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NORM ESTIMATES FOR FUNCTIONS OF TWO NON-COMMUTING OPERATORS

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ABSTRACT. Analytic functions of two non-commuting bounded operators in a Banach space are considered. Sharp norm estimates are established. Applications to operator equations and differential equations in a Banach space are discussed.

1. Introduction and statement of the main result. In the book **[9]**, Gel'fand and Shilov have established an estimate for the norm of a regular matrix-valued function in connection with their investigations of partial differential equations. However, that estimate is not sharp, and it is not attained for any matrix. The problem of obtaining a sharp estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature, cf., **[3]**. In the paper **[10]** (see also **[11]**) the author has derived an estimate for regular matrix-valued functions, which is attained in the case of normal matrices. In **[16]**, the results of the paper **[10]** were extended to functions of two noncommuting matrices. The aim of this paper is to generalize the main result from **[16]** to some classes of functions of two noncommuting operators in a Banach space. It should be noted that functions of many operators were investigated by many mathematicians, (cf., **[1, 21, 24]** and references therein) however the norm estimates were not considered, but as it is well-known, operator valued functions give us representations of solutions of various differential, difference equations and operator equations. This fact allows us to investigate stability, well-posedness and perturbations of these equations by norm estimates for operator valued functions, cf., **[3]**.

Let X be a complex Banach space with a Schauder basis $\{d_k\}$, the identity operator I and a norm $\|.\|$. For a linear operator A, $\|A\|$ is the

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operator norm, $\sigma(A)$ is the spectrum, $R_z(A) = (A - zI)^{-1}$ ($z \notin \sigma(A)$) is the resolvent, $r_s(A)$ denotes the spectral radius.

Everywhere below, $A, \widetilde{A}$ and K are bounded linear operators in X. Let Ω_A and $\Omega_{\tilde{A}}$ be open simple connected supersets of $\sigma(A)$ and $\sigma(\widetilde{A})$, respectively, and f be a scalar function analytic on $\Omega_A \times \Omega_{\tilde{A}}$. We define the operator valued function

$$F(f, A, K, \widetilde{A}) := -\frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} f(z,w) R_z(A) K R_w(\widetilde{A})\, dw\, dz, \tag{1.1}$$

where $C_A \subset \Omega_A, C_{\tilde{A}} \subset \Omega_{\tilde{A}}$ are closed contours surrounding $\sigma(A)$ and $\sigma(\widetilde{A})$, respectively.

Such functions play an essential role in the theory of operator equations. More specifically, consider the operator equation

$$\sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} A^j Z \widetilde{A}^k = K \quad (m_1, m_2 < \infty), \tag{1.2}$$

where Z should be found and c_{jk} are complex numbers. Put

$$p(z,w) = \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} z^j \widetilde{w}^k.$$

Then by Theorem 3.1 from **[3,** Chapter 1] a unique solution of equation (1.2) is given by the formula

$$Z = F\left(\frac{1}{p(z,w)}, A, K, \widetilde{A}\right), \tag{1.3}$$

provided $p(z,w) \neq 0$ ($z \in \sigma(A), w \in \sigma(B)$). Equations of the type (1.2) naturally arose in various applications, cf., [**3, 19, 20**], for example, the Lyapunov equation $A^*Z + ZA = K$, cf., [**3**], and the Lyapunov type equation

$$Z - A^*ZA = K \tag{1.4}$$

play important roles in the theories of differential and difference equations, respectively, cf., [**13**]. These equations recently attracted the attention of many mathematicians. Mainly numerical methods for the solutions of operator and matrix equations were developed, cf., [**27**].

