehat{B}$ contains a unique irreducible arc $\widehat{B}_*$ connecting $f(p_B)$ with $f(A)$. The continuum $\widehat{B}_* \cup f(A)$, being the union of two locally connected continua, is locally connected. Thus, it contains an arc $L(A)$ from $f(p_B)$ to $f(q)$. We fix a homeomorphism from the arc p_Bq in G to $L(A)$ that sends p_B to $f(p_B)$ and q to $f(q)$, and we denote by $h(x)$ the image of any $x \in p_Bq$ under this homeomorphism. If $x \in p_Bq$, then $h(x) = f(y)$ for some $y \in A \cup B$. Thus, $d(x, h(x)) \leq d(x, y) + d(y, f(y)) < \operatorname{diam}(A \cup B) + \delta < \epsilon/6 + \epsilon/6 + \epsilon/6 < \epsilon$.

Let A_1 and A_2 form a cluster of exactly two members of $\mathcal{A}$. There are exactly two members, B_1 and B_2, of $\mathcal{B}$ intersecting the union $A_1 \cup A_2$. Assume $A_1 \cap B_1 \neq \emptyset \neq A_2 \cap B_2$ (the other case is similar). The arcs $\widehat{B}_1$ and $\widehat{B}_2$ contain irreducible arcs $\widehat{B}_{1*}$ and $\widehat{B}_{2*}$, respectively, connecting $f(p_{B_1})$ with $f(A_1)$, and $f(p_{B_2})$ with $f(A_2)$, correspondingly. The continuum $\widehat{B}_{1*} \cup f(A_1) \cup f(A_2) \cup \widehat{B}_{2*}$, being the union of four locally connected continua, contains an arc $L(A_1, A_2)$ from $f(p_{B_1})$ to $f(p_{B_2})$. We fix a homeomorphism from the unique arc $p_{B_1}p_{B_2}$ in $B_1 \cup A_1 \cup A_2 \cup B_2$ to $L(A_1, A_2)$ that sends p_{B_1} to $f(p_{B_1})$, and we denote by $h(x)$ the image of any $x \in p_{B_1}p_{B_2}$ under this homeomorphism. If $x \in p_{B_1}p_{B_2}$, then $h(x) = f(y)$ for some $y \in B_1 \cup A_1 \cup A_2 \cup B_2$. Thus, $d(x, h(x)) \leq d(x, y) + d(y, f(y)) < \operatorname{diam}(B_1 \cup A_1 \cup A_2 \cup B_2) + \delta < 4\epsilon/6 + \epsilon/6 < \epsilon$.

The case of a regular member A of $\mathcal{A}$ is similar to the case of a cluster of two members of A but simpler. The role of $A_1 \cup A_2$ from the previous case is played by A. Again, we have members B_1 and B_2 of $\mathcal{B}$ on both sides of A. The definition of h on the arc $p_{B_1}p_{B_2}$ is almost identical as in the previous case, and so is an estimation of the supremum of $d(x, h(x))$. We leave the details to the reader.

Let A_1, A_2 and A_3 form a cluster of exactly three members of $\mathcal{A}$. There are exactly three members, B_1, B_2 and B_3, of $\mathcal{B}$ intersecting the union $A_1 \cup A_2 \cup A_3$. Without loss of generality, assume $A_1 \cap B_1 \neq$

