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ESSENTIAL SPECTRAL RADIUS OF QUASICOMPACT ENDOMORPHISMS OF LIPSCHITZ ALGEBRAS

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ABSTRACT. We establish a formula for the essential spectral radius of an endomorphism T of Lipschitz algebras under a condition which is equivalent to the quasicompactness of the endomorphism T. We also conclude a necessary and sufficient condition for an endomorphism of these algebras to be Riesz. Finally, we get a relation for the spectrum and the set of eigenvalues of a quasicompact and Riesz endomorphism of these algebras.

1. Introduction. Let (X, d) be a compact metric space with infinitely many points and $0 < \alpha \leq 1$. The Lipschitz algebra of order α, $\mathrm{Lip}(X, \alpha)$, is the algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X \quad \text{and} \quad x \neq y\right\} < \infty.$$

The subalgebra of those functions f with

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} = 0, \tag{1.1}$$

is denoted by $\mathrm{lip}(X, \alpha)$. These Lipschitz algebras were first studied by Sherbert **[12, 13]**. The algebras $\mathrm{Lip}(X, \alpha)$ for $0 < \alpha \leq 1$ and $\mathrm{lip}(X, \alpha)$ for $0 < \alpha < 1$ are natural Banach function algebras on X under the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X} |f(x)|$. Recall that a function algebra A on a compact Hausdorff space X is called *natural* if every nonzero complex homomorphism on A is an evaluation homomorphism at some point of X **[3,** Definition 4.1.3]. We note that $\mathrm{Lip}(X, 1) \subseteq \mathrm{lip}(X, \alpha) \subseteq \mathrm{Lip}(X, \alpha)$ (see **[1, 7]**).

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It is known that, if A is a natural Banach function algebra on a compact Hausdorff space X and T is a unital endomorphism of A, then there exists a self-map φ on X such that $Tf = f \circ \varphi$ for all $f \in A$. The converse does not hold in general. That is, given a continuous self-map $\varphi : X \to X$, the mapping T defined on A by $Tf = f \circ \varphi$ does not in general take A into A. However, if φ is a self-map on X such that, for every $f \in A$, $f \circ \varphi \in A$, then $T : f \mapsto f \circ \varphi$ is a unital endomorphism of A. In each case, we say that T is induced by φ. Thus, any unital endomorphism T of A can be regarded as a composition operator C_φ, and conversely any composition operator on A is a unital endomorphism. Sherbert in [**12,** Theorem 5.1] showed that a linear map T on $\mathrm{Lip}(X,\alpha)$ is a unital endomorphism if and only if there exists a self-map $\varphi : X \to X$ such that $Tf = f \circ \varphi$ for all $f \in \mathrm{Lip}(X,\alpha)$ and $d(\varphi(x),\varphi(y)) \leq Cd(x,y)$ for some constant $C > 0$ and for all $x, y \in X$. In this case, the self-map φ is called *Lipschitz function*, and we write

$$p(\varphi) = \sup_{x \neq y} \frac{d(\varphi(x),\varphi(y))}{d(x,y)}.$$

Kamowitz and Shenberg in [**8**] showed that an endomorphism T of $\mathrm{Lip}(X,\alpha)$ or of $\mathrm{lip}(X,\alpha)$ induced by a self-map φ on X is compact if and only if φ is a *supercontraction*, that is,

$$\lim_{d(x,y)\to 0} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} = 0.$$

In this note, we consider endomorphisms of Lipschitz algebras which are quasicompact or Riesz. For convenience, we give the definition of these notions.

Definition 1.1. Let E be an infinite dimensional Banach space. We denote by $\mathcal{B}(E)$ and $\mathcal{K}(E)$ the Banach algebra of all bounded linear operators and compact linear operators on E, respectively. The essential norm $\|T\|_e$ of $T \in \mathcal{B}(E)$ is the norm of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$, i.e.,

$$\|T\|_e = \|T - \mathcal{K}(E)\| = \mathrm{dist}\,(T, \mathcal{K}(E)) = \inf\{\|T - K\| : K \in \mathcal{K}(E)\}.$$

The essential spectral radius $r_e(T)$ of $T \in \mathcal{B}(E)$ is given by the formula

$$r_e(T) = \lim_{n\to\infty} (\|T^n\|_e)^{1/n} = \lim_{n\to\infty} \|T^n - \mathcal{K}(E)\|^{1/n}.$$

The operator $T \in \mathcal{B}(E)$ is called *Riesz* if $r_e(T) = 0$ and *quasicompact* if $r_e(T) < 1$.

Clearly, T is compact if and only if its essential norm is zero and T is quasicompact if and only if $\|T^n\|_e < 1$ for some positive integer n. Every Riesz operator is also quasicompact.

Recall that if T is an endomorphism of a Banach function algebra A on X induced by the self-map $\varphi : X \to X$, then T^n is an endomorphism of A induced by the self-map $\varphi_n : X \to X$ for each $n \in \mathbb{N}$, where φ_n is the nth iterate of φ. We also set $\varphi_0 = id$.

