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GENERALIZED U-FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO-DIVISORS

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ABSTRACT. Recently, substantial progress has been made on generalized factorization techniques in integral domains, in particular, τ-factorization. There have also been advances made in investigating factorization in commutative rings with zero-divisors. One approach which has been found to be very successful is that of U-factorization introduced by Fletcher. We seek to synthesize work done in these two areas by generalizing τ-factorization to rings with zero-divisors by using the notion of U-factorization.

1. Introduction. Much work has been done on generalized factorization techniques in integral domains. There is an excellent overview in [**6**], where particular attention is paid to τ-factorization. Several authors have investigated ways to extend factorization to commutative rings with zero-divisors, for instance, Anderson, Valdez-Leon, Ağargün, Chun [**5, 8, 1**]. One particular method was that of U-factorization introduced by Fletcher in [**11, 12**]. This method of factorization has been studied extensively by Axtell and others in [**2, 9, 10**]. We synthesize the work done into a single study of what we will call τ-U-*factorization*.

In this paper, we will assume R is a commutative ring with 1. Let $R^* = R - \{0\}$, let $U(R)$ be the set of units of R, and let $R^{\#} = R^* - U(R)$ be the non-zero, non-units of R. As in [**10**], we define U-factorization as follows. Let $a \in R$ be a non-unit. If $a = \lambda a_1 \cdots a_n b_1 \cdots b_m$ is a factorization with $\lambda \in U(R)$, $a_i, b_i \in R^{\#}$, then we will call $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ a U-*factorization* of a if (1) $a_i(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for all $1 \leq i \leq n$ and (2) $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \neq (b_1 \cdots \widehat{b_j} \cdots b_m)$ for $1 \leq j \leq m$ where $\widehat{b_j}$ means b_j is omitted from the product. Here $(b_1 \cdots b_m)$ is the principal ideal generated by $b_1 \cdots b_m$. The b_i's in this particular U-factorization above will be referred to as *essential divisors.*

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The a_i's in this particular U-factorization above will be referred to as *inessential divisors*. A U-factorization is said to be *trivial* if there is only one essential divisor.

Note. We have added a single unit factor in front with the inessential divisors which was not in Axtell's original paper. This is added for consistency with the τ-factorization definitions, and it is evident that a unit is always inessential. We allow only one unit factor, so it will not affect any of the finite factorization properties.

Remark. If $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a U-factorization, then for any $1 \leq i_0 \leq m$, we have $(a) = (b_1 \cdots b_m) \subsetneq (b_1 \cdots \widehat{b_{i_0}} \cdots b_m)$. This is immediate from the definition of U-factorization.

In **[9]**, Axtell defines a non-unit a and b to be associate if $(a) = (b)$ and a non-zero non-unit a is said to be irreducible if $a = bc$ implies a is associate to b or c. R is commutative ring R to be U-*atomic* if every non zero non-unit has a U-factorization in which every essential divisor is irreducible. R is said to be a U-*finite factorization ring* if every non zero non-unit has a finite number of distinct U-factorizations. R is said to be a U-*bounded factorization ring* if every non zero non-unit has a bound on the number of essential divisors in any U-factorization. R is said to be a U-*weak finite factorization ring* if every non zero non-unit has a finite number of non-associate essential divisors. R is said to be a U-*atomic* idf-*ring* if every non zero non-unit has a finite number of non-associate irreducible essential divisors. R is said to be a U-*half factorization ring* if R is U-atomic and every U-atomic factorization has the same number of irreducible essential divisors. R is said to be a U-*unique factorization ring* if it is a U-HFR and, in addition, each U-atomic factorization can be arranged so the essential divisors correspond up to associate. In **[10,** Theorem 2.1], it is shown that this definition of U-UFR is equivalent to the one given by Fletcher in **[11, 12]**.

In Section 2, we begin with some preliminary definitions and results about τ-factorization in integral domains as well as factorization in rings with zero-divisors. In Section 3, we state definitions for τ-U-irreducible elements and τ-U-finite factorization properties. We also prove some preliminary results using these new definitions. In Section 4, we demonstrate the relationship between rings satisfying the various τ-U finite factorization properties. Furthermore, we compare these properties with the rings satisfying τ-finite factorization properties studied in [**13**]. In the final section, we investigate direct products of rings. We introduce a relation $\tau_\times$ which carries many τ-U-finite factorization properties of the component rings through the direct product.