$\emptyset \neq A_2 \cap B_2$ and $A_3 \cap B_3 \neq \emptyset$. The arcs $\widehat{B}_1$, $\widehat{B}_2$ and $\widehat{B}_3$ contain irreducible arcs $\widehat{B}_{1*}$, $\widehat{B}_{2*}$ and $\widehat{B}_{3*}$, respectively, connecting $f(p_{B_1})$ with $f(A_1)$, $f(p_{B_2})$ with $f(A_2)$, and $f(p_{B_3})$ with $f(A_3)$, correspondingly. The continuum $\widehat{B}_{1*} \cup f(A_1) \cup f(A_2) \cup \widehat{B}_{2*}$, being the union of four locally connected continua, contains an arc K from $f(p_{B_1})$ to $f(p_{B_2})$. This arc intersects $f(A_1) \cup f(A_2)$. The continuum $\widehat{B}_{3*} \cup f(A_1) \cup f(A_2) \cup f(A_3)$ is also locally connected. Thus, it contains an irreducible arc L connecting $f(p_{B_3})$ with K. The junction point with K may be neither $f(p_{B_1})$ nor $f(p_{B_2})$. Thus, the union $T = K \cup L$ is a simple triod in $f(A_1) \cup f(A_2) \cup f(A_3) \cup \widehat{B}_{1*} \cup \widehat{B}_{2*} \cup \widehat{B}_{3*}$ having $f(p_{B_1})$, $f(p_{B_2})$ and $f(p_{B_3})$ as its end points.

Let T_0 be the simple triod in $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3$ having p_{B_1}, p_{B_2} and p_{B_3} as its end points. We fix a homeomorphism from T_0 to T that sends p_{B_1} to $f(p_{B_1})$, p_{B_2} to $f(p_{B_2})$ and p_{B_3} to $f(p_{B_3})$. We denote by $h(x)$ the image of any $x \in T_0$ under this homeomorphism. If $x \in T_0$, then $h(x) = f(y)$ for some $y \in A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3$. Thus, $d(x, h(x)) \leq d(x, y) + d(y, f(y)) < \operatorname{diam}(A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3) + \delta < 4\epsilon/6 + \epsilon/6 < \epsilon$.

Combining together this construction for all members and clusters of $\mathcal{A}$, a homeomorphism $h : G \to h(G)$ is defined with $h(G) \subset H \subset f(G)$ and $\widetilde{d}(h) < \epsilon$, as needed. $\square$

Below, we apply this theorem to the study of internal inverse limits. However, this result seems to be of interest in its own right. What other spaces may have the property concluded in the above theorem? First, notice that G cannot be a graph having order greater than or equal 4. Indeed, a 4-od in the plane admits, for each $\epsilon > 0$, a map onto a graph homeomorphic to the letter **H** such that each point is moved to the ϵ-neighborhood of itself. Obviously, this last graph contains no 4-od, and thus the conclusion of the theorem does not hold for a 4-od.

Question 3.3. *Suppose G is a graph satisfying the conclusion of the above theorem. That is, for every copy G' embedded in a space X, and for every $\epsilon > 0$, there is a $\delta > 0$ such that if $f : G' \to X$ is a map with $\widetilde{d}(f) < \delta$, then there is an embedding $h : G' \to f(G')$ with $\widetilde{d}(h) < \epsilon$. Is G a graph of order at most 3?*

Theorem 3.4. *Suppose X is a continuum and $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a sequence of graphs in X, each having order at most three. Suppose also that $\{r_k \colon X \to G_k\}$ is a sequence of retractions converging uniformly to the identity map on X. Then X can be represented as an inverse limit of members of $\mathcal{G}$ with retractions as bonding maps.*

Proof. Given positive integers m and n, let $g_n^m : G_m \to G_n$ be the restriction $r_n|_{G_m}$. Given an n, the maps $g_m^n : G_n \to G_m$, for $m > n$, satisfy $\lim_{m\to\infty} \widetilde{d}(g_m^n) = 0$. By Theorem 3.2, for fixed n and sufficiently large m, there are embeddings $h_m^n : G_n \to G_m$ such that $\lim_{m\to\infty} \widetilde{d}(h_m^n) = 0$. Note that it suffices to prove the conclusion for a subsequence of $\{G_n\}$. Thus, by replacing $\{G_n\}$ with an inductively selected subsequence, without loss of generality, we assume the embeddings $h_m^n : G_n \to G_m$ are defined for all n and $m > n$. Using the embeddings h_m^n, for each n, we want to slightly modify maps g_n^m to r-maps $f_n^m : G_m \to G_n$, and then apply Theorem 2 to complete the proof.