Some results have been obtained concerning quasicompact and Riesz endomorphisms of certain Lipschitz subalgebras in [**9, 10, 11**]. Behrouzi [**2**] studied quasicompact and Riesz endomorphisms of $\mathrm{Lip}(X, \alpha)$ and gave an estimate for the essential spectral radius of an endomorphism of $\mathrm{lip}(X, \alpha)$ under certain conditions. In this note, we assume that T is an endomorphism of Lipschitz algebras either $\mathrm{Lip}(X, \alpha)$ or $\mathrm{lip}(X, \alpha)$ induced by the self-map φ on X. We first show that the essential spectral radius of T satisfies

$$r_e(T) = \lim_{n\to\infty} p(\varphi_n)^{\alpha/n}$$

when $0 < \alpha < 1$ and

$$r_e(T) \leq \lim_{n\to\infty} p(\varphi_n)^{\alpha/n}$$

when $\alpha = 1$, provided $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$. We conclude that the condition $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ is sufficient for the endomorphism T to be quasicompact. Also, this condition is necessary for the quasicompactness of T, when X is connected. In addition, we show that T is Riesz if $\lim_{n\to\infty} p(\varphi_n)^{1/n} = 0$, and this is also a necessary condition provided X is connected and $0 < \alpha < 1$. We then generalize these results by establishing a formula for the essential spectral radius $r_e(T)$ under a condition which is equivalent to the quasicompactness of T without connectedness assumption on X. As an immediate consequence of the latter result we obtain a necessary and sufficient condition for the endomorphism T to be Riesz when $0 < \alpha < 1$. Moreover, when $\alpha = 1$, this condition is also sufficient. At the end, using the definition of Riesz point [**6,** page 217], we get a relation for the spectrum and the set of eigenvalues of a quasicompact and Riesz endomorphism of these algebras.

2. Results. Let X be a compact metric space with infinitely many points, and let the self-map $\varphi : X \to X$ be continuous. Then we have a nested sequence $\varphi_{n+1}(X) \subseteq \varphi_n(X)$ of nonempty compact sets, whence the intersection $\cap_{n=1}^{\infty}\varphi_n(X)$ is also nonempty. Moreover, if $p(\varphi_n) \to 0$ as $n \to 0$, then $\operatorname{diam}(\varphi_n(X)) \to 0$; hence, $\cap_{n=1}^{\infty}\varphi_n(X)$ is a singleton, say $\{x_0\}$. Using Banach's contraction principle, one can see that x_0 is the unique fixed point of φ. Therefore, if one defines the constant function $\theta : X \to X$ by $\theta(x) = x_0$, then

$$d(\varphi_n(x), \theta(x)) = d(\varphi_n(x), x_0) = d(\varphi_n(x), \varphi_n(x_0)) \le p(\varphi_n)\operatorname{diam}(X),$$

for all $x \in X$. Hence,

$$\lim_{n\to\infty} \sup_{x\in X} d(\varphi_n(x), \theta(x)) = \lim_{n\to\infty} \sup_{x\in X} d(\varphi_n(x), x_0) = 0,$$

for some $x_0 \in X$, if $p(\varphi_n) \to 0$. Note also that, for $n \in \mathbb{N}$ and $x, y \in X$, with $\varphi_n(x) \ne \varphi_n(y)$, we have $\varphi_k(x) \ne \varphi_k(y)$ for each $k = 0, 1, \dots, n$ and therefore,

$$\frac{d(\varphi_n(x), \varphi_n(y))}{d(x, y)} = \prod_{k=1}^{n} \frac{d(\varphi_k(x), \varphi_k(y))}{d(\varphi_{k-1}(x), \varphi_{k-1}(y))} \le p(\varphi)^n,$$

from which one obtains $p(\varphi_n) \le p(\varphi)^n$ for all $n \in \mathbb{N}$. It follows that $p(\varphi_n) \to 0$ if $p(\varphi) < 1$. Conversely, if $p(\varphi_n) \to 0$, then $p(\varphi_{n_0}) < 1$ for some positive integer n_0.

Remark 2.1. Let (X, d) be a compact pointed metric space, that is, a compact metric space with a base point $e \in X$. The Lipschitz space $\mathrm{Lip}_0(X, \alpha)$ is the space of all Lipschitz functions $f : X \to \mathbb{C}$ of order α $(0 < \alpha \le 1)$ which are zero at the base point $e \in X$. The space $\mathrm{Lip}_0(X, \alpha)$ is a Banach space under the Lipschitz norm $p_\alpha(\cdot)$. The space $\mathrm{lip}_0(X, \alpha)$, $0 < \alpha < 1$, is the closed subspace consisting of those functions $f \in \mathrm{Lip}_0(X, \alpha)$ that satisfy (1.1) (see [**15**]). Vargas et al. in [**14,** Theorem 3.1] showed that, if $\varphi : X \to X$ is a base point preserving Lipschitz mapping, then the essential norm of the composition operator $C_\varphi : \mathrm{lip}_0(X, \alpha) \to \mathrm{lip}_0(X, \alpha)$ satisfies the lower estimate

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \le \|C_\varphi\|_e.$$

Their proof is valid for the Banach algebras $\mathrm{lip}(X, \alpha)$ and $\mathrm{Lip}(X, \alpha)$ with the norm $\|\cdot\|_\alpha$ when $0 < \alpha < 1$. Using this fact, we obtain a formula for the essential spectral radius of a unital endomorphism of Lipschitz algebras.

Considering $p(\varphi_n) \leq p(\varphi)^n$, and using the fact that $p(\varphi_{m+n}) \leq p(\varphi_m)p(\varphi_n)$, $\lim_{n\to\infty} p(\varphi_n)^{1/n}$ exists and

$$\lim_{n\to\infty} p(\varphi_n)^{1/n} = \inf_n p(\varphi_n)^{1/n}$$

(see, for example [**3,** Proposition A.1.26(iii)]). Therefore, in the next theorem we can replace $\lim_{n\to\infty} p(\varphi_n)^{1/n}$ with $\inf_n p(\varphi_n)^{1/n}$.