In the paper [**4**], reflexive and anti-reflexive solutions of a linear equation were explored. Furthermore, suppose that

$$T(t) := -\frac{1}{4\pi^2}\int_{C_{\widetilde{A}}}\int_{C_A} e^{t(z+w)} R_z(A) K R_w(\widetilde{A})\, dw\, dz. \tag{1.5}$$

Take into account that $zR_z(A) = AR_z(A) - I$. Then simple calculations show that

$$\begin{aligned} T'(t) &= -\frac{1}{4\pi^2}\int_{C_{\widetilde{A}}}\int_{C_A} (z+w) e^{t(z+w)} R_z(A) K R_w(\widetilde{A})\, dw\, dz \\ &= -\frac{1}{4\pi^2}\int_{C_{\widetilde{A}}}\int_{C_A} e^{t(z+w)} [A R_z(A) K R_w(\widetilde{A}) \\ &\quad + R_z(A) K R_w(\widetilde{A})\widetilde{A}]\, dw\, dz. \end{aligned}$$

So

$$T'(t) = AT(t) + T(t)\widetilde{A}. \tag{1.6}$$

Such equations arise in numerous applications, in particular, in the theory of vector differential equations, cf., [**18,** page 509], [**3,** Section VI.4, equation (4.15) and Section VI.2], [**8,** Section XV.10]. The literature on operator equations is rather rich. In particular, the paper [**23**] deals with necessary and sufficient conditions for the existence of solutions to systems of the general solution to a system of adjointable operator equations over Hilbert C^* modules.

The paper [**6**] should be mentioned. In that paper, nonlinear operator equations of the form $ABA = A^2$ and $BAB = B^2$ are considered. For other recent results on operator equations see [**2, 5**]. Certainly, we could not survey the whole object here. In the above-mentioned papers, no estimates were established for solutions of the equations considered above. In the present paper, sharp norm estimates for functions of the type (1.1) are established. They give us estimates for solutions of equations (1.2) and (1.6).

Let A and $\widetilde{A}$ be represented in basis $\{d_k\}$ by matrices $(a_{jk})_{j,k=1}^{\infty}$ and $(\widetilde{a}_{jk})_{j,k=1}^{\infty}$, respectively. So $A = S + W$ where $S = \operatorname{diag}[a_{11}, a_{22}, \ldots]$ and $W := A - S$ is the off diagonal matrix. That is, the entries w_{jk} of W are $w_{jk} = a_{jk}$ $(j \neq k)$ and $w_{jj} = 0$ $(j, k = 1, 2, \ldots)$. Similarly, $\widetilde{A} = \widetilde{S} + \widetilde{W}$, where $\widetilde{S}, \widetilde{W}$ are the diagonal and off-diagonal parts of $\widetilde{A}$, respectively. We put $|A| = (|a_{jk}|)_{j,l=1}^{\infty}$, i.e., $|A|$ is the matrix whose entries are absolute values of A in basis $\{d_k\}$. We also write $C \geq 0$ if all the entries of a matrix C are nonnegative. In the same vein, we have the symbols $|h|$, $h \geq 0$ and $h \leq g$ for vectors $h, g \in X$. Denote by $\mathrm{co}\,(S)$ and $\mathrm{co}\,(\widetilde{S})$ the closed convex hulls of the diagonal entries $a_{11}, a_{22}, \ldots$ and $\widetilde{a}_{11}, \widetilde{a}_{22}, \ldots$, respectively, and let

$$\eta_{j,k} := \frac{1}{k!j!} \sup_{\substack{z \in \mathrm{co}\,(S) \\ w \in \mathrm{co}\,(\tilde{S})}} \left| \frac{\partial^{j+k} f(z,w)}{\partial z^j \partial w^k} \right| \quad (j,k = 0,1,2,\ldots).$$

Finally, put $\Omega(r) := \{z \in \mathbb{C} : |z| \leq r\}$ for an $r > 0$. Now we are in a position to formulate our main result.