2. Preliminary definitions and results. As in [**8**], we let $a \sim b$ if $(a) = (b)$, $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$, and $a \cong b$ if (1) $a \sim b$ and (2) $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$. We say a and b are *associates* (respectively, *strong associates, very strong associates*) if $a \sim b$ (respectively, $a \approx b$, $a \cong b$). As in [**4**], a ring R is said to be a *strongly associate* (respectively, *very strongly associate*) ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (respectively, $a \cong b$).

Let τ be a relation on $R^\#$, that is, $\tau \subseteq R^\# \times R^\#$. We will always assume further that τ is symmetric. Let a be a non-unit, $a_i \in R^\#$ and $\lambda \in U(R)$. Then $a = \lambda a_1 \cdots a_n$ is said to be a *τ-factorization* if $a_i \tau a_j$ for all $i \neq j$. If $n = 1$, then this is said to be a *trivial τ-factorization.* Each a_i is said to be a τ-factor, or that a_i τ-divides a, written $a_i \mid_\tau a$.

We say that τ is *multiplicative* (respectively, *divisive*) if, for $a, b, c \in R^\#$ (respectively, $a, b, b' \in R^\#$), $a\tau b$ and $a\tau c$ imply $a\tau bc$ (respectively, $a\tau b$ and $b' \mid b$ imply $a\tau b'$). We say τ is *associate* (respectively *strongly associate*, *very strongly associate*) *preserving* if, for $a, b, b' \in R^\#$ with $b \sim b'$ (respectively, $b \approx b'$, $b \cong b'$) $a\tau b$ implies $a\tau b'$. We define a *τ-refinement* of a τ-factorization $\lambda a_1 \cdots a_n$ to be a factorization of the form

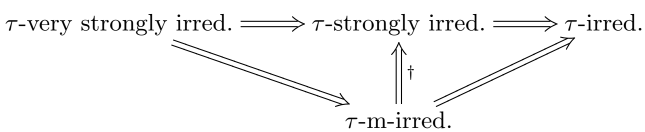
$$(\lambda\lambda_1 \cdots \lambda_n) \cdot b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$$

where $a_i = \lambda_i b_{i1} \cdots b_{im_i}$ is a τ-factorization for each i. This is slightly different from the original definition in [**6**] where no unit factor was allowed, and one can see they are equivalent when τ is associate preserving. We then say that τ is *refinable* if every τ-refinement of a τ-factorization is a τ-factorization. We say τ is *combinable* if, whenever $\lambda a_1 \cdots a_n$ is a τ-factorization, then so is each $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_n$.

We now summarize several of the definitions given in [**13**]. Let $a \in R$ be a non-unit. Then a is said to be *τ-irreducible* or *τ-atomic* if, for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some i. We will say a

is τ-*strongly irreducible* or τ-*strongly atomic* if, for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some a_i. We will say that a is τ-*m-irreducible* or τ-*m-atomic* if, for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all i. Note that the m is for *maximal* since such an a is maximal among principal ideals generated by elements which occur as τ-factors of a. We will say that a is τ-*very strongly irreducible* or τ-*very strongly atomic* if $a \cong a$ and a has no non-trivial τ-factorizations. See **[13]** for more equivalent definitions of these various forms of τ-irreducibility.

From [**13,** Theorem 3.9], we have the following relations where † represents the implication requires a strongly associate ring:



3. τ-U-irreducible elements. A τ-*U-factorization* of a non-unit $a \in R$ is a U-factorization $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ for which $\lambda a_1 \cdots a_n b_1 \cdots b_m$ is also a τ-factorization.

Given a symmetric relation τ on $R^{\#}$, we say R is τ-*U-refinable* if, for every τ-U-factorization of any non-unit $a \in U(R)$, $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$, any τ-U-factorization of an essential divisor, $b_i = \lambda' c_1 \cdots c_{n'} \lceil d_1 \cdots d_{m'} \rceil$ satisfies

$$a = \lambda \lambda' a_1 \cdots a_n c_1 \cdots c_{n'} \lceil b_1 \cdots b_{i-1} d_1 \cdots d_{m'} b_{i+1} \cdots \ b_m \rceil$$

is a τ-U-factorization.

Example 3.1. Let $R = \mathbb{Z}/20\mathbb{Z}$, and let $\tau = R^{\#} \times R^{\#}$.

