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NOTE ON IGUSA'S CUSP FORM OF WEIGHT 35

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ABSTRACT. A congruence relation satisfied by Igusa's cusp form of weight 35 is presented. As a tool to confirm the congruence relation, a Sturm-type theorem for the case of odd-weight Siegel modular forms of degree 2 is included.

1. Introduction. In **[5]**, Igusa gave a set of generators of the graded ring of degree 2 Siegel modular forms. In these generators, there are four even-weight forms φ_4, φ_6, χ_{10}, χ_{12}, and only one odd-weight form, χ_{35}. Here φ_k is the normalized Eisenstein series of weight k, and χ_k is a cusp form of weight k.

The purpose of this paper is to introduce a strange congruence relation of the odd-weight cusp form X_{35}, which is a suitable normalization of χ_{35} (for the precise definition, see subsection 2.2).

Main result. Denote by $a(T;X_{35})$ the T-th Fourier coefficient of the cusp form X_{35}. If T satisfies $\det(T) \not\equiv 0 \pmod{23}$, then

$$a(T;X_{35}) \equiv 0 \pmod{23},$$

or equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23},$$

where Θ is the theta operator on Siegel modular forms (for the precise definition, see subsection 2.4).

This result shows that *almost* all the Fourier coefficients $a(T;X_{35})$ are divisible by 23.

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2. Preliminaries.

2.1. Notation. First we confirm the notation. Let $\Gamma_n = Sp_n(\mathbb{Z})$ be the Siegel modular group of degree n and $\mathbb{H}_n$ the Siegel upper-half space of degree n. We denote by $M_k(\Gamma_n)$ the $\mathbb{C}$-vector space of all Siegel modular forms of weight k for Γ_n, and $S_k(\Gamma_n)$ is the subspace of cusp forms.

Any $F(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in L_n} a(T;F)q^T, \qquad q^T := e^{2\pi i \mathrm{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where T runs over all elements of L_n, and

$$\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbb{Z}\},$$
$$L_n := \{T \in \Lambda_n \mid T \text{ is semi-positive definite}\}.$$

In this paper, we deal mainly with the case of $n = 2$. For simplicity, we write

$$T = (m, n, r) \quad \text{for} \quad T = \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix} \in \Lambda_2.$$

For a subring R of $\mathbb{C}$, let $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$ denote the R-module of all modular forms whose Fourier coefficients lie in R.

2.2. Igusa's generators. Let

$$M(\Gamma_2) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)$$

be the graded ring of Siegel modular forms of degree 2. Igusa [**5**] gave a set of generators of the ring $M(\Gamma_2)$. The set consists of five generators

$$\varphi_4, \quad \varphi_6, \quad \chi_{10}, \quad \chi_{12}, \quad \chi_{35},$$

where φ_k is the normalized Eisenstein series on Γ_2 and χ_k is a cusp form of weight k. Moreover he showed that the even-weight generators φ_4, φ_6, χ_{10} and χ_{12} are algebraically independent. Later, he extended the result to the integral case ([**6**]). Namely, he gave a minimal set of generators over $\mathbb{Z}$ of the ring

$$M(\Gamma_2)_{\mathbb{Z}} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}}.$$

The set of generators consists of 15 modular forms including the following forms:

$$X_4 := \varphi_4, \qquad X_6 := \varphi_6,$$
$$X_{10} := -2^{-2}\chi_{10}, \qquad X_{12} := 2^2 \cdot 3\chi_{12},$$
$$X_{35} := 2^2 i\chi_{35}.$$

Of course, these forms have rational integral Fourier coefficients under the following normalization:

$$a((0,0,0);X_4) = a((0,0,0);X_6) = 1$$
$$a((1,1,1);X_{10}) = a((1,1,1);X_{12}) = 1$$
$$a((2,3,-1);X_{35}) = 1.$$

2.3. Order and the p-minimum matrix. We define a lexicographical order "$\succ$" for two different elements $T = (m,n,r)$ and $T' = (m',n',r')$ of Λ_2 by

$$T \succ T' \Longleftrightarrow (1) \quad \mathrm{tr}\,(T) > \mathrm{tr}\,(T')$$

or

$$(2) \quad \mathrm{tr}\,(T) = \mathrm{tr}\,(T'), \quad m > m'$$

or

$$(3) \quad \mathrm{tr}\,(T) = \mathrm{tr}\,(T'), \quad m = m',\ r > r'.$$

Let p be a prime and $\mathbb{Z}_{(p)}$ the local ring consisting of p-integral rational numbers. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we define the *$p$-minimum matrix* $m_p(F)$ of F by

$$m_p(F) := \min\{T \in L_2 \mid a(T;F) \not\equiv 0 \pmod{p}\},$$

where the "min" is defined in the sense of the above order. If $F \equiv 0 \pmod{p}$, then we define $m_p(F) = (\infty)$.