Theorem 1.1. *Let $f(z,w)$ be holomorphic on a neighborhood of $\Omega(r_A) \times \Omega(r_{\tilde{A}})$ for some $r_A > r_s(S) + r_s(|W|)$ and $r_{\tilde{A}} > r_s(\widetilde{S}) + r_s(|\widetilde{W}|)$. Then*

$$|F(A,K,\widetilde{A})| \leq \sum_{j,k=0}^{\infty} \eta_{j,k} |W|^j |K| |\widetilde{W}|^k. \tag{1.7}$$

The proof of this theorem is presented in the next section. Below we check that the series in (1.7) really strongly converges. Note that estimates for functions of two commuting infinite matrices were established in **[15]**.

Theorem 1.1 supplements the recent results on matrix valued functions **[7, 14, 25]**. About the recent results on infinite matrices and their applications, see the interesting paper **[26]**.

2. Proof of Theorem 1.1.

Lemma 2.1. *Under the hypothesis of Theorem* 1.1, *let A and $\widetilde{A}$ have n-dimensional ranges $(n < \infty)$. Then inequality* (1.7) *is valid.*

Proof. We have $R_\lambda(A) = (S + W - I\lambda)^{-1} = (I + R_\lambda(S)W)R_\lambda(S)$. Consequently,

$$R_\lambda(A) = \sum_{k=0}^{\infty} (-1)^k (R_\lambda(S)W)^k R_\lambda(S),$$

provided the spectral radius $r_s(R_\lambda(S)W)$ of the matrix $R_\lambda(S)W$ is less than one. The entries of this matrix are

$$\frac{a_{jk}}{a_{jj}-\lambda} \quad (\lambda \neq a_{jj},\ j \neq k)$$

and the diagonal entries are zero. Clearly,

$$|R_\lambda(S)W| \leq \frac{|W|}{\min_k |a_{kk}-\lambda|}.$$

But $|a_{kk}-\lambda| \geq |\lambda| - |a_{kk}| \geq |\lambda| - r_s(S)$, provided $|\lambda| > r_s(S)$. So in this case

$$|R_\lambda(S)W| \leq \frac{|W|}{|\lambda| - r_s(S)}.$$

Therefore, if $|\lambda| > r_s(|W|) + r_s(S)$, then

$$r_s(R_\lambda(S)W) \leq \frac{r_s(|W|)}{|\lambda| - r_s(S)} < 1$$

and the series

$$\sum_{k=0}^{\infty} (R_\lambda(S)W)^k (-1)^k$$

converges. Similarly,

$$R_\lambda(\widetilde{A}) = \sum_{k=0}^{\infty} (-1)^k (R_\lambda(\widetilde{S})\widetilde{W})^k R_\lambda(\widetilde{S}),$$

provided $|\lambda| > r_s(|\widetilde{W}|) + r_s(\widetilde{S})$. So by (1.1) we have

$$F(A, K, \widetilde{A}) = \sum_{m,k=0}^{\infty} C_{mk} \tag{2.1}$$

where

$$C_{mk} = (-1)^{k+m+1} \frac{1}{4\pi^2} \int_{|w|=r_{\widetilde{A}}} \int_{|\lambda|=r_A} f(\lambda, w)(R_\lambda(S)W)^m R_\lambda(S)K \\ \times (R_w(\widetilde{S})\widetilde{W})^k R_w(\widetilde{S})\, d\lambda\, dw.$$

Since $S, \widetilde{S}$ are diagonal matrices with respect to basis $\{d_k\}$, we can write out

$$R_\lambda(S) = \sum_{j=1}^{n} \frac{Q_j}{a_{jj} - \lambda}, \qquad R_\lambda(\widetilde{S}) = \sum_{j=1}^{n} \frac{Q_j}{\widetilde{a}_{jj} - \lambda},$$

where $Q_j h = h_j d_j$ for a

$$h = \sum_{k=1}^{\infty} h_k d_k \in X.$$

Consequently,

$$C_{mk} = \sum_{i_1=1}^{n} Q_{i_1} W \sum_{i_2=1}^{n} Q_{i_2} W \cdots W \sum_{i_{m+1}=1}^{n} Q_{i_{m+1}} K \sum_{j_1=1}^{n} Q_{j_1} \widetilde{W} \sum_{j_2=1}^{n} Q_{j_2} \widetilde{W} \cdots \widetilde{W} \sum_{j_{k+1}=1}^{n} Q_{j_{k+1}} I_{\substack{i_1,i_2,\dots,i_{m+1} \\ j_1 j_2 \cdots j_{k+1}}}$$