Consider the product $\widehat{X} = X \times \{0, \frac{1}{1}, \frac{1}{2}, \ldots\}$ with $X_0 = X \times \{0\}$ and the projection π_0 to X. To define f_n^m's, we fix an n. For $m > n$, let $U_{m,n}$ be a sequence of open ϵ_m-neighborhoods of $H_m = h_m^n(G_n)$ in G_m, respectively, with $\epsilon_m < 1/n$ and $\lim \epsilon_m = 0$. Let $F_m = G_m \setminus U_{m,n}$, and let Y and Z be the following two subsets of $\widehat{X}$:

$$Y = X_0 \cup \bigcup \{G_{n+k} \times \{1/k\} \mid k \in \{1, 2, \ldots\}\}$$

$$Z = X_0 \cup \bigcup \{(H_{n+k} \cup F_{n+k}) \times \{1/k\} \mid k \in \{1, 2, \ldots\}\}.$$

We define a retraction $r_0 : Z \to G_n \times \{0\}$ as follows. For $(x, 0) \in X_0$, let $r_0(x, 0) = (0, r_n(x))$. For $p = (x, 1/k)$ with $x \in H_m$ and $m = n + k$, we let $r_0(p) = ((h_m^n)^{-1}(x), 0)$. For $p = (x, 1/k)$ with $y \in F_m$ and $m = n + k$, we let $r_0(p) = (g_n^m(y), 0)$. Verifying that r_0 is well defined and continuous is straightforward. Since G_n is an absolute retract for one-dimensional compact spaces [**7,** page 354, Theorem 1], and Y and Z are one-dimensional and compact, the retraction r_0 can be extended to a retraction $r : Y \to G_n \times \{0\}$.

For any $k \in \{1, 2, \ldots\}$, let $q_k : G_{n+k} \to Y$ be defined as $p \mapsto (p, 1/k)$. Given $p \in G_m = G_{n+k}$, we let $f_n^m(p) = \pi_0(r(q_k(p)))$. Note that $f_n^m : G_m \to G_n$ are well defined r-maps, and the sequence $\{f_n^m\}$ is uniformly equicontinuous.

We are almost ready to apply Theorem 1.4 to complete the proof. A careful reader notices, however, that by taking an arbitrary extension r of r_0, we have possibly lost the uniform convergence of the sizes $\widetilde{d}(f_n^m)$ to 0 with respect to n, which is a condition of Theorem 1.4. Nevertheless, observe that, for each n, we have $\lim_{m\to\infty} \sup\{d(g_n^m(x), f_n^m(x)) \mid x \in G_m\} = 0$ and $\lim_{n\to\infty} \sup\{\widetilde{d}(g_n^m) \mid m > n\} = 0$. By choosing a correct subsequence of $\{G_n\}$ and the corresponding maps, we can ensure that $\lim_{n\to\infty} \sup\{\widetilde{d}(f_n^m) \mid m > n\} = 0$. Applying Theorems 1.4 and 2.1, the conclusion follows. $\square$

We have the following corollary to Theorem 3.4.

Corollary 3.5. *Let X be a compactum, and let $\mathcal{G}$ be a class of graphs of order at most three. The following are equivalent.*

(i) *X is retractably $\mathcal{G}$-like.*
(ii) *X is retractably $\mathcal{G}$-representable.*
(iii) *$X \overset{T}{\approx} \varprojlim\{X_n, r_n^{n+1}\}$, where each X_n is in $\mathcal{G}$ and each r_n^{n+1} is an r-map.*

Corollary 3.5 provides another partial answer to Question 5.3 in [**5**]. Namely, if the images $r_k(X)$, for a decreasing sequence $\{\epsilon_k\} \to 0$, can be chosen to be graphs of order at most three, then 5.2.2 implies 5.2.1.

We do not know whether, in our results, the assumption that the graphs have order at most three is essential. We end the paper with the following two natural questions.

Question 3.6. *Given a class $\mathcal{G}$ of graphs, is every retractably $\mathcal{G}$-like continuum retractably $\mathcal{G}$-representable?*

Question 3.7. *Does there exist a class $\mathcal{P}$ of polyhedra and a retractably $\mathcal{P}$-like continuum which is not retractably $\mathcal{P}$-representable?*

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