Remark 2.1. The p-minimum matrices of Igusa's generators are

$$\begin{aligned} m_p(X_4) &= m_p(X_6) = (0,0,0), \\ m_p(X_{10}) &= m_p(X_{12}) = (1,1,-1), \\ m_p(X_{35}) &= (2,3,-1), \end{aligned}$$

for any prime number p.

The following properties are essential.

Lemma 2.2. (1) $T_1 \succ T_2$, $S_1 \succ S_2$ *implies* $T_1 + S_1 \succ T_2 + S_2$.
(2) $T_1 \succ T_2$ *implies* $T_1 \pm S \succ T_2 \pm S$.
(3) $T + S = T' + S'$, $T \succ T'$ *implies* $S \prec S'$.
(4) $m_p(F \cdot G) = m_p(F) + m_p(G)$.

Proof. (1), (2) Trivial.

(3) We use (2) without notice. By the assumption $T + S = T' + S'$, we have $T - T' = S' - S$. Then $0_2 \prec T - T' = S' - S$ because of $T \succ T'$. Hence, $S \prec S'$.

(4) Let $m_p(F) = T_0$ and $m_p(G) = T_0'$. Then, for all $T \prec T_0$ (respectively, $T \prec T_0'$), $a(T;F) \equiv 0 \pmod p$ and $a(T_0;F) \not\equiv 0 \pmod p$ (respectively, $a(T;G) \equiv 0 \pmod p$ and $a(T_0';G) \not\equiv 0 \pmod p$). Now, recall that the T-th Fourier coefficient $a(T;F\cdot G)$ of $F \cdot G$ is given by

$$a(T;F\cdot G) = \sum_{\substack{S,S' \in L_2 \\ S+S'=T}} a(S;F)a(S';G).$$

If $T \prec T_0 + T_0'$, then $T = S + S' \prec T_0 + T_0'$, and hence $S \prec T_0$ or $S' \prec T_0'$ because of (1). In this case, $a(S;F) \equiv 0 \pmod p$ or $a(S';G) \equiv 0 \pmod p$. Therefore, $a(S;F)a(S';G) \equiv 0 \pmod p$ for each S, S' with $S + S' \prec T_0 + T_0'$. This implies $a(T;F) \equiv 0 \pmod p$ for all $T \prec T_0 + T_0'$.

In order to complete the proof of (4), we need to prove that $a(T_0 + T_0'; F \cdot G) \not\equiv 0 \pmod p$. If $S + S' = T_0 + T_0'$, then we have by (3) that $S \prec T_0$, $S' \succ T_0'$ or $S \succ T_0$, $S' \prec T_0'$ or $S = T_0$, $S' = T_0'$. In the first two cases, since $a(S;F) \equiv 0 \pmod p$ or $a(S';G) \equiv 0 \pmod p$, we get $a(S;F)a(S';G) \equiv 0 \pmod p$. In the third case,

$a(T_0;F)a(T_0';G) \not\equiv 0 \pmod{p}$. Thus, $a(T_0+T_0';F\cdot G) \not\equiv 0 \pmod{p}$, namely, $m_p(F\cdot G) = T_0+T_0'$. This completes the proof of (4). □

Sturm-type theorem. A Sturm-type theorem for the Siegel modular forms was first given by Poor and Yuen in [**7**]. Recently Choi, Choie and the first author [**4**] investigated such a problem in the case of degree 2 and proved some theorems.

We introduce the statement of this theorem for the case of level 1.

Theorem 2.3 (Choi, Choie and Kikuta [**4**])**.** *Let p be a prime with $p \geq 5$ and k an even positive integer. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with Fourier expansion*

$$F = \sum_{T\in L_2} a(T;F)q^T,$$

we assume that $a((m,n,r);F) \equiv 0 \pmod{p}$ for all m, n, r such that $0 \leq m$, $n \leq k/10$ and $4mn-r^2 \geq 0$. Then $F \equiv 0 \pmod{p}$.