Certainly $0 = \lceil 10 \cdot 10 \rceil$ is a τ-U-factorization. But $10 = \lceil 2 \cdot 5 \rceil$ is a τ-U-factorization; however, $0 = \lceil 2 \cdot 5 \cdot 2 \cdot 6 \rceil$ is not a U-factorization since 5 becomes inessential after a τ-U-refinement. It will sometimes be important to ensure the essential divisors of a τ-U-refinement of a τ-U-factorization's essential divisors remain essential. We will see that, in a présimplifiable ring, there are no inessential divisors, so for τ-refinable, R will be τ-U-refinable.

As stated in **[9]**, the primary benefit of looking at U-factorizations is the elimination of troublesome idempotent elements that ruin many of the finite factorization properties. For instance, even $\mathbb{Z}_6$ is not a BFR (a ring in which every non-unit has a bound on the number of non-unit factors in any factorization) because we have $3 = 3^2$. Thus, 3 is an idempotent, so $3 = 3^n$ for all $n \geq 1$ which yields arbitrarily long factorizations. When we use U-factorization, we see any of these factorizations can be rearranged to $3 = 3^{n-1} \lceil 3 \rceil$, which has only one essential divisor.

Let $\alpha \in$ {irreducible, strongly irreducible, m-irreducible, very strongly irreducible}. Let a be a non-unit. If

$$a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$$

is a τ-U-factorization, then this factorization is said to be a *τ-U-α-factorization* if it is a τ-U-factorization and the essential divisors b_i are τ-α for $1 \leq i \leq m$.

One must be somewhat more careful with U-factorizations as there is a loss of uniqueness in the factorizations. For instance, if we let $R = \mathbb{Z}_6 \times \mathbb{Z}_8$, then we can factor $(3,4)$ as $(3,1)\lceil (3,3)(1,4) \rceil$ or $(3,3)\lceil (3,1)(1,4) \rceil$. On the bright side, we have **[2,** Proposition 4.1].

Theorem 3.2. *Every factorization can be rearranged into a U-factorization.*

Corollary 3.3. *Let R be a commutative ring with* 1 *and τ a symmetric relation on $R^{\#}$. Let $\alpha \in$ {irreducible, strongly irreducible, m-irreducible, very strongly irreducible}. For every τ-α factorization of a non-unit $a \in R$, $a = \lambda a_1 \cdots a_n$, we can rearrange this factorization into a τ-U-α-factorization.*

Proof. Let $a = \lambda a_1 \cdots a_n$ be a τ-α-factorization. By Theorem 3.2, we can rearrange this to form a U-factorization. This remains a τ-factorization since τ is assumed to be symmetric. Lastly, each a_i is τ-α, so the essential divisors are τ-α. □

This leads us to another characterization of τ-irreducible.

Theorem 3.4. *Let $a \in R$ be a non-unit. If any τ-U-factorization of a has only one essential divisor, then a is τ-irreducible.*

Proof. Suppose $a = \lambda a_1 \cdots a_n$. Then this can be rearranged into a U-factorization, and hence a τ-U-factorization. By hypothesis, there can only be one essential divisor. Suppose it is a_n. We have $a = \lambda a_1 \cdots a_{n-1} \lceil a_n \rceil$ is a τ-U-factorization and $a \sim a_n$ as desired. □

We now define the finite factorization properties using the τ-U-factorization approach. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible $\}$, and let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. R is said to be *τ-U-α* if, for all non-units $a \in R$, there is a τ-U-α-factorization of a. R is said to satisfy *τ-U-ACCP* (ascending chain condition on principal ideals) if every properly ascending chain of principal ideals $(a_1) \subsetneq (a_2) \subsetneq \cdots$ such that a_{i+1} is an essential divisor in some τ-U-factorization of a_i, for each i terminates after finitely many principal ideals. R is said to be a *τ-U-BFR* if, for all non-units $a \in R$, there is a bound on the number of essential divisors in any τ-U-factorization of a.

R is said to be a *τ-U-β-FFR* if for all non-units $a \in R$, there are only finitely many τ-U-factorizations up to rearrangement of the essential divisors and β. R is said to be a *τ-U-β-WFFR* if, for all non-units $a \in R$, there are only finitely many essential divisors among all τ-U-factorizations of a up to β. R is said to be a *τ-U-α-β-divisor finite (df) ring* if, for all non-units $a \in R$, there are only finitely many essential τ-α divisors up to β in the τ-U-factorizations of a.