In the remainder of this paper, we regard $\mathfrak{L}(\alpha)$ as being either the algebra $\mathrm{Lip}(X,\alpha)$ for $0<\alpha\leq 1$ or the algebra $\mathrm{lip}(X,\alpha)$ for $0<\alpha<1$.

Theorem 2.2. *Let X be a compact metric space, $0<\alpha<1$ and T an endomorphism of $\mathrm{Lip}(X,\alpha)$ or of $\mathrm{lip}(X,\alpha)$ induced by the self-map φ on X. If $p(\varphi_{n_0}) < 1$ for some positive integer n_0, then $r_e(T) = \lim_{n\to\infty} p(\varphi_n)^{\alpha/n}$.*

Proof. By Remark 2.1, we have

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \left(\frac{d(\varphi_n(x),\varphi_n(y))}{d(x,y)}\right)^{\alpha} \leq \|T^n\|_e,$$

for every $n\in\mathbb{N}$. By the assumption that $p(\varphi_{n_0})<1$ and by the definition of essential spectral radius $r_e(T) = \lim_{n\to\infty}\|T^n\|_e^{1/n}$, for given $\varepsilon>0$, one can choose a positive integer j such that $p(\varphi_j)<1$ and $\|T^j\|_e^{1/j} < r_e(T)+\varepsilon/2$. Fix a positive integer j with such a property. It follows that

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \left(\frac{d(\varphi_j(x),\varphi_j(y))}{d(x,y)}\right)^{\alpha/j} \leq \|T^j\|_e^{1/j} < r_e(T)+\frac{\varepsilon}{2},$$

and therefore,

$$\sup_{0<d(x,y)<\delta} \left(\frac{d(\varphi_j(x),\varphi_j(y))}{d(x,y)}\right)^{\alpha/j} < r_e(T)+\frac{\varepsilon}{2},$$

for some $\delta>0$.

Furthermore, $p(\varphi_{kj}) \leq p(\varphi_j)^k < 1$ and then $d(\varphi_{kj}(x),\varphi_{kj}(y)) \leq d(x,y)$ for each $x,y\in X$ and for each positive integer k. Let $n\in\mathbb{N}$,

$x, y \in X$ with $0 < d(x,y) < \delta$ and $\varphi_{nj}(x) \neq \varphi_{nj}(y)$. Then $0 < d(\varphi_{kj}(x), \varphi_{kj}(y)) < \delta$ for each k, from which we obtain

$$\begin{aligned}\left(\frac{d(\varphi_{nj}(x),\varphi_{nj}(y))}{d(x,y)}\right)^{\alpha/(nj)} &= \left(\prod_{k=0}^{n-1}\frac{d(\varphi_{(k+1)j}(x),\varphi_{(k+1)j}(y))}{d(\varphi_{kj}(x),\varphi_{kj}(y))}\right)^{\alpha/(nj)}\\ &= \prod_{k=0}^{n-1}\left(\frac{d(\varphi_j(\varphi_{kj}(x),\varphi_j(\varphi_{kj}(y))}{d(\varphi_{kj}(x),\varphi_{kj}(y))}\right)^{\alpha/(nj)}\\ &\leq \prod_{k=0}^{n-1}\sup_{0<d(u,v)<\delta}\left(\frac{d(\varphi_j(u),\varphi_j(v))}{d(u,v)}\right)^{\alpha/(nj)}\\ &= \sup_{0<d(x,y)<\delta}\left(\frac{d(\varphi_j(x),\varphi_j(y))}{d(x,y)}\right)^{\alpha/j}\\ &< r_e(T)+\frac{\varepsilon}{2}.\end{aligned}$$

Therefore,

$$\sup_{0<d(x,y)<\delta}\left(\frac{d(\varphi_{nj}(x),\varphi_{nj}(y))}{d(x,y)}\right)^{\alpha/(nj)} \leq r_e(T)+\frac{\varepsilon}{2},$$

for each $n \in \mathbb{N}$.

Also, since

$$\lim_{n\to\infty}\left(r_e(T)+\frac{\varepsilon}{2}\right)^{(n-1)/n} = r_e(T)+\frac{\varepsilon}{2},$$

and $\lim_{n\to\infty} p(\varphi_{nj}) = 0$, there exists $N \in \mathbb{N}$ such that $(r_e(T) + (\varepsilon/2))^{(n-1)/n} < r_e(T)+\varepsilon$ for every $n \geq N$, and $p(\varphi_{Nj}) < \delta/(\mathrm{diam}(X))$. It follows that

$$d(\varphi_{Nj}(x),\varphi_{Nj}(y)) < \frac{\delta}{\mathrm{diam}(X)}d(x,y) \leq \delta$$

for each $x, y \in X$.