Here

$$I_{\substack{i_1,i_2,\dots,i_{m+1} \\ j_1 j_2 \cdots j_{k+1}}} = \frac{(-1)^{k+m+1}}{4\pi^2} \int_{|w|=r_{\tilde{A}}} \int_{|\lambda|=r_A} \frac{f(\lambda,w)\,d\lambda}{(a_{i_1 i_1}-\lambda)\cdots(a_{i_{m+1} i_{m+1}}-\lambda)(\widetilde{a}_{j_1 j_1}-w)\dots(\widetilde{a}_{j_{k+1} j_{k+1}}-w)}.$$

As it is proved in [**12**],

$$|I_{\substack{i_1,i_2,\dots,i_{m+1} \\ j_1 j_2 \cdots j_{k+1}}}| \le \eta_{m,k}.$$

Hence,

$$|C_{mk}| \le \eta_{mk} \sum_{j_1=1}^{n} Q_{j_1} |W| \sum_{j_2=1}^{n} Q_{j_2} |W| \cdots |W| \sum_{j_k=1}^{n} Q_{j_m} |K| \sum_{l_1=1}^{n} Q_{l_l} |\widetilde{W}| \dots |\widetilde{W}| \sum_{l_k=1}^{n} Q_{l_m}.$$

Thus, $|C_{mk}| \le \eta_{mk} |W|^m |K| |\widetilde{W}|^k$. Now (2.1) implies the required result. $\square$

Proof of Theorem 1.1. Passing to the limit as $n \to \infty$ in the previous lemma we get required result due to the Banach-Steinhaus theorem. □

3. Norm estimates. *In this section the norm* $\|.\|$ *in* X *is a lattice norm.* That is, $\|f\| \le \|h\|$ whenever $|f| \le |h|$ and $\|A\| \le \||A|\|$. Theorem 1.1 implies

Corollary 3.1. *Under the hypothesis of Theorem* 1.1, *we have*

$$\|F(A,K,\widetilde{A})\| \le \||K|\| \sum_{j,k=0}^{\infty} \eta_{j,k} \||W|^j\| \||\widetilde{W}|^k\|. \tag{3.1}$$

If A and $\widetilde{A}$ are diagonal: $W = \widetilde{W} = 0$, inequality (3.1) takes the form

$$\|F(A,K,\widetilde{A})\| \le \||K|\| \sup_{\substack{z \in \text{co}\,(S) \\ w \in \text{co}\,(\tilde{S})}} |f(z,w)|.$$

Furthermore, since $|W| \le |A|$, from (3.1), it follows

$$\||F(A,K,\widetilde{A})|\| \le \||K|\| \sum_{j,k=0}^{\infty} \eta_{j,k} \||W|\|^j \||\widetilde{W}|\|^k \tag{3.2}$$

$$\le \||K|\| \sum_{j,k=0}^{\infty} \eta_{j,k} \||A|\|^j \||\widetilde{A}|\|^k. \tag{3.3}$$

For instance, let $X = l^p$ for some integer $p \ge 2$, and let W and $\widetilde{W}$ be Hille-Tamarkin matrices, cf., **[22]**. Namely,
(3.4)

$$M_p(W) := \left(\sum_{j=1}^{\infty} \left[\sum_{k=1,\ k \ne j}^{\infty} |a_{jk}|^q \right]^{p/q} \right)^{1/p} < \infty \quad \text{and} \quad M_p(\widetilde{W}) < \infty$$

with $1/p + 1/q = 1$. So, under (3.4), $A = (\tilde{a}_{jk})$ and $\widetilde{A} = (\tilde{a}_{jk})$ represent bounded linear operators in l^p, provided S and $\widetilde{S}$ are bounded. As it is well-known, $\||W|\| \le M_p(|W|) = M_p(W)$, cf., **[22]**. Now (3.2) yields