R is said to be a *τ-U-α-HFR* if R is τ-U-α and, for all non-units $a \in R$, the number of essential divisors in any τ-U-α-factorization of a is the same. R is said to be a *τ-U-α-β-UFR* if R is a τ-U-α-HFR, and the essential divisors of any two τ-U-α-factorizations can be rearranged to match up to β.

R is said to be *présimplifiable* if, for every $x \in R$, $x = xy$ implies $x = 0$ or $y \in U(R)$. This is a condition which has been well studied and is satisfied by any domain or local ring. We introduce two slight modifications of this. R is said to be *τ-présimplifiable* if, for every $x \in R$, the only τ-factorizations of x which contain x as a τ-factor are of the form $x = \lambda x$ for a unit λ. R is said to be *τ-U-présimplifiable* if, for every non zero non-unit $x \in R$, all τ-U-factorizations have no non-unit inessential divisors.

Theorem 3.5. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. We have the following*:

(1) *If R is présimplifiable, then R is τ-U-présimplifiable.*
(2) *If R is τ-U-présimplifiable, then R is τ-présimplifiable.*

That is, présimplifiable $\Rightarrow$ τ-U-présimplifiable $\Rightarrow$ τ-présimplifiable. If $\tau = R^{\#} \times R^{\#}$, then all are equivalent.

Proof.

(1) Let R be présimplifiable, and let $x \in R^{\#}$. Suppose
$$x = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$$
is a τ-U-factorization. Then $(x) = (b_1 \cdots b_m)$. R présimplifiable implies that all the associate relations coincide, so in fact $x \cong b_1 \cdots b_m$ implies that $\lambda a_1 \cdots a_n \in U(R)$, and hence all inessential divisors are units.
(2) Let R be τ-U-présimplifiable, and let $x \in R$ be such that $x = \lambda x a_1 \cdots a_n$ is a τ-factorization. We claim that $x = \lambda a_1 \cdots a_n \lceil x \rceil$ is a τ-U-factorization. For any $1 \leq i \leq n$, $x \mid a_i x$ and $(a_i x)(\lambda a_1 \cdots \widehat{a_i} \cdots a_n) = x$ shows $a_i x \mid x$, proving the claim. This implies $\lambda a_1 \cdots a_n \in U(R)$ as desired.

Let $\tau = R^{\#} \times R^{\#}$, and suppose R is τ-présimplifiable. Suppose $x = xy$; for $x \neq 0$, we show $y \in U(R)$. If $x \in U(R)$, then multiplying through by x^{-1} yields $1 = x^{-1}x = x^{-1}xy = y$ and $y \in U(R)$ as desired. We may now assume $x \in R^{\#}$. If $y = 0$, then $x = 0$, a contradiction. If $y \in U(R)$, we are already done, so we may assume $y \in R^{\#}$. Thus, $x \tau y$, and $x = xy$ is a τ-factorization, so $y \in U(R)$, as desired. $\square$

4. τ-U-finite factorization relations. We now would like to show the relationship between rings with various τ-U-α-finite factorization properties as well as compare these rings with the τ-α-finite factorization properties of [**13**].

Theorem 4.1. *Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$. Consider the following statements.*

(1) *R is a* τ-BFR.
(2) *R is* τ*-présimplifiable and, for every non-unit* $a_1 \in R$*, there is a fixed bound on the length of chains of principal ideals* (a_i) *ascending from* a_1 *such that at each stage* $a_{i+1} \mid_\tau a_i$.
(3) *R is* τ*-présimplifiable and a* τ-U-BFR.
(4) *For every non-unit* $a \in R$*, there are natural numbers* $N_1(a)$ *and* $N_2(a)$ *such that, if* $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ *is a* τ*-U-factorization, then* $n \leq N_1(a)$ *and* $m \leq N_2(a)$.