Let $n > N$ and $x, y \in X$ with $\varphi_{nNj}(x) \neq \varphi_{nNj}(y)$. Then,

$$\left(\frac{d(\varphi_{nNj}(x),\varphi_{nNj}(y))}{d(x,y)}\right)^{\alpha/(nNj)}$$

$$
\begin{aligned}
&= \left(\frac{d(\varphi_{nNj}(x), \varphi_{nNj}(y))}{d(\varphi_{Nj}(x), \varphi_{Nj}(y))} \frac{d(\varphi_{Nj}(x), \varphi_{Nj}(y))}{d(x,y)} \right)^{\alpha/(nNj)} \\
&\leq \left(\frac{d(\varphi_{(n-1)Nj}(\varphi_{Nj}(x)), \varphi_{(n-1)Nj}(\varphi_{Nj}(y)))}{d(\varphi_{Nj}(x), \varphi_{Nj}(y))} \right)^{\alpha/(nNj)} \\
&\leq \left(\sup_{0<d(x,y)<\delta} \left(\frac{d(\varphi_{(n-1)Nj}(x), \varphi_{(n-1)Nj}(y))}{d(x,y)} \right)^{\alpha/[(n-1)Nj]} \right)^{(n-1)/n} \\
&< \left(r_e(T) + \frac{\varepsilon}{2} \right)^{(n-1)/n} < r_e(T) + \varepsilon.
\end{aligned}
$$

Therefore, $p(\varphi_{nNj})^{\alpha/(nNj)} \leq r_e(T) + \varepsilon$, for each $n > N$. Hence, $\lim_{n\to\infty} p(\varphi_n)^{\alpha/n} = \inf p(\varphi_n)^{\alpha/n} \leq r_e(T)$.

For the converse inequality, using the well-known relations $r_e(T^n) = r_e(T)^n$ and $p(\varphi_n) \leq p(\varphi)^n$, one may assume that $n_0 = 1$ and $p(\varphi) < 1$. Then $p(\varphi_n) \to 0$ and $\cap_{n=1}^{\infty} \varphi_n(X) = \{x_0\}$, where x_0 is the unique fixed point of φ. Define rank one endomorphism $S : \mathfrak{L}(\alpha) \to \mathfrak{L}(\alpha)$ by $Sf = f \circ \theta = f(x_0)1$ for $f \in \mathfrak{L}(\alpha)$ where $\theta : X \to X$ is the constant function $\theta(x) = x_0$. Let $n \in \mathbb{N}$ and $f \in \mathfrak{L}(\alpha)$ with $\|f\|_\alpha \leq 1$. Then,

$$
\begin{aligned}
|T^n f(x) - Sf(x)| &= |f(\varphi_n(x)) - f(x_0)| \leq p_\alpha(f)\, d(\varphi_n(x), x_0)^\alpha \\
&\leq \|f\|_\alpha p(\varphi_n)^\alpha d(x, x_0)^\alpha \leq p(\varphi_n)^\alpha (\operatorname{diam}(X))^\alpha,
\end{aligned}
$$

for each $x \in X$. Hence, $\|T^n f - Sf\|_X \leq p(\varphi_n)^\alpha (\operatorname{diam}(X))^\alpha$. On the other hand,

$$
\begin{aligned}
&|(T^n f - Sf)(x) - (T^n f - Sf)(y)| \\
&= |f(\varphi_n(x)) - f(\varphi_n(y))| \\
&\leq p_\alpha(f) d(\varphi_n(x), \varphi_n(y))^\alpha \leq p(\varphi_n)^\alpha d(x,y)^\alpha,
\end{aligned}
$$

for every $x, y \in X$. Thus, $p_\alpha(T^n f - Sf) \leq p(\varphi_n)^\alpha$. Therefore,

$$
\begin{aligned}
\|T^n f - Sf\|_\alpha &= \|T^n f - Sf\|_X + p_\alpha(T^n f - Sf) \\
&\leq (1 + (\operatorname{diam}(X))^\alpha) p(\varphi_n)^\alpha,
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $f \in \mathfrak{L}(\alpha)$ with $\|f\|_\alpha \leq 1$. Hence, $\|T^n - S\| \leq (1 + (\operatorname{diam}(X))^\alpha) p(\varphi_n)^\alpha$ for each $n \in \mathbb{N}$. Therefore,

$$
\|T^n\|_e = \|T^n - \mathcal{K}(\mathfrak{L}(\alpha))\| \leq \|T^n - S\| \leq (1 + (\operatorname{diam}(X))^\alpha) p(\varphi_n)^\alpha,
$$

and then,

$$r_e(T) = \lim_{n\to\infty} \|T^n\|_e^{1/n} \leq \lim_{n\to\infty} p(\varphi_n)^{\alpha/n}. \qquad \square$$

Considering the last part of the proof of the previous theorem, we note that the converse inequality is true even for $\alpha = 1$. In fact, we have the following proposition.

Proposition 2.3. *Let X be a compact metric space, and let T be an endomorphism of* $\mathrm{Lip}(X,1)$ *induced by the self-map φ on X. If $p(\varphi_{n_0}) <$ 1 for some positive integer n_0, then $r_e(T) \leq \lim_{n\to\infty} p(\varphi_n)^{1/n}$.*

It was shown in [**2,** Theorem 2.1] that an endomorphism T of $\mathrm{Lip}(X,\alpha)$ induced by a self-map φ on X is quasicompact if $p(\varphi_n) \to 0$ and φ_n converges uniformly on X to the constant function $\theta(x) = x_0$ for some $x_0 \in X$. Here, as a consequence of Theorem 2.2 and Proposition 2.3, we obtain that $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ is sufficient for an endomorphism T of $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$ or of $\mathrm{lip}(X,\alpha)$ $(0 < \alpha < 1)$ to be quasicompact (Corollary 2.4 (i)). Also, the function defined in the proof of the converse part of [**2,** Theorem 2.1] does not belong to $\mathrm{lip}(X,\alpha)$. As Corollary 2.4 (ii), defining a suitable function, a slightly modified argument establishes the converse part of [**2,** Theorem 2.1] for the Lipschitz algebras $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$ and $\mathrm{lip}(X,\alpha)$ $(0 < \alpha < 1)$.