We rewrite this theorem for later use:

Theorem 2.4. *Let p be a prime with $p \geq 5$. Assume that $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ satisfies $m_p(F) \succ ([k/10],[k/10],r_0)$ for the maximum $r_0 \in \mathbb{Z}$ such that $([k/10],[k/10],r_0) \in L_2$. Then $m_p(F) = (\infty)$, i.e., $F \equiv 0 \pmod{p}$.*

Proof. The assertion follows immediately from the inclusion

$$\left\{T\in L_2 \,\middle|\, T \preceq \left(\left[\frac{k}{10}\right],\left[\frac{k}{10}\right],r_0\right)\right\} \supset \left\{(m,n,r)\in L_2 \,\middle|\, m,n \leq \frac{k}{10}\right\}. \tag{2.1}$$ □

Remark 2.5. In general, the converse of inclusion (2.1) is not true. For example, $([k/10]+1,0,0) \prec ([k/10],[k/10],r_0)$ (for $k \geq 20$). We need a statement of this type to aid the proof of the next proposition.

In order to prove our main result, we need a Sturm-type theorem for the *odd-weight case*:

Proposition 2.6. *Let p be a prime with $p \geq 5$ and k an odd positive integer. For $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we assume that*

$$m_p(F) \succ \left(\left[\frac{k-35}{10} \right] + 2, \left[\frac{k-35}{10} \right] + 3, r_0 - 1 \right),$$

where $r_0 \in \mathbb{Z}$ is the maximum number such that

$$\left(\left[\frac{k-35}{10} \right], \left[\frac{k-35}{10} \right], r_0 \right) \in L_2.$$

Then $m_p(F) = (\infty)$, namely, $F \equiv 0 \pmod{p}$.

Remark 2.7. When $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ is of odd weight, $X_{35} \cdot F \in M_{k+35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ is of even weight. Using Theorem 2.3 directly, we have the following statement: If $a((m, n, r); F) \equiv 0 \pmod{p}$ for all m, n and r such that $0 \leq m$, $n \leq \frac{k+35}{10}$ and $4mn - r^2 \geq 0$, then $F \equiv 0 \pmod{p}$.

For our purposes, however, the estimation of Proposition 2.6 is better than this estimation.

Proof of Proposition 2.6. First note that

$$M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = X_{35} M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$$

for odd k. Hence, there exists $G \in M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $F = X_{35} \cdot G$. Using Lemma 2.2 (4), we get $m_p(F) = m_p(X_{35}) + m_p(G)$. Since $m_p(X_{35}) = (2, 3, -1)$, we have

$$m_p(G) = m_p(F) - (2, 3, -1) \succ \left(\left[\frac{k-35}{10} \right], \left[\frac{k-35}{10} \right], r_0 \right).$$

It should be noted that Lemma 2.2 (2) is used to get the last inequality. Since G is of even weight, we can apply Theorem 2.4 to G. This shows that $F = X_{35} \cdot G \equiv 0 \pmod{p}$. □

2.4. Theta operator. In [**8**], Serre used the theta operator θ on elliptic modular forms to develop the theory of p-adic modular forms:

$$\theta = q\frac{d}{dq} : f = \sum a(t; f)q^t \longmapsto \theta(f) := \sum t \cdot a(t; f)q^t.$$

Later the operator was generalized to the case of Siegel modular forms:

$$\Theta : F = \sum a(T; F)q^T \longmapsto \Theta(F) := \sum \det(T) \cdot a(T; F)q^T$$

(e.g., cf., [**3**])). Moreover, the following fact was proven:

Theorem 2.8 (Böcherer-Nagaoka [**3**]). *Assume that a prime p satisfies $p \geq n+3$. Then, for any Siegel modular form F in $M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, there exists a Siegel cusp form G in $S_{k+p+1}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ satisfying*

$$\Theta(F) \equiv G \pmod{p}.$$

Example 2.9. Under the notation in subsection 2.2, we have

$$\Theta(X_6) \equiv 4X_{12} \pmod{5}.$$

3. Main result. On the basis of the previous preparation, we can now describe our main result.

Theorem 3.1. *Let $a(T; X_{35})$ denote the Fourier coefficient of X_{35}. If $det(T) \not\equiv 0 \pmod{23}$, then*

$$a(T; X_{35}) \equiv 0 \pmod{23},$$

or, equivalently,

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$

Proof. Our proof mainly depends on Proposition 2.6 and numerical calculation of the Fourier coefficients of X_{35}. If we use the theta operator, this assertion is equivalent to showing that