Then (4) $\Rightarrow$ (1) *and* (2) $\Rightarrow$ (3). *For* τ *refinable,* (1) $\Rightarrow$ (2) *and, for R* τ*-U-présimplifiable,* (3) $\Rightarrow$ (4). *Thus, all are equivalent if R is* τ*-U-présimplifiable and* τ *is refinable.*

Let $\star$ *represent* τ *being refinable, and let* $\dagger$ *represent R being* τ*-U-présimplifiable. Then the following diagram summarizes the theorem.*

$$\begin{array}{ccc} (1) & \overset{\star}{\Longrightarrow} & (2) \\ \Uparrow & & \Downarrow \\ (4) & \underset{\dagger}{\Longleftarrow} & (3) \end{array}$$

Proof. (1) $\Rightarrow$ (2). Let τ be refinable. Suppose there were a non-trivial τ-factorization $x = \lambda x a_1 \cdots a_n$ with $n \geq 1$. Since τ is assumed to be refinable, we can continue to replace the τ-factor x with this factorization.

$$\begin{aligned} x &= \lambda x a_1 \cdots a_n = (\lambda\lambda) x a_1 \cdots a_n a_1 \cdots a_n \\ &= \cdots = (\lambda\lambda\lambda) x a_1 \cdots a_n a_1 \cdots a_n a_1 \cdots a_n \\ &= \cdots \end{aligned}$$

yields an unbounded series of τ-factorizations of increasing length.

Let a_1 be a non-unit in R. Suppose N is the bound on the length of any τ-factorization of a_1. We claim that N satisfies the requirement of (2). Let $(a_1) \subsetneq (a_2) \subsetneq \cdots$ be an ascending chain of principal ideals generated by elements which satisfy $a_{i+1} \mid_\tau a_i$ for each i. Say $a_i = \lambda_i a_{i+1} a_{i1} \cdots a_{in_i}$ for each i. Furthermore, we can assume $n_i \geq 1$ for each i or else the containment would not be proper. Then we can write

$$a_1 = \lambda_1 a_2 a_{11} \cdots a_{1n_1} = \lambda_1 \lambda_2 a_3 a_{21} \cdots a_{2n_2} a_{11} \cdots a_{1n_1} = \cdots .$$

Each remains a τ-factorization since τ is refinable, and we have added at least one factor at each step. If the chain were greater than length N, we would contradict R being a τ-BFR.

(2) $\Rightarrow$ (3). Let $a \in R$ be a non-unit. Let N be the bound on the length of any properly ascending chain of principle ideals ascending from a such that $a_{i+1} \mid_\tau a_i$. If $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a τ-U-factorization, then we get an ascending chain with $b_1 \cdots b_{i-1} \mid_\tau b_1 \cdots b_i$ for each i:

$$\begin{aligned}(a) = (b_1 \cdots b_m) &\subsetneq (b_1 \cdots b_{m-1}) \subsetneq (b_1 \cdots b_{m-2}) \subsetneq \cdots \\ &\subsetneq (b_1 b_2) \subsetneq (b_1).\end{aligned}$$

Hence, $m \leq N$, and we have found a bound on the number of essential divisors in any τ-U-factorization of a, making R a τ-U-BFR.

(3) $\Rightarrow$ (4). Let $a \in R$ be a non-unit. Let $N_e(a)$ be the bound on the number of essential divisors in any τ-U-factorization of a. Since R is τ-U-présimplifiable, there are no inessential τ-U-divisors of a. We can set $N_1(a) = 0$, and $N_2(a) = N_e(a)$ and see that this satisfies the requirements of the theorem.

(4) $\Rightarrow$ (1). Let $a \in R$ be a non-unit. Then any τ-factorization $a = \lambda a_1 \cdots a_n$ can be rearranged into a τ-U-factorization, say $a = \lambda a_{s_1} \cdots a_{s_i} \left\lceil a_{s_{i+1}} \cdots a_{s_n} \right\rceil$. But then $n = i + (n - i) \leq N_1(a) + N_2(a)$. Hence, the length of any τ-factorization must be less than $N_1(a)+N_2(a)$ proving R is a τ-BFR as desired. $\square$

The way we have defined our finite factorization properties on only the essential divisors causes a slight problem. Given a τ-U-factorization $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$, we only know that $a \sim b_1 \cdots b_m$. This may no longer be a τ-factorization of a, but rather only some associate of a. This is easily remedied by insisting that our rings are strongly associate.