Corollary 2.4. *Let X be a compact metric space and T an endomorphism of* $\mathrm{Lip}(X,\alpha)$, $0 < \alpha \leq 1$ *or of* $\mathrm{lip}(X,\alpha)$, $0 < \alpha < 1$ *induced by the self-map φ on X.*

- (i) *If $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$, then T is quasicompact.*
- (ii) *If X is connected and T is quasicompact, then $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$.*

Proof.

(i) Let $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$. Then by Theorem 2.2 and Proposition 2.3, we have

$$r_e(T) \leq \lim_{n\to\infty} p(\varphi_n)^{\alpha/n} = \lim_{k\to\infty} p(\varphi_{kn_0})^{\alpha/(kn_0)} \leq p(\varphi_{n_0})^{\alpha/n_0} < 1,$$

which implies that T is quasicompact.

(ii) Let X be connected and T be quasicompact. Using [**4,** Theorem 1.2], there exists $x_0 \in X$ such that the operators T^n converge, in operator norm, to a rank-one endomorphism S_0 of $\mathfrak{L}(\alpha)$ defined by $S_0(f) = f(x_0)1$. The point x_0 is the unique fixed point of φ.

In the case $\alpha = 1$, take $\beta = 1$; otherwise, choose any $\beta \in (\alpha, 1]$. For each $y \in Y$ and $n \in \mathbb{N}$, define

$$f_n(x) = \frac{d(x, \varphi_n(y))^\beta}{(\operatorname{diam}(X))^\beta + (\operatorname{diam}(X))^{\beta-\alpha}},$$

for $x \in X$. Then $f_n \in \mathfrak{L}(\alpha)$, $\|f_n\|_\alpha \leq 1$ and

$$\begin{aligned}\|T^n - S_0\| &\geq \|T^n f_n - S_0 f_n\|_\alpha \geq p_\alpha(T^n f_n - S_0 f_n)\\ &= p_\alpha(f_n \circ \varphi_n) \geq \frac{|f_n \circ \varphi_n(x) - f_n \circ \varphi_n(y)|}{d(x,y)^\alpha}\\ &= \frac{1}{(\operatorname{diam}(X))^\beta + (\operatorname{diam}(X))^{\beta-\alpha}} \frac{d(\varphi_n(x), \varphi_n(y))^\beta}{d(x,y)^\alpha},\end{aligned}$$

for every $x, y \in X$ with $x \neq y$ and any $\beta \in (\alpha, 1]$ or $\beta = \alpha = 1$. Taking limit as $\beta \to \alpha$, we conclude that

$$\|T^n - S_0\| \geq \frac{1}{(\operatorname{diam}(X))^\alpha + 1} \frac{d(\varphi_n(x), \varphi_n(y))^\alpha}{d(x,y)^\alpha},$$

for every $x, y \in X$ with $x \neq y$. Hence

$$\begin{aligned}\|T^n - S_0\| &\geq \frac{1}{(\operatorname{diam}(X))^\alpha + 1} \sup_{x \neq y} \frac{d(\varphi_n(x), \varphi_n(y))^\alpha}{d(x,y)^\alpha}\\ &= \frac{1}{(\operatorname{diam}(X))^\alpha + 1}\, p(\varphi_n)^\alpha.\end{aligned}$$

Therefore, $\lim_{n\to\infty} p(\varphi_n) = 0$ and $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$. □

From the proof of Corollary 2.4, one can obtain the following interesting relation for any endomorphism T of $\mathfrak{L}(\alpha)$ induced by the self-map φ on X:

$$\max\left\{1, \frac{1}{(\operatorname{diam}(X))^\alpha + 1}\, p(\varphi)^\alpha\right\} \leq \|T\| \leq \max\{1, p(\varphi)^\alpha\}.$$

In [**2,** Proposition 2.3], it was shown that an endomorphism T of $\operatorname{Lip}(X, \alpha)$ induced by a self-map φ on X is Riesz, if $\lim_{n\to\infty} p(\varphi_n)^{1/n} =$

0. As an immediate consequence of Theorem 2.2 and Proposition 2.3 one can get this result for the Lipschitz algebras $\mathrm{Lip}(X,\alpha)$ and $\mathrm{lip}(X,\alpha)$. Also, using Theorem 2.2 and Corollary 2.4 (ii), one can show that the condition $\lim_{n\to\infty} p(\varphi_n)^{1/n} = 0$ is necessary for the endomorphism T of $\mathrm{Lip}(X,\alpha)$ or of $\mathrm{lip}(X,\alpha)$ to be Riesz whenever $0 < \alpha < 1$ and X is connected.

Corollary 2.5. *Let X be a compact metric space, and let T be an endomorphism of* $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$ *or of* $\mathrm{lip}(X,\alpha)$ $(0 < \alpha < 1)$ *induced by the self-map φ on X.*

(i) *If* $\lim_{n\to\infty} p(\varphi_n)^{1/n} = 0$*, then T is Riesz.*
(ii) *If X is connected,* $0 < \alpha < 1$ *and T is Riesz, then* $\lim_{n\to\infty} p(\varphi_n)^{1/n} = 0$.