Corollary 3.2. *Under the hypothesis of Theorem* 1.1, *let* $X = l^p, 2 \leq p < \infty$, *and conditions* (3.4) *hold. Then*

$$\|F(A, K, \widetilde{A})\| \leq \||K|\|| \sum_{j,k=0}^{\infty} \eta_{j,k} M_p^j(W) M_p^k(\widetilde{W}),$$

provided the series converges.

4. Functions of operators in a Hilbert space. Let $X = H$ be a separable Hilbert space with a scalar product $(.,.)$ and the norm $\|.\| = \sqrt{(.,.)}$. In this section we improve Theorem 1.1 in the case of so-called triagonalizable operators acting in H.

Let SN_p be the Schatten-von Neumann ideal of operators K in H with the finite norm $N_p(K) := [\text{Trace}\,(KK^*)^{p/2}]^{1/p}$ $(1 \leq p < \infty)$. So SN_2 is the ideal of Hilbert-Schmidt operators, and SN_1 is the ideal of nuclear operators.

Recall that a linear operator V is called *quasinilpotent* if $\sigma(V) = \{0\}$. A compact quasinilpotent operator will be called a *Volterra operator*. Let $E(t)$ be an orthogonal resolution of the identity in H, defined on a real segment $[a, b]$. E is called a maximal resolution of the identity (m.r.i.), if its every gap $E(t_0 + 0) - E(t_0 - 0)$ (if it exists) is one-dimensional, cf., **[11]**. We will say that a bounded linear operator A is *triangularizable*, if there are an m.r.i. $E(t)$, a normal operator D and a Volterra one V, such that

$$A = D + V, \tag{4.1}$$

and

$$E(t)VE(t) = VE(t) \quad \text{and} \quad DE(t) = E(t)D \ (t \in [a, b]). \tag{4.2}$$

A triangularizable operator A has the property

$$\sigma(A) = \sigma(D), \tag{4.3}$$

cf., **[11,** Lemma 7.5.1**]**. Each compact operator is triangularizable, and each operator having the Schatten-von Neumann Hermitian component is triangularizable; for more details, see **[11,** Chapter 7**]**. We will call D and V *the diagonal part and nilpotent part of* A, respectively. We have $V \in SN_2$, provided $A_I = (A - A^*)/2i \in SN_2$ or $AA^* - I \in SN_1$.

Indeed, due to [**11,** Lemma 7.7.2], we have

$$N_2(V) = \chi(A), \tag{4.4}$$

where

$$\chi(A) := \left[2N_2^2(A_I) - 2\sum_{k=1}^{\infty} |\mathrm{Im}\,\lambda_k(A)|^2\right]^{1/2}, \quad \text{if } A_I \in SN_2.$$

Due to [**11,** Lemma 7.15.2],

$$N_2(V) = \theta(A) \tag{4.5}$$

where

$$\theta(A) := \left[\mathrm{Trace}\,(AA^* - I) - \sum_{k=1}^{\infty}(|\lambda_k(A)|^2 - 1)\right]^{1/2}, \quad \text{if } AA^* - I \in SN_1.$$

Moreover, due to [**11,** Lemmas 6.3.6 and 2.3.2], we can write
(4.6)

$$N_2(V) = g(A), \text{ where } g(A) := \left[N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2\right]^{1/2}, \text{ if } A \in SN_2.$$

Obviously, $\chi(A) \le \sqrt{2}N_2(A_I)$ and $g^2(A) \le N_2^2(A) - |\mathrm{Trace}\,A^2|$. It is not hard to show also that $\theta(A) \le 2\sqrt{|\mathrm{Trace}\,(AA^* - I)|}$.