Lemma 4.2. *Let R be a strongly associate ring with τ a symmetric relation on $R^\#$, and let $\alpha \in$ {irreducible, strongly irreducible, m-irreducible, very strongly irreducible}. Let $a \in R$ be a non-unit. If $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ is a τ-U-α-factorization, then there is a unit $\mu \in U(R)$ such that $a = \mu b_1 \cdots b_m$ is a τ-α-factorization.*

Proof. Let $a = \lambda a_1 a_2 \cdots a_n \lceil b_1 b_2 \cdots b_m \rceil$ be a τ-U-α-factorization. By definition, $(a) = (b_1 \cdots b_m)$, and R strongly associate implies that

$a \approx b_1 \cdots b_m$. Let $\mu \in U(R)$ be such that $a = \mu b_1 \cdots b_m$. We still have $b_i \tau b_j$ for all $i \neq j$, and b_i is τ-α for every i. Hence, $a = \mu b_1 \cdots b_m$ is the desired τ-factorization, proving the lemma. □

Theorem 4.3. *Let R be a commutative ring with* 1*, and let τ be a symmetric relation on $R^{\#}$. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$, and $\beta \in \{$associate, strongly associate, very strongly associate$\}$. We have the following*:

(1) *If R is τ-α, then R is τ*-U-*α.*
(2) *If R satisfies τ*-ACCP*, then R satisfies τ*-U-ACCP*.*
(3) *If R is a τ*-BFR*, then R is a τ*-U-BFR*.*
(4) *If R is a τ-β*-FFR*, then R is a τ*-U-*β*-FFR*.*
(5) *Let R be a τ-β*-WFFR*. Then R is a τ*-U-*β*-WFFR*.*
(6) *Let R be a τ-α-β-divisor finite ring. Then R is a τ-U-α-β-divisor finite ring.*
(7) *Let R be a strongly associate τ-α*-HFR *(respectively, τ-α-β*-UFR*). Then R is τ-U-α*-HFR *(respectively, τ-U-α-β*-UFR*).*

Proof. (1) This is immediate from Corollary 3.3.

(2) Suppose there were an infinite properly ascending chain of principal ideals $(a_1) \subsetneq (a_2) \subsetneq \cdots$ such that a_{i+1} is an essential divisor in some τ-U-factorization of a_i, for each i. Every essential τ-U-divisor is certainly a τ-divisor. This would contradict the fact that R satisfies τ-ACCP.

(3) We suppose that there is a non-unit $a \in R$ with τ-U-factorizations having arbitrarily large numbers of essential τ-U-divisors. Each is certainly a τ-factorization, having at least as many τ-factors as there are essential τ-divisors, so this would contradict the hypothesis.

(4) Every τ-U-factorization is certainly among the τ-factorizations. If the latter is finite, then so is the former.

(5) For any given non-unit $a \in R$, every essential τ-U-divisor of a is certainly a τ-factor of a which has only finitely many up to β. Hence, there can be only finitely many essential τ-U-factors up to β.

(6) Let $a \in R$ be a non-unit. Every essential τ-U-α-divisor of a is a τ-α-factor of a. There are only finitely many τ-α-divisors up to β, so then there can be only finitely many τ-U-α-divisors of a up to β.

(7) We have already seen that R being τ-α implies R is τ-U-α. Let $a \in R$ be a non-unit. We suppose for a moment there are two τ-α-U-factorizations:

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil = \lambda' a_1' \cdots a_{n'}' \lceil b_1' \cdots b_{m'}' \rceil$$

such that $m \neq m'$ (respectively, $m \neq m'$ or there is no rearrangement such that b_i and b_i' are β for each i). Lemma 4.2 implies there are $\mu, \mu' \in U(R)$ with $a = \mu b_1 \cdots b_m = \mu' b_1' \cdots b_{m'}'$ are two τ-α-factorizations of a, so $m = m'$ (respectively, $m = m'$ and there is a rearrangement so that b_i and b_i' are β for each $1 \leq i \leq m$), a contradiction, proving R is indeed a τ-U-α-HFR (respectively, -β-UFR) as desired. □

Theorem 4.4. *Let R be a commutative ring with* 1 *and τ a symmetric relation on $R^{\#}$. Let $\alpha \in$ {irreducible, strongly irreducible,* m-*irreducible, very strongly irreducible}, and let $\beta \in$ {associate, strongly associate, very strongly associate}.*

1. *If R is a τ-U-α-β-*UFR*, then R is a τ-α-*U-HFR.
2. *If R is τ-*U*-refinable and R is a τ-*U*-α-β-*UFR*, then R is a τ-*U*-β-*FFR.
3. *If R is τ-*U*-refinable and R is a τ-*U*-α-*HFR*, then R is a τ-*U-BFR.
4. *If R is a τ-*U*-β-*FFR*, then R is a τ-*U-BFR.
5. *If R is a τ-*U*-β-*FFR*, then R is a τ-*U*-β-*WFFR.
6. *If R is a τ-*U*-β-*WFFR*, then R is a τ-*U*-α-β-divisor finite ring.*
7. *If R is τ-*U*-refinable and R is a τ-*U*-α-*BFR*, then R satisfies τ-*U-ACCP.
8. *If R is τ-*U*-refinable and R satisfies τ-*U-ACCP*, then R is τ-*U*-atomic.*

Proof. (1) This is immediate from the definitions.

