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GRÖBNER-SHIRSHOV BASES OF SOME WEYL GROUPS

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ABSTRACT. In this paper, we obtain Gröbner-Shirshov (non-commutative) bases for the n-extended affine Weyl group $\widetilde{W}$ of type A_1, elliptic Weyl groups of types $A_1^{(1,1)}$, $A_1^{(1,1)^*}$ and the 2-extended affine Weyl group of type $A_2^{(1,1)}$ with a generator system of a 2-toroidal sense. It gives a new algorithm for getting normal forms of elements of these groups and hence a new algorithm for solving the word problem in these groups.

1. Introduction. The Gröbner basis theory for commutative algebras was introduced by Buchberger [**8**] and provides a solution to the reduction problem for commutative algebras. In [**3**], Bergman generalized the Gröbner basis theory to associative algebras by proving the "Diamond lemma." On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [**14**]. In [**4**], Bokut noticed that Shirshov's method also works for associative algebras. Hence, for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner-Shirshov basis* theory. There are some important studies on this subject related to the groups (see, for instance, [**7, 9**]). We may finally refer to the papers [**2, 5, 6, 10, 11**] for some other recent studies on Gröbner-Shirshov bases.

Algorithmic problems such as the *word, conjugacy* and *isomorphism problems* have played an important role in group theory since the work of Dehn in the early 1900's. These problems are called *decision problems* which ask for a yes or no answer to a specific question. Among these decision problems, the word problem especially has been studied widely in groups (see [**1**]). It is well known that the word problem for finitely presented groups is not solvable in general; that is, given

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any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group.

The method of Gröbner-Shirshov bases which is the main theme of this paper gives a new algorithm for getting normal forms of elements of groups, and hence a new algorithm for solving the word problem in these groups. By considering this fact, our aim in this paper is to find Gröbner-Shirshov bases of the n-extended affine Weyl group of type A_1, elliptic Weyl groups of types $A_1^{(1,1)}$, $A_1^{(1,1)^*}$ and the 2-extended affine Weyl group of type $A_2^{(1,1)}$.

Extended affine root systems and the associated Weyl groups were introduced and studied by Saito [**12**]. In particular, 2-extended affine root systems are also called elliptic root systems from the point of view of the elliptic singularities. The defining relations of generators of the elliptic Weyl groups associated to the elliptic root systems were obtained by Saito and Takebayashi [**13**].

Throughout this paper, we order words in given alphabet in the deg-lex way comparing two words first by their lengths and then lexicographically when the lengths are equal. Additionally, $(i) \cap (j)$ and $(i) \cup (j)$ denote the intersection and inclusion compositions of relations (i), (j), respectively.

2. Gröbner-Shirshov bases and composition-diamond lemma. Let K be a field and $K\langle X\rangle$ the free associative algebra over K generated by X. Denote X^* as the free monoid generated by X, where the empty word is the identity denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$. Suppose that X^* is a well-ordered set. Then every nonzero polynomial $f \in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Let f and g be two monic polynomials in $K\langle X\rangle$. We then have two compositions as follows:

- If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.
- If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial

$(f,g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

If g is monic, $\overline{f} = a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then the transformation $f \mapsto f - \alpha agb$ is called elimination (ELW) of the leading word of g in f.

Let $S \subseteq K\langle X\rangle$ with each $s \in S$ monic. Then the composition $(f,g)_w$ is called *trivial modulo* (S,w) if $(f,g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K, a_i, b_i \in X^*, s_i \in S$ and $a_i\overline{s_i}b_i < w$. If this is the case, then we write $(f,g)_w \equiv 0 \bmod (S,w)$.

We call the set S endowed with the well ordering $<$ a *Gröbner-Shirshov basis* for $K\langle X \mid S\rangle$ if any composition $(f,g)_w$ of polynomials in S is trivial modulo S and corresponding w.

The following lemma was proved by Shirshov **[14]** for free Lie algebras with deg-lex ordering.