In the sequel, we generalize the above obtained results to possibly unconnected metric spaces.

Theorem 2.6. *Let X be a compact metric space,* $0 < \alpha < 1$*, and T an endomorphism of* $\mathrm{Lip}(X,\alpha)$ *or of* $\mathrm{lip}(X,\alpha)$ *induced by the self-map φ on X. If there exists a decomposition of X into a finite number of mutually disjoint clopen subsets, say $X_1, X_2, \ldots, X_m$, such that, for each $i \in \{1,\ldots,m\}$, there exists $n_i \in \mathbb{N}$ with $\varphi_{n_i}(X_i) \subseteq X_i$ and $p(\varphi_{n_i}|_{X_i}) < 1$, then* $r_e(T) = \max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)}$.

Proof. By Remark 2.1, we have

$$\lim_{t\to 0} \sup_{\substack{0<d(x,y)<t\\ x,y\in X_i}} \left(\frac{d(\varphi_n(x),\varphi_n(y))}{d(x,y)}\right)^{\alpha} \leq \lim_{t\to 0} \sup_{\substack{0<d(x,y)<t\\ x,y\in X}} \left(\frac{d(\varphi_n(x),\varphi_n(y))}{d(x,y)}\right)^{\alpha} \leq \|T^n\|_e,$$

for each $i \in \{1,2,\ldots,m\}$ and every $n \in \mathbb{N}$. Similar to the proof of Theorem 2.2, one can easily deduce that $\lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)} = \inf_n p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)} \leq r_e(T)$. Hence,

$$\max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)} \leq r_e(T).$$

We now show the converse inequality. By the hypotheses, we have $\varphi_{kn_i}(X_i) \subseteq X_i$ and $p(\varphi_{kn_i}|_{X_i}) < 1$ for each positive integer k. Therefore, if we set $n_0 = n_1 n_2 \cdots n_m$, then $\varphi_{n_0}(X_i) \subseteq X_i$ and $p(\varphi_{n_0}|_{X_i}) < 1$ for each $i \in \{1, 2, \ldots, m\}$. As in the proof of Theorem 2.2, we may assume that $n_0 = 1$, and, in a similar way, we have $\lim_{n\to\infty} p(\varphi_n|_{X_i}) = 0$ and $\bigcap_{n=1}^{\infty} \varphi_n(X_i) = \{x_i\}$ for each $i \in \{1, \ldots, m\}$, where $x_i \in X_i$ is the unique fixed point of $\varphi|_{X_i}$. Define the continuous self-map $\theta : X \to X$ by $\theta(x) = x_i$, $(x \in X_i)$ and consider the finite rank endomorphism $S : \mathfrak{L}(\alpha) \to \mathfrak{L}(\alpha)$ by $Sf = f \circ \theta = \sum_{i=1}^{m} f(x_i)\chi_{X_i}$, where χ_{X_i} is the characteristic function of X_i.

Let $n \in \mathbb{N}$ and $f \in \mathfrak{L}(\alpha)$ with $\|f\|_\alpha \leq 1$. Then

$$\|T^n f - Sf\|_X \leq (\mathrm{diam}(X))^\alpha \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha.$$

Set $\mu = \min_{1\leq i<j\leq m} d(X_i, X_j)$. Then

$$\frac{|(T^n f - Sf)(x) - (T^n f - Sf)(y)|}{d(x,y)^\alpha} \leq \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha,$$

when x, y belong to the same X_i, and

$$\frac{|(T^n f - Sf)(x) - (T^n f - Sf)(y)|}{d(x,y)^\alpha} \leq \frac{2}{\mu^\alpha}(\mathrm{diam}(X))^\alpha \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha,$$

when x, y are in the different X_i. Hence,

$$p_\alpha(T^n f - Sf) \leq \left(1 + \frac{2}{\mu^\alpha}(\mathrm{diam}(X))^\alpha\right) \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha.$$

Therefore,

$$\|T^n - S\| \leq \left(1 + \left(\frac{2}{\mu^\alpha} + 1\right)(\mathrm{diam}(X))^\alpha\right) \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha,$$

for each $n \in \mathbb{N}$. Whence,

$$\|T^n\|_e \leq \left(1 + \left(\frac{2}{\mu^\alpha} + 1\right)(\mathrm{diam}(X))^\alpha\right) \max_{1\leq i\leq m} p(\varphi_n|_{X_i})^\alpha,$$

and then $r_e(T) \leq \max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_n|_{X_i})^{\alpha/n}$. □

Remark 2.7. Similar to Proposition 2.3, the inequality

$$r_e(T) \leq \max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)},$$

holds for $\alpha = 1$.

Now we would like to generalize Corollaries 2.4 and 2.5 for possibly unconnected X. For this purpose, we shall need the following results due to Feinstein and Kamowitz [**5**]. We recall that a complex algebra A is semiprime if $J = \{0\}$ is the only ideal in A such that the product of every pair of elements in J is 0. Clearly, Banach function algebras, in particular, Lipschitz algebras, are semiprime.

Lemma 2.8. [**5,** Lemma 3.1]. *Let B be a unital commutative semiprime Banach algebra, and let T be a bounded unital quasicompact endomorphism of B. Suppose that*

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\},$$

and that the eigenvalue 1 of T has multiplicity 1. Then the operators T^n converge in operator norm to a rank-one unital endomorphism S of B.