(2) Let $a \in R$ be a non-unit. Let $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ be the unique τ-α-U-factorization up to rearrangement and β. Given any other τ-U-factorization, we can τ-U-refine each essential τ-U-divisor into a τ-U-α-factorization of a. There is a rearrangement of the essential divisors to match up to β with b_i for each $1 \leq i \leq m$. Thus, the essential divisors in any τ-U-factorization come from some combination of products of β of the m τ-U-α essential factors in our

original factorization. Hence, there are at most 2^m possible distinct τ-U-factorizations up to β, making this a τ-U-β-FFR as desired.

(3) For a given non-unit $a \in R$, the number of essential divisors in any τ-U-α-factorization is the same, say N. We claim this is a bound on the number of essential divisors of any τ-U-factorization. Suppose there were a τ-U-factorization $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ with $m > N$. For every i, b_i has a τ-U-α-factorization with at least one essential divisor. Since R is τ-U-refinable, we can τ-U-refine the factorization yielding a τ-U-α-factorization of a with at least m τ-U-α essential factors. This contradicts the assumption that R is a τ-U-α-HFR.

(4) Let R be a τ-U-β-FFR. Let $a \in R$ be a non-unit. There are only finitely many τ-U-factorizations of a up to rearrangement and β of the essential divisors. We can simply take the maximum of the number of essential divisors among all of these factorizations. This is an upper bound for the number of essential divisors in any τ-U-factorization.

(5) Let R be a τ-U-β-FFR. Then, for any non-unit $a \in R$, let S be the collection of essential divisors in the finite number of representative τ-U-factorizations of a up to β. This gives us a finite collection of elements up to β. Every essential divisor up to β in a τ-U-factorization of a must be among these, so this collection is finite as desired.

(6) If every non-unit $a \in R$ has a finite number of proper essential τ-U divisors, then certainly there are a finite number of essential τ-α-U-divisors.

(7) Suppose R is a τ-U-BFR, but $(a_1) \subsetneq (a_2) \subsetneq \cdots$ is a properly ascending chain of principal ideals such that a_{i+1} is an essential factor in some τ-U-factorization of a_i, say

$$a_i = \lambda_i a_{i1} \cdots a_{in_i} \lceil a_{i+1} b_{i1} \cdots b_{im_i} \rceil$$

for each i. Furthermore, $m_i \geq 1$, for each i; otherwise, we would have $(a_{i+1}) = (a_i)$, contrary to our assumption that our chain is properly increasing. Our assumption that R is τ-U refinable allows us to factor a_1 as follows:

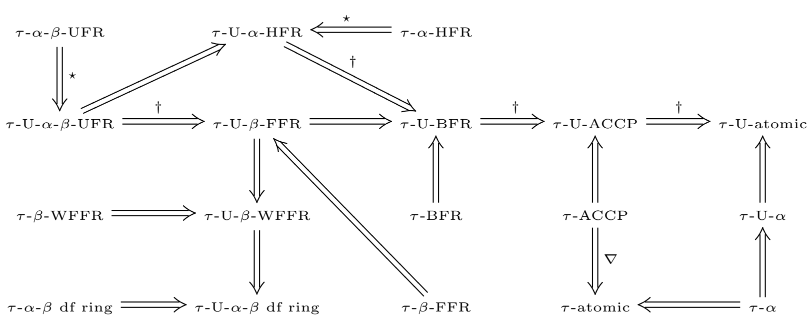
$$\begin{aligned} a_1 &= \lambda_1 a_{11} \cdots a_{1n_1} \lceil a_2 b_{11} \cdots b_{1m_1} \rceil \\ &= \lambda_1 \lambda_2 a_{11} \cdots a_{1n_1} a_{21} \cdots a_{2n_2} \lceil a_3 b_{21} \cdots b_{2m_2} b_{11} \cdots b_{1m_1} \rceil \end{aligned}$$

and so on. At each iteration i, we have at least $i + 1$ essential factors in our τ-U-factorization. This contradicts the assumption that a_1

should have a bound on the number of essential divisors in any τ-U-factorization.