Lemma 2.1. (Composition-Diamond lemma). *Let K be a field, let $A = K\langle X \mid S\rangle = K\langle X\rangle/\mathrm{Id}\,(S)$, and let $<$ be a monomial ordering on X^*, where $\mathrm{Id}\,(S)$ is the ideal of $K\langle X\rangle$ generated by S. Then the following statements are equivalent*:

(i) *S is a Gröbner-Shirshov basis.*
(ii) *$f \in \mathrm{Id}\,(S) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in S$ and $a,b \in X^*$.*
(iii) *$\mathrm{Irr}\,(S) = \{u \in X^* \mid u \neq a\overline{s}b, s \in S, a,b \in X^*\}$ is a basis for the algebra $A = K\langle X \mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^{comp}. Such a process is called the *Shirshov algorithm.*

2.1. Gröbner-Shirshov basis for the n-extended affine Weyl group $\widetilde{W}$ of type A_1. In **[17]**, the author calculated the growth series of the n-extended affine Weyl group $\widetilde{W}$ of type A_1 with a generator system of an n-toroidal sense. To do that the author had the following result.

Proposition 2.2. [**17**]. *The* n*-extended affine Weyl group* $\widetilde{W}$ *of type* A_1 *is presented as follows*:
Generators: w_i $(0 \leq i \leq n)$,
Relations: $w_i^2 = 1$ $(0 \leq i \leq n)$, $(w_i w_1 w_j)^2 = 1$ $(i, j \neq 1,\ 0 \leq i \neq j \leq n)$.

Let us order the generators as in the following:

- for $i < j$, we have $w_i > w_j$, i.e., $w_0 > w_1 > w_2 > \cdots > w_n$.

Now we give the first main result of this section.

Theorem 2.3. *A Gröbner-Shirshov basis of the* n*-extended affine Weyl group* $\widetilde{W}$ *of type* A_1 *consists of the following polynomials*:

$$(1)\ w_i^2 - 1 \quad (0 \leq i \leq n),$$
$$(2)\ w_i w_1 w_j - w_j w_1 w_i \quad (i, j \neq 1, 0 \leq i \neq j \leq n),$$

relative to the deg-lex order of words in the generators.

Proof. We need to prove that all compositions of polynomials (1)–(2) are trivial. To do that, firstly, we consider the intersection compositions of these polynomials. Hence, we have the following ambiguities w:

$$(1) \cap (1) : w = w_i^3 \quad (0 \leq i \leq n),$$
$$(1) \cap (2) : w = w_i^2 w_1 w_j \quad (i, j \neq 1, 0 \leq i \neq j \leq n),$$
$$(2) \cap (1) : w = w_i w_1 w_j^2 \quad (i, j \neq 1, 0 \leq i \neq j \leq n),$$
$$(2) \cap (2) : w = w_i w_1 w_j w_1 w_k \quad (i, j, k \neq 1, 0 \leq i < j < k \leq n).$$

All these ambiguities are trivial. Let us show one of them.

$$\begin{aligned}
(2) \cap (2) : w &= w_i w_1 w_j w_1 w_k \quad (i, j, k \neq 1, 0 \leq i < j < k \leq n),\\
(f, g)_w &= (w_i w_1 w_j - w_j w_1 w_i) w_1 w_k - w_i w_1 (w_j w_1 w_k - w_k w_1 w_j)\\
&= w_i w_1 w_j w_1 w_k - w_j w_1 w_i w_1 w_k - w_i w_1 w_j w_1 w_k\\
&\quad + w_i w_1 w_k w_1 w_j\\
&= w_i w_1 w_k w_1 w_j - w_j w_1 w_i w_1 w_k\\
&\qquad\qquad (\text{for } i < k \text{ we have } w_i w_1 w_k - w_k w_1 w_i)\\
&\equiv w_k w_1 w_i w_1 w_j - w_j w_1 w_k w_1 w_i\\
&\qquad\qquad (\text{for } j < k \text{ we have } w_j w_1 w_k - w_k w_1 w_j)
\end{aligned}$$

$$\equiv w_k w_1 w_j w_1 w_i - w_k w_1 w_j w_1 w_i \equiv 0.$$

It is seen that there are no inclusion compositions of polynomials (1)–(2). Hence, the proof. □

By considering Lemma 2.1 and Theorem 2.3, we have the following result.