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$

From Theorem 2.8, there exists a Siegel cusp form $G \in S_{59}(\Gamma_2)_{\mathbb{Z}_{(23)}}$ such that

$$\Theta(X_{35}) \equiv G \pmod{23}.$$

Therefore, the proof is reduced to showing that

$$G \equiv 0 \pmod{23}. \tag{3.1}$$

We now apply Proposition 2.6 to the form G. It then suffices to show that

$$a((m,n,r); G) \equiv 0 \pmod{23} \quad \text{for } T = (m,n,r)$$
$$\text{with } \operatorname{tr}(T) = m + n \leq 10.$$

Since $a((m,n,r);G) = -a((n,m,r);G)$ for the odd-weight form G, this statement is equivalent to

$$a((m,n,r);\Theta(X_{35})) \equiv 0 \pmod{23} \quad \text{for } T=(m,n,r) \\ \text{with } \operatorname{tr}(T) = m+n \le 9.$$

We then write down the first part the Fourier expansion of X_{35} following the order introduced in subsection 2.3. For this, we set

$$q_{jk} := \exp(2\pi i z_{jk}) \quad \text{for} \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2.$$

The terms corresponding to $T=(m,n,r)$ with $\operatorname{tr}(T) = m+n \le 9$ are as follows:

$$\begin{aligned}
X_{35} = {} & (q_{12}^{-1} - q_{12})q_{11}^2 q_{22}^3 + (-q_{12}^{-1} + q_{12})q_{11}^3 q_{22}^2 \\
& + (-q_{12}^{-3} - 69q_{12}^{-1} + 69q_{12} + q_{12}^3)q_{11}^2 q_{22}^4 + (q_{12}^{-3} + 69q_{12}^{-1} - 69q_{12} - q_{12}^3)q_{11}^4 q_{22}^2 \\
& + (69q_{12}^{-3} + 2277q_{12}^{-1} - 2277q_{12} - 69q_{12}^3)q_{11}^2 q_{22}^5 \\
& + (q_{12}^{-5} - 32384q_{12}^{-2} - 129421q_{12}^{-1} + 129421q_{12} + 32384q_{12}^2 - q_{12}^5)q_{11}^3 q_{22}^4 \\
& + (-q_{12}^{-5} + 32384q_{12}^{-2} + 129421q_{12}^{-1} - 129421q_{12} - 32384q_{12}^2 + q_{12}^5)q_{11}^4 q_{22}^3 \\
& + (-69q_{12}^{-3} - 2277q_{12}^{-1} + 2277q_{12} + 69q_{12}^3)q_{11}^5 q_{22}^2 \\
& + (q_{12}^{-5} - 2277q_{12}^{-3} - 47702q_{12}^{-1} + 47702q_{12} + 2277q_{12}^3 - q_{12}^5)q_{11}^2 q_{22}^6 \\
& + (32384q_{12}^{-4} - 2184448q_{12}^{-2} - 3203072q_{12}^{-1} + 3203072q_{12} + 2184448q_{12}^2 \\
& \qquad - 32384q_{12}^4)q_{11}^3 q_{22}^5 \\
& + (-32384q_{12}^{-4} + 2184448q_{12}^{-2} + 3203072q_{12}^{-1} - 3203072q_{12} - 2184448q_{12}^2 \\
& \qquad + 32384q_{12}^4)q_{11}^5 q_{22}^3 \\
& + (-q_{12}^{-5} + 2277q_{12}^{-3} + 47702q_{12}^{-1} - 47702q_{12} - 2277q_{12}^3 + q_{12}^5)q_{11}^6 q_{22}^2 \\
& + (-69q_{12}^{-5} + 47702q_{12}^{-3} + 709665q_{12}^{-1} - 709665q_{12} - 47702q_{12}^3 + 69q_{12}^5)q_{11}^2 q_{22}^7 \\
& + (-q_{12}^{-7} + 129421q_{12}^{-5} + 2184448q_{12}^{-4} + 41321984q_{12}^{-2} + 105235626q_{12}^{-1} \\
& \quad - 105235626q_{12} - 41321984q_{12}^2 - 2184448q_{12}^4 - 129421q_{12}^5 + q_{12}^7)q_{11}^3 q_{22}^6 \\
& + (-69q_{12}^{-7} - 32384q_{12}^{-6} + 107121810q_{12}^{-3} - 31380096q_{12}^{-2} + 759797709q_{12}^{-1} \\
& \quad - 759797709q_{12} + 31380096q_{12}^2 - 107121810q_{12}^3 + 32384q_{12}^6 + 69q_{12}^7)q_{11}^4 q_{22}^5 \\
& + (69q_{12}^{-7} + 32384q_{12}^{-6} - 107121810q_{12}^{-3} + 31380096q_{12}^{-2} - 759797709q_{12}^{-1} \\
& \quad + 759797709q_{12} - 31380096q_{12}^2 + 107121810q_{12}^3 - 32384q_{12}^6 - 69q_{12}^7)q_{11}^5 q_{22}^4 \\
& + (q_{12}^{-7} - 129421q_{12}^{-5} - 2184448q_{12}^{-4} - 41321984q_{12}^{-2} - 105235626q_{12}^{-1} \\
& \quad + 105235626q_{12} + 41321984q_{12}^2 + 2184448q_{12}^4 + 129421q_{12}^5 - q_{12}^7)q_{11}^6 q_{22}^3 \\
& + (69q_{12}^{-5} - 47702q_{12}^{-3} - 709665q_{12}^{-1} + 709665q_{12} + 47702q_{12}^3 - 69q_{12}^5)q_{11}^7 q_{22}^2 + \cdots .
\end{aligned}$$