By $\mathrm{co}\,(A)$, we denote the closed convex hull of $\sigma(A)$. Let $f(z,w)$ be regular on a neighborhood of $\mathrm{co}\,(A) \times \mathrm{co}\,(\widetilde{A})$, and let the numbers $\psi_{jk} = \psi_{jk}(f, A, \widetilde{A})$ be defined by

$$\psi_{00} = \sup_{\substack{z\in\sigma(A)\\ w\in\sigma(\tilde{A})}} |f(z,w)|;$$

$$\psi_{jk} = \frac{1}{(j!k!)^{3/2}} \sup_{\substack{z\in\mathrm{co}(A)\\ w\in\mathrm{co}(\tilde{A})}} \left|\frac{\partial^{j+k} f(z,w)}{\partial z^k \partial w^j}\right|;$$

$$\psi_{0j} := \frac{1}{(j!)^{3/2}} \sup_{\substack{z\in\sigma(A)\\ w\in\mathrm{co}\,(\tilde{A})}} \left|\frac{\partial^j f(z,w)}{\partial w^j}\right|,$$

and

$$\psi_{j0} := \frac{1}{(j!)^{3/2}} \sup_{\substack{z \in \mathrm{co}\,(A) \\ w \in \sigma(\tilde{A})}} \left| \frac{\partial^j f(z,w)}{\partial z^j} \right| \quad (j, k \geq 1).$$

Theorem 4.1. *Let both A and $\widetilde{A}$ be triangularizable operators whose nilpotent parts V and $\widetilde{V}$ are nonzero Hilbert-Schmidt operators. If, in addition, $f(z,w)$ is regular on a neighborhood of* $\mathrm{co}\,(A) \times \mathrm{co}\,(\widetilde{A})$, *then*

$$\|F(f, A, K, \widetilde{A})\| \leq N_2(K) \sum_{j,k=0}^{\infty} \psi_{jk} N_2^j(V) N_2^k(\widetilde{V}).$$

If $V = 0$, $\widetilde{V} \neq 0$ is in SN_2, and $f(z,w)$ is regular on a neighborhood of $\sigma(A) \times co\,(\widetilde{A})$, then

$$\|F(f, A, K, \widetilde{A})\| \leq N_2(K) \sum_{j=0}^{\infty} \psi_{0j} N_2^j(\widetilde{V}).$$

If both A and $\widetilde{A}$ are normal and $f(z,w)$ is regular on a neighborhood of $\sigma(A) \times \sigma(\widetilde{A})$, then

$$\|F(f, A, K, \widetilde{A})\| \leq N_2(K) \sup_{\substack{z \in \sigma(A) \\ w \in \sigma(\widetilde{A})}} |f(z,w)|.$$

For example, consider the equation

$$AX - X\widetilde{A} = K, \tag{4.7}$$

assuming that A_I, $\widetilde{A}_I = (\widetilde{A} - \widetilde{A}^*)/(2i)$ and K are Hilbert-Schmidt operators, and

$$\delta := \mathrm{dist}\,(\mathrm{co}\,(A), \mathrm{co}\,(\widetilde{A})) > 0.$$

Take $f(z,w) = \frac{1}{z-w}$. Then

$$\psi_{jk} \leq \frac{(k+j)!}{\delta^{j+k+1}(k!j!)^{3/2}} \quad (j, k = 0, 1, \ldots, n-1).$$

Hence, by Theorem 4.1 and (4.4), a solution of (4.7) satisfies the inequality

$$\|X\| \le N_2(K) \sum_{j,k=0}^{\infty} \frac{(k+j)!}{\delta^{j+k+1}(k!j!)^{3/2}} \chi^j(A)\chi^k(\widetilde{A}).$$