Corollary 2.4. *Let $C(u)$ be a normal form of a word $u \in A_1$. Then $C(u)$ has a form*

$$w_{i_1}^{\epsilon_{i_1}} w_{i_2}^{\epsilon_{i_2}} \cdots w_{i_k}^{\epsilon_{i_k}} w_1 w_{j_1}^{\epsilon_{j_1}} w_{j_2}^{\epsilon_{j_2}} \cdots w_{j_t}^{\epsilon_{j_t}} w_1 w_{s_1}^{\epsilon_{s_1}} w_{s_2}^{\epsilon_{s_2}} \cdots w_{s_r}^{\epsilon_{s_r}} w_1 \cdots w_1 w_{m_1}^{\epsilon_{m_1}} w_{m_2}^{\epsilon_{m_2}} \cdots w_{m_p}^{\epsilon_{m_p}} w_1 w_{n_1}^{\epsilon_{n_1}} w_{n_2}^{\epsilon_{n_2}} \cdots w_{n_l}^{\epsilon_{n_l}},$$

where $i_k < j_1$, $j_t < s_1$, $m_p < n_1$ and $\epsilon_q = 0, 1$ for $q \in \{i_1, \ldots, i_k, j_1, \ldots, j_t, s_1, \ldots, s_r, m_1, \ldots, m_p, n_1, \ldots, n_l\}$.

2.2. Gröbner-Shirshov basis for the elliptic Weyl groups of types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$. The generators and relations of the elliptic Weyl group W of type $A_1^{(1,1)}$ are given as follows ([**13**], [**16**]):
Generators: w_i, w_i^* $(i = 0, 1)$,
Relations: $w_i^2 = {w_i^*}^2 = 1$ $(i = 0, 1)$, $w_0 w_0^* w_1 w_1^* = 1$.
The relation $w_0 w_0^* w_1 w_1^* = 1$ is also written as $w_0^* w_1 = w_0 w_1^* (\Leftrightarrow w_1^* w_0 = w_1 w_0^*)$.

Now we order the generators as $w_0^* > w_0 > w_1^* > w_1$. Therefore, we have the following result.

Theorem 2.5. *A Gröbner-Shirshov basis of the elliptic Weyl group W of type $A_1^{(1,1)}$ consists of the following polynomials*:

(1) $w_0^2 - 1$,	(2) $w_1^2 - 1$,	(3) ${w_0^*}^2 - 1$,
(4) ${w_1^*}^2 - 1$,	(5) $w_0^* w_1 - w_0 w_1^*$,	(6) $w_0^* w_0 - w_1 w_1^*$,

relative to deg-lex order of words in the generators.

Proof. We need to prove that all compositions of polynomials (1)–(6) are trivial. To do that, firstly, we consider the intersection compositions of these polynomials. Thus we have the following ambiguities:

$(1)\cap(1): w=w_0^3$, $(2)\cap(2): w=w_1^3$, $(3)\cap(3): w={w_0^*}^3$,
$(4)\cap(4): w={w_1^*}^3$, $(3)\cap(5): w={w_0^*}^2w_1$, $(5)\cap(2): w=w_0^*w_1^2$,
$(3)\cap(6): w={w_0^*}^2w_0$, $(6)\cap(1): w=w_0^*w_0^2$.

All of these ambiguities are trivial. Let us show some of them.

$$\begin{aligned}
(1)\cap(1): w &= w_0^3,\\
(f,g)_w &= (w_0^2-1)w_0 - w_0(w_0^2-1)\\
&= w_0^3 - w_0 - w_0^3 + w_0 \equiv 0.\\
(3)\cap(5): w &= {w_0^*}^2w_1,\\
(f,g)_w &= ({w_0^*}^2-1)w_1 - w_0^*(w_0^*w_1 - w_0w_1^*)\\
&= {w_0^*}^2w_1 - w_1 - {w_0^*}^2w_1 + w_0^*w_0w_1^*\\
&= w_0^*w_0w_1^* - w_1 \equiv w_1{w_1^*}^2 - w_1 \equiv 0.
\end{aligned}$$

It is seen that there are no inclusion compositions of polynomials (1)–(6). Hence, the proof. □

By Lemma 2.1 and Theorem 2.5, we have the following result.