Theorem 2.9. [**5,** Theorem 3.2]. *Let B be a unital commutative semiprime Banach algebra, and let T be a bounded unital quasicompact endomorphism of B. Then there exists an $n \in \mathbb{N}$ such that $\sigma(T^n) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\}$. For such n, the unital quasicompact endomorphism T^n of B has the following properties*:

(i) *The eigenspace of T^n corresponding to eigenvalue 1 is a finite dimensional, unital subalgebra of B isomorphic to $\mathbb{C}^m$ for some $m \in \mathbb{N}$, and hence spanned by m orthogonal idempotents, say $e_1, e_2, \ldots, e_m$.*

(ii) *Set $B_i = e_iB$ $(1 \leq i \leq m)$. Then (under an equivalent norm) each B_i is a commutative, unital semiprime Banach algebra, with identity e_i, and*

$$B = \bigoplus_{i=1}^{m} B_i.$$

(iii) *For $1 \leq i \leq m$, $T^n|_{B_i}$ is a unital quasicompact endomorphism of B_i, and $T^n|_{B_i}$ satisfies the conditions of Lemma 2.8. The operators $\{T^{kn}|_{B_i}\}_{k=1}^{\infty}$ converge in operator norm to a rank-1 unital endomorphism of B_i, say S_i.*

(iv) *The operators $\{T^{kn}\}_{k=1}^{\infty}$ converge in operator norm to the rank-m*

endomorphism S of B given by

$$S(b) = \sum_{i=1}^{m} S_i(be_i) \quad (b \in B).$$

We are now in a position to prove the generalization of Corollaries 2.4 and 2.5.

Theorem 2.10. *Let X be a compact metric space and T be an endomorphism of $\mathrm{Lip}(X,\alpha)$, $0 < \alpha \leq 1$, or of $\mathrm{lip}(X,\alpha)$, $0 < \alpha < 1$ induced by the self-map φ on X. Then T is quasicompact if and only if there exists a decomposition of X into a finite number of mutually disjoint clopen subsets, say $X_1, X_2, \ldots, X_m$ such that, for each $i \in \{1, 2, \ldots, m\}$, there exists $n_i \in \mathbb{N}$ with $\varphi_{n_i}(X_i) \subseteq X_i$ and $p(\varphi_{n_i}|_{X_i}) < 1$.*

Proof. If there exists a decomposition of X with such properties in the statement, then by Theorem 2.6 and Remark 2.7, $r_e(T) \leq \max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/(nn_i)} < 1$. Hence, T is quasicompact.

Conversely, suppose that T is quasicompact. By Theorem 2.9 (i), there exists $n_0 \in \mathbb{N}$ such that $\{f : T^{n_0}f = f\} = \{f : f \circ \varphi_{n_0} = f\}$ is a finite dimensional, unital subalgebra of $\mathfrak{L}(\alpha)$ spanned by m orthogonal idempotents, say $e_1, e_2 \ldots, e_m$. Therefore, there exists a finite number of mutually disjoint clopen subsets of X, say $X_1, X_2 \ldots, X_m$ with union X and

$$\{f : T^{n_0}f = f\} = \{f : f \circ \varphi_{n_0} = f\} = \left\{\sum_{i=1}^{m} \lambda_i \chi_{X_i} : \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}\right\}.$$

Then $\varphi_{n_0}(X_i) \subseteq X_i$ for each $i \in \{1, 2, \ldots, m\}$.

Set $\mathfrak{L}_i(\alpha) = \chi_{X_i}\mathfrak{L}(\alpha)$. In fact, either $\mathfrak{L}_i(\alpha) \simeq \mathrm{Lip}(X_i, \alpha)$ for $0 < \alpha \leq 1$ or $\mathfrak{L}_i(\alpha) \simeq \mathrm{lip}(X_i, \alpha)$ for $0 < \alpha < 1$. Also, by Theorem 2.9 (iii), $T^{n_0}|_{\mathfrak{L}_i(\alpha)}$ is a quasicompact endomorphism of $\mathfrak{L}_i(\alpha)$ induced by the self-map $\varphi_{n_0}|_{X_i}$, for each $i \in \{1, 2, \ldots, m\}$, and the operators $\{T^{nn_0}|_{\mathfrak{L}_i(\alpha)}\}_{n=1}^{\infty}$ converge, in operator norm, to a rank-1 unital endomorphism of $\mathfrak{L}_i(\alpha)$, say S_i. Since S_i is a rank-1 unital endomorphism of $\mathfrak{L}_i(\alpha)$, there exists $x_i \in X_i$ such that $S_i(f|_{X_i}) = f(x_i)1$ for $f \in \mathfrak{L}(\alpha)$, similar to the proof of Corollary 2.4, one can show that

$$\|T^{nn_0} - S_i\| \geq \frac{1}{(\mathrm{diam}(X_i))^\alpha + 1}\, p(\varphi_{nn_0}|_{X_i})^\alpha.$$

Therefore, $\lim_{n\to\infty} p(\varphi_{nn_0}|_{X_i}) = 0$ and $p(\varphi_{n_i}|_{X_i}) < 1$ for some $n_i \in \mathbb{N}$. □