The Fourier coefficients different from ± 1 are as follows:
$a((4,1,2);X_{35}) = -69 = -3\cdot\underline{23}$, $a((5,1,2);X_{35}) = 2277 = 3^2\cdot 11\cdot\underline{23}$, $a((4,1,3);X_{35}) = -1294121 = -17\cdot\underline{23}\cdot 331$, $a((4,2,3);X_{35}) = -32384 = -2^7\cdot 11\cdot\underline{23}$, $a((6,1,2);X_{35}) = -47702 = -2\cdot 17\cdot\underline{23}\cdot 61$, $a((5,1,3);X_{35}) = -3203072 = -2^{13}\cdot 17\cdot\underline{23}$, $a((5,2,3);X_{35}) = -2184448 = -2^8\cdot 7\cdot\underline{23}\cdot 53$, $a((7,1,2);X_{35}) = 709665 = 3\cdot 5\cdot 11^2\cdot 17\cdot\underline{23}$, $a((6,1,3);X_{35}) = 105235626 = 2\cdot 3\cdot\underline{23}\cdot 762577$, $a((6,2,3);X_{35}) = 41321984 = 2^9\cdot 11^2\cdot\underline{23}\cdot 29$, $a((5,1,4);X_{35}) = 759797709 = 3\cdot 11\cdot\underline{23}\cdot 29\cdot 34519$, $a((5,2,4);X_{35}) = -31380096 = -2^7\cdot 3\cdot 11\cdot 17\cdot 19\cdot\underline{23}$, $a((5,3,4);X_{35}) = 107121810 = 2\cdot 3\cdot 5\cdot 19\cdot\underline{23}\cdot 8171$.

All of these Fourier coefficients are divisible by 23. On the other hand, if $a(T;X_{35}) = \pm 1$ for T in this range, then $\det(T) = 23/4 \equiv 0 \pmod{23}$. This fact implies that

$$a((m,n,r);\Theta(X_{35})) \equiv 0 \pmod{23}$$

for $T = (m,n,r)$ with $\mathrm{tr}\,(T) = m+n \leq 9$. Therefore, we obtain

$$a((m,n,r);G) \equiv 0 \pmod{23}$$

for $T = (m,n,r)$ with $\mathrm{tr}\,(T) = m+n \leq 9$. Consequently, we have (3.1). This completes the proof of our theorem. □

Remark 3.2.

(1) The numerical examples of the Fourier coefficients $a(T;X_{35})$ in the above are calculated by using Ibukiyama's determinant expression of X_{35} (cf. [**1,** page 253]).
(2) The converse statement of the theorem is not true in general. In fact,

$$a((1,6,1);X_{35}) = 0 \quad \text{and} \quad \det((1,6,1)) = 23/4 \equiv 0 \pmod{23}.$$

(3) There are other "modulo 23" congruences for the Siegel modular forms in [**2,** Satz 5,(a)]. In that case, the congruence is concerned with the Eisenstein lifting of the Ramanujan delta function.

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