Corollary 2.6. *Let $C(u)$ be a normal form of a word $u\in A_1^{(1,1)}$. Then $C(u)$ has a form $U_1(w_0^*)^{n_1}(w_1^*)^{n_1'}U_2(w_0^*)^{n_2}(w_1^*)^{n_2'}\cdots U_k(w_0^*)^{n_k}(w_1^*)^{n_k'}$, where $n_i, n_i' = 0,1$ and $U_i = w_0^{\epsilon_{i_1}}w_1^{\delta_{i_1}}w_0^{\epsilon_{i_2}}w_1^{\delta_{i_2}}\cdots w_0^{\epsilon_{i_s}}w_1^{\delta_{i_s}}$, where $\epsilon_{i_j},\delta_{i_j}=0,1$ for $1\le i\le k$ and $1\le j\le s$.*

Another elliptic Weyl group $A_1^{(1,1)*}$ has the following generators and relations:
Generators: w_0, w_1 w_1^*,
Relations: $w_0^2=w_1^2={w_1^*}^2=(w_0w_1w_1^*)^2=1$.
This Weyl group is obtained from the Weyl group of type $A_1^{(1,1)}$ by removing one generator w_0^*.

Now we order the generators as $w_0>w_1^*>w_1$. According to this order, we have the following result, the proof of which can be done easily and similarly to the proof of Theorem 2.5.

Theorem 2.7. *A Gröbner-Shirshov basis of the elliptic Weyl group W of type $A_1^{(1,1)*}$ consists of the following polynomials*:

(1) w_0^2-1, (2) w_1^2-1, (3) ${w_1^*}^2-1$, (4) $w_0w_1w_1^*-w_1^*w_1w_0$,

relative to deg-lex order of words in the generators.

By Lemma 2.1 and Theorem 2.7, we have the following result.

Corollary 2.8. *Let $C(u)$ be a normal form of a word $u \in A_1^{(1,1)*}$. Then $C(u)$ has a form $U_1(w_1^*)^{n_1}U_2(w_1^*)^{n_2}\cdots U_t(w_1^*)^{n_t}$, where $n_i = 0,1$ and $U_i = w_0^{\epsilon_{i_1}}w_1^{\delta_{i_1}}w_0^{\epsilon_{i_2}}w_1^{\delta_{i_2}}\cdots w_0^{\epsilon_{i_p}}w_1^{\delta_{i_p}}$, where $\epsilon_{i_j},\delta_{i_j} = 0,1$, for $1 \le i \le t$ and $1 \le j \le p$.*

2.3. Gröbner-Shirshov basis for the 2-extended affine Weyl group $\widetilde{W}$ of type $A_2^{(1,1)}$. The Weyl groups, the elliptic Weyl group of type $A_2^{(1,1)}$ in **[18]** and the 2-extended affine Weyl group of type $A_2^{(1,1)}$ in this paper are isomorphic, but their generator systems are different, and the latter is obtained by removing two generators from the former.

Proposition 2.9. **[15]**. *The 2-extended affine Weyl group $\widetilde{W}$ of type $A_2^{(1,1)}$ is presented as follows*:
Generators: w_i $(0 \le i \le 3)$,
Relations: $w_i^2 = 1$ $(0 \le i \le 3)$,

$$\begin{aligned} w_0w_1w_0 &= w_1w_0w_1, & w_0w_2w_0 &= w_2w_0w_2,\\ w_1w_2w_1 &= w_2w_1w_2, & w_1w_3w_1 &= w_3w_1w_3,\\ w_2w_3w_2 &= w_3w_2w_3, & w_0w_1w_0w_2w_3 &= w_3w_1w_0w_2w_0. \end{aligned}$$

Let us order the generators as $w_0 > w_1 > w_2 > w_3$. Now we have the following result.