Corollary 2.11. *Let X be a compact metric space and T be an endomorphism of $\mathrm{Lip}(X,\alpha)$, $0 < \alpha \leq 1$ or of $\mathrm{lip}(X,\alpha)$, $0 < \alpha < 1$ induced by the self-map φ on X. Then, for $0 < \alpha < 1$, T is Riesz if and only if there exists a decomposition of X into a finite number of mutually disjoint clopen subsets, say $X_1, X_2, \ldots, X_m$ such that, for each $i \in \{1, 2, \ldots, m\}$, there exists $n_i \in \mathbb{N}$ with $\varphi_{n_i}(X_i) \subseteq X_i$ and $\lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{1/n} = 0$. Moreover, when $\alpha = 1$, these conditions also imply that T is Riesz.*

Proof. If there exists a decomposition of X with such properties, then one can say, $\lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i}) = 0$, whence $p(\varphi_{nn_i}|_{X_i}) < 1$ for some $n \in \mathbb{N}$. Then using Theorem 2.6, Remark 2.7 and the hypothesis, we have $r_e(T) \leq \max_{1\leq i\leq m} \lim_{n\to\infty} p(\varphi_{nn_i}|_{X_i})^{\alpha/n} = 0$ which implies that T is Riesz.

Conversely, suppose that T is a Riesz endomorphism. Then it is also quasicompact and $r_e(T) = 0$. Therefore, using Theorems 2.6 and 2.10, the result is concluded. □

We conclude this paper by establishing some results about $\sigma(T)$ the spectrum of T and $\sigma_p(T)$ the set of eigenvalues of T.

Theorem 2.12. *Let X be a compact metric space, $0 < \alpha < 1$ and T be a quasicompact endomorphism of $\mathrm{Lip}(X,\alpha)$ or of $\mathrm{lip}(X,\alpha)$ induced by the self-map φ on X. Then*

$$\sigma_p(T) \subseteq \{\lambda : |\lambda| \leq r_e(T)\} \cup \{\lambda : \lambda^n = 1\}, \tag{2.1}$$

$$\sigma(T) \subseteq \{\lambda : |\lambda| \leq r_e(T)\} \cup \{\lambda : \lambda^n = 1\}, \tag{2.2}$$

for some positive integer n. In particular, 1 is an isolated point of the spectrum of T.

Proof. According to the proof of Theorem 2.10 there exist positive integer n and a finite number of mutually disjoint clopen subsets of X, say $X_1, X_2 \ldots, X_m$, with union X such that $\varphi_n(X_i) \subseteq X_i$ and $\lim_{k\to\infty} p(\varphi_{kn|_{X_i}}) = 0$, and so there is the unique fixed point of $\varphi_{n|_{X_i}}$ say x_i, for each $i \in \{1, 2, \ldots, m\}$. Take any $\lambda \in \mathbb{C}$ with $\lambda^n \neq 1$. For each $f \in \ker(\lambda I - T)$, we have $f \circ \varphi_= \lambda f$, and then $f(x_i) = f \circ \varphi_n(x_i) = \lambda^n f(x_i)$, which implies $f(x_i) = 0$ for each $i \in \{1, 2, \ldots, m\}$. If f is non-zero then there exists a point $x \in X_i$ for some $i \in \{1, 2, \ldots, m\}$ such that $f(x) \neq 0$ and, for each positive integer k,

$$|\lambda^{kn} f(x)| = |f \circ \varphi_{kn}(x) - f \circ \varphi_{kn}(x_i)| \leq d(x, x_i)^\alpha p_\alpha(f) p(\varphi_{kn}|_{X_i})^\alpha,$$

and then

$$|\lambda||f(x)|^{1/(kn)} \leq (\mathrm{diam}(X))^\alpha p_\alpha(f)^{1/(kn)} (p(\varphi_{kn}|_{X_i}))^{\alpha/(kn)}.$$

Taking the limit as $k \to \infty$,

$$|\lambda| \leq \lim_{k\to\infty} p(\varphi_{kn}|_{X_i}))^{\alpha/(kn)} \leq r_e(T).$$

Hence, for each $\lambda \in \mathbb{C}$ with $\lambda^n \neq 1$, if $|\lambda| > r_e(T)$, then $\ker(\lambda I - T) = \{0\}$, which implies (2.1).

Moreover, if $|\lambda| > r_e(T)$, then also $\ker(\lambda I - T) = \{0\}$. Using [**6,** Propositions 51.8 and 52.1], if $|\lambda| > r_e(T)$, then λ is a Riesz point of T, and, by the definition of the Riesz point [**6,** page 217], the operator $\lambda I - T$ is invertible, that is, $\lambda \notin \sigma(T)$. Therefore, the relation (2.2) follows. □

As an immediate consequence, we have

Corollary 2.13. *Let X be a compact metric space, $0 < \alpha < 1$, and T an endomorphism of* $\mathrm{Lip}(X, \alpha)$ *or of* $\mathrm{lip}(X, \alpha)$ *induced by the self-map φ on X.*

(i) *If T is Riesz, then $\sigma(T) \subseteq \{0\} \cup \{\lambda : \lambda^n = 1\}$ for some positive integer n.*
(ii) *If T is quasicompact and $\sigma(T) \subseteq \{0\} \cup \{\lambda : |\lambda| = 1\}$, then T is Riesz.*

We remark that, in Theorem 2.12 and Corollary 2.13, if we assume the connectedness of X, we get $n = 1$. Therefore, by Corollary 2.13, if X is a connected compact metric space and the endomorphism T is Riesz, then $\sigma(T) = \{0, 1\}$.

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