5. Proof of Theorem 4.1. First, let A and $\widetilde{A}$ have n-dimensional ranges, $n < \infty$. Let $e = \{e_k\}_{k=1}^n$ and $\widetilde{e} = \{\widetilde{e}_k\}_{k=1}^n$ be the orthogonal normal bases of the triangular representation (Schur's bases) to A and $\widetilde{A}$, respectively. So

$$Ae_k = \sum_{j=1}^{k} a_{jk}e_j, \qquad \widetilde{A}\widetilde{e}_k = \sum_{j=1}^{k} \widetilde{a}_{jk}\widetilde{e}_j.$$

We can write

$$A = D + V, \qquad \widetilde{A} = \widetilde{D} + \widetilde{V}, \tag{5.1}$$

where $D, \widetilde{D}$ are the diagonal parts, V and $\widetilde{V}$ are the nilpotent parts of A and $\widetilde{A}$, respectively. Namely,

$$De_k = \lambda_k e_k; \quad Ve_k = \sum_{j=1}^{k-1} a_{jk}e_j \ (\lambda_j \in \sigma(A)).$$

Similarly, $\widetilde{D}$ and $\widetilde{V}$ are defined. Furthermore, let $|V|_e$ be the operator whose entries in $e = \{e_k\}$ are the absolute values of the entries of the matrix V. That is, $(|V|_e e_j, e_k) = |(V_j, e_k)|$ and

$$|V|_e = \sum_{k=1}^{n}\sum_{j=1}^{k-1} |a_{jk}|(., e_k)e_j.$$

Similarly $|\widetilde{V}|_{\widetilde{e}}$ is defined with respect to $\widetilde{e} = \{\widetilde{e}_k\}$. In addition, $|K|$ is defined by

$$|K|\widetilde{e}_j = \sum_{k=1}^{n} |(K\widetilde{e}_j, e_k)|e_k.$$

We need the following result proved in [**16,** Lemma 2.2].

Lemma 5.1. *Let A and $\widetilde{A}$ have range $n < \infty$, and let $f(z,w)$ be regular on a neighborhood of* $\operatorname{co}(A) \times \operatorname{co}(\widetilde{A})$. *Then*

$$\|F(f, A, K, \widetilde{A})\| \leq \||K|\| \sum_{j,k=0}^{n-1} \sqrt{k!j!}\psi_{jk} \||V|_e^j\| \| |\widetilde{V}|_{\tilde{e}}^k\|.$$

Theorem 2.5.1 from **[11]** implies

$$\|\widehat{V}^k\| \leq \frac{1}{\sqrt{k!}} N_2^k(\widehat{V}) \tag{5.2}$$

for any nilpotent matrix $\widehat{V}$. Take into account that $N_2(|V|_e) = N_2(V)$, then

$$\||V|_e^k\| \leq \frac{1}{\sqrt{k!}} N_2(V) \quad (k = 1, \ldots, n-1).$$

A similar inequality holds for $\widetilde{V}$. In addition,

$$\begin{aligned} N_2^2(|K|) &= \sum_{j=1}^{n} \||K|\widetilde{e}_j\|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} |(K\widetilde{e}_j, e_k)|^2 \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \|K\widetilde{e}_j\|^2 = N_2^2(K). \end{aligned}$$

Now the previous lemma yields the inequality

$$\|F(f, A, K, \widetilde{A})\| \leq N_2(K) \sum_{j,k=0}^{n-1} \psi_{jk} N_2^j(V) N_2^k(\widetilde{V}). \tag{5.3}$$

Proof of Theorem 4.1. We need the following result proved in **[17,** Lemma 2.2].

Lemma 5.2. *Let A be a triangularizable operator. Then there is a sequence of m-dimensional operators B_m $(m = 1, 2, \ldots)$ strongly converging to A, such that $\sigma(B_m) \subseteq \sigma(A)$. Moreover, the nilpotent parts of B_m tend to the nilpotent part V of A in the operator norm.*

This lemma and (5.3) prove the theorem. □

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