Theorem 2.10. *A Gröbner-Shirshov basis of the 2-extended affine Weyl group $\widetilde{W}$ of type $A_2^{(1,1)}$ consists of the following polynomials:*

(1) $w_0^2 - 1$, (2) $w_1^2 - 1$, (3) $w_2^2 - 1$, (4) $w_3^2 - 1$,

(5) $w_0w_1w_0 - w_1w_0w_1$, (6) $w_0w_2w_0 - w_2w_0w_2$,

(7) $w_1w_2w_1 - w_2w_1w_2$, (8) $w_1w_3w_1 - w_3w_1w_3$,

(9) $w_2w_3w_2 - w_3w_2w_3$, (10) $w_0w_1w_0w_2w_3 - w_3w_1w_0w_2w_0$,

(11) $w_0w_3w_1w_3w_2 - w_1w_3w_2w_3w_0$,

(12) $w_1w_0w_1w_3w_2 - w_3w_1w_2w_0w_3$,

relative to deg-lex order of words in the generators.

Proof. We need to prove that all compositions of polynomials (1)–(12) are trivial. To do that, firstly, we consider the intersection compositions of these polynomials. Thus we have the following ambiguities:

$(1) \cap (1) : w = w_0^3$, $(1) \cap (5) : w = w_0^2w_1w_0$,

$(1) \cap (6) : w = w_0^2w_2w_0$,

$(1) \cap (10) : w = w_0^2w_1w_0w_2w_3$, $(1) \cap (11) : w = w_0^2w_3w_1w_3w_2$,

$(2) \cap (2) : w = w_1^3$, $(2) \cap (7) : w = w_1^2w_2w_1$,

$(2) \cap (8) : w = w_1^2w_3w_1$, $(2) \cap (12) : w = w_1^2w_0w_1w_3w_2$,

$(3) \cap (3) : w = w_2^3$, $(3) \cap (9) : w = w_2^2w_3w_2$,

$(4) \cap (4) : w = w_3^3$, $(5) \cap (1) : w = w_0w_1w_0^2$,

$(5) \cap (5) : w = w_0w_1w_0w_1w_0$, $(5) \cap (6) : w = w_0w_1w_0w_2w_0$,

$(5) \cap (10) : w = w_0w_1w_0w_1w_0w_2w_3$,

$(5) \cap (11) : w = w_0w_1w_0w_3w_1w_3w_2$,

$(6) \cap (1) : w = w_0w_2w_0^2$, $(6) \cap (5) : w = w_0w_2w_0w_1w_0$,

$(6) \cap (6) : w = w_0w_2w_0w_2w_0$, $(6) \cap (10) : w = w_0w_2w_0w_1w_0w_2w_3$,

$(6) \cap (11) : w = w_0w_2w_0w_3w_1w_3w_2$, $(7) \cap (2) : w = w_1w_2w_1^2$,

$(7) \cap (7) : w = w_1w_2w_1w_2w_1$, $(7) \cap (8) : w = w_1w_2w_1w_3w_1$,

$(7) \cap (12) : w = w_1w_2w_1w_0w_1w_3w_2$, $(8) \cap (2) : w = w_1w_3w_1^2$,

$(8) \cap (7) : w = w_1w_3w_1w_2w_1$, $(8) \cap (8) : w = w_1w_3w_1w_3w_1$,

$(8) \cap (12) : w = w_1w_3w_1w_0w_1w_3w_2$, $(9) \cap (3) : w = w_2w_3w_2^2$,

$$(9)\cap(9): w = w_2w_3w_2w_3w_2, \quad (10)\cap(4): w = w_0w_1w_0w_2w_3^2,$$
$$(11)\cap(3): w = w_0w_3w_1w_3w_2^2, \quad (11)\cap(9): w = w_0w_3w_1w_3w_2w_3w_2,$$
$$(12)\cap(3): w = w_1w_0w_1w_3w_2^2, \quad (12)\cap(9): w = w_1w_0w_1w_3w_2w_3w_2.$$

All of these ambiguities are trivial. Let us show some of them.

$$\begin{aligned}
(2)\cap(7): w &= w_1^2w_2w_1,\\
(f,g)_w &= (w_1^2-1)w_2w_1 - w_1(w_1w_2w_1 - w_2w_1w_2)\\
&= w_1^2w_2w_1 - w_2w_1 - w_1^2w_2w_1 + w_1w_2w_1w_2\\
&= w_1w_2w_1w_2 - w_2w_1 \equiv w_2w_1w_2^2 - w_2w_1 \equiv 0.\\
(12)\cap(9): w &= w_1w_0w_1w_3w_2w_3w_2,\\
(f,g)_w &= (w_1w_0w_1w_3w_2 - w_3w_1w_2w_0w_3)w_3w_2\\
&\quad - w_1w_0w_1w_3(w_2w_3w_2 - w_3w_2w_3)\\
&= w_1w_0w_1w_3w_2w_3w_2 - w_3w_1w_2w_0w_3^2w_2\\
&\quad - w_1w_0w_1w_3w_2w_3w_2 + w_1w_0w_1w_3^2w_2w_3\\
&\equiv w_1w_0w_1w_2w_3 - w_3w_1w_2w_0w_2 \equiv w_3w_1w_2w_0w_2\\
&\quad - w_3w_1w_2w_0w_2 \equiv 0.
\end{aligned}$$

Now we consider the ambiguity $(5)\cap(6): w = w_0w_1w_0w_2w_0$. Then we get

$$\begin{aligned}
(f,g)_w &= (w_0w_1w_0 - w_1w_0w_1)w_2w_0 - w_0w_1(w_0w_2w_0 - w_2w_0w_2)\\
&= w_0w_1w_0w_2w_0 - w_1w_0w_1w_2w_0 - w_0w_1w_0w_2w_0 + w_0w_1w_2w_0w_2\\
&= w_0w_1w_2w_0w_2 - w_1w_0w_1w_2w_0.
\end{aligned}$$

The polynomial $w_0w_1w_2w_0w_2 - w_1w_0w_1w_2w_0$ is written as the relator $w_0w_1w_2w_0w_2 = w_1w_0w_1w_2w_0$. Since we have studied group structure and we have the relator $w_0^2 = 1$, we can multiply both sides of this relation by w_0. Hence, we obtain the relation $w_0w_1w_2w_0w_2w_0 = w_1w_0w_1w_2$, and thus the polynomial $w_0w_1w_2w_0w_2w_0 - w_1w_0w_1w_2$. Then we get

$$\begin{aligned}
w_0w_1w_2w_0w_2w_0 - w_1w_0w_1w_2 &\equiv w_0w_1w_2^2w_0w_2 - w_1w_0w_1w_2\\
&\equiv w_0w_1w_0w_2 - w_1w_0w_1w_2\\
&\equiv w_1w_0w_1w_2 - w_1w_0w_1w_2 \equiv 0.
\end{aligned}$$

The ambiguities $(5)\cap(11), (6)\cap(5), (6)\cap(10), (6)\cap(11), (7)\cap(8), (7)\cap(12), (8)\cap(7), (8)\cap(12), (11)\cap(3), (11)\cap(9), (12)\cap(3)$ are also trivial by the same process.

Now we check inclusion compositions of polynomials (1)–(12). In this case, we have one inclusion composition $(5)\cup(10) : w = w_0w_1w_0w_2w_3$ which is trivial. Let us show it.

$$\begin{aligned}(5)\cup(10) : w &= w_0w_1w_0w_2w_3,\\ (f,g)_w &= (w_0w_1w_0 - w_1w_0w_1)w_2w_3\\ &\quad - 1(w_0w_1w_0w_2w_3 - w_3w_1w_0w_2w_0)\\ &= w_0w_1w_0w_2w_3 - w_1w_0w_1w_2w_3\\ &\quad - w_0w_1w_0w_2w_3 + w_3w_1w_0w_2w_0\\ &= w_3w_1w_0w_2w_0 - w_1w_0w_1w_2w_3\\ &\equiv w_3w_1w_0w_2w_0 - w_3w_1w_0w_2w_0 \equiv 0.\end{aligned}$$

Hence, the proof. □

By considering Theorems 2.3, 2.5, 2.7 and 2.10, we have the following result.

Corollary 2.11. *The word problem for the n-extended affine Weyl group of type A_1, elliptic Weyl groups of types $A_1^{(1,1)}$, $A_1^{(1,1)^*}$ and the 2-extended affine Weyl group of type $A_2^{(1,1)}$ is solvable.*

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