(8) Suppose R were not τ-U-atomic. Then there exists $a_1 \in R$ such that there is no τ-U-atomic factorization of a_1. If a_1 were τ-atomic, we would be done as $a_1 = 1\lceil a_1 \rceil$ is a τ-U-atomic factorization. Thus, there exists a τ-factorization $a_1 = \lambda_1 a_2 a_{21} \cdots a_{2_{n2}}$ such that a_1 is not associated with any factor. Moreover, this can be rearranged into a U-factorization. If every essential factor were τ-atomic, we would be done as we have found a τ-U-atomic factorization. Thus, at least one essential divisor is not τ-atomic. Suppose this is a_2 after reordering if necessary. We know $a_1 \not\sim a_2$, so $(a1) \subsetneq (a_2)$. We may now continue this process with a_2. The assumption of τ-U-refinability would allow us to replace a_2 with the essential factors of a_2 in the τ-U-factorization of a_1. Again, at each stage there must be an essential divisor which is not τ-atomic or else we would have found a τ-U-atomic factorization of a_1. Thus, we are able to produce an infinite properly ascending chain of principal ideals such that each is generated by an essential τ-divisor of the previous generator as desired. This is a contradiction of the fact that R satisfies τ-U-ACCP. Thus, a_1 must have a τ-U-atomic factorization after finitely many steps, and R has been shown to be τ-U-atomic as desired. $\square$

The following diagram summarizes our results from Theorems 4.3 and 4.4 where $\star$ represents R being strongly associate, ∇ represents τ is refinable and associate preserving, and $\dagger$ represents R is τ-U-refinable:



We have left off the relations which were proven in [**13,** Theorem 4.1] and focused instead on the rings satisfying the U-finite factorization properties. Examples given in [**4, 6, 9, 10**] show that arrows can neither be reversed nor added to the diagram with a few exceptions.

Question 4.5. *Does U-atomic imply atomic?*

Anderson and Valdez-Leon show in [**8,** Theorem 3.13] that if R has a finite number of non-associate irreducibles, then U-atomic and atomic are equivalent. This remains open in general.

Question 4.6. *Does U-ACCP imply ACCP?*

We can modify Axtell's proof of [**9,** Theorem 2.9] to add a partial converse to Theorem 4.4 (5) if τ is combinable and associate preserving. The idea is the same, but slight adjustments are required to adapt it to τ-factorizations and to allow uniqueness up to any type of associate.

Theorem 4.7. *Let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Let R be a commutative ring with 1, and let τ be a symmetric relation on $R^{\#}$ which is both combinable and associate preserving. R is a τ-*U*-β-*FFR *if and only if R is a τ-*U*-β-*WFFR.

Proof. ($\Rightarrow$) was already shown, so we need only prove the converse.

($\Leftarrow$). Suppose R is not a τ-U-β-FFR. Let $a \in R$ be a non-unit which has infinitely many τ-U-factorizations up to β. Let $b_1, b_2, \ldots, b_m$ be a complete list of essential τ-U-divisors of a up to β. Let

$$a = a_1 \cdots a_n \lceil c_1 \cdots c_k \rceil = a'_1 \cdots a'_{n'} \lceil d_1 \cdots d_n \rceil$$

be two τ-U-factorizations of a, and assume we have re-ordered the essential divisors in both factorizations above so that the β of b_1 appear first, followed by β of b_2, etc. Let $A = \langle (c_1), (c_2), \ldots, (c_k) \rangle$ and $B = \langle (d_1), (d_2), \ldots, (d_n) \rangle$ be sequences of ideals. We call the factorizations *comparable* if A is a subsequence of B or vice versa.

Suppose A is a proper subsequence of B

$$B = \langle (d_1), \ldots, (d_{i_1}) = (c_1), \ldots, (d_{i_2}) = (c_2), \ldots, (d_{i_k}) = (c_k), \ldots, (d_n) \rangle$$

with $n > k$. Because τ is combinable and symmetric,

$$a = a'_1 \cdots a'_{n'} \left[d_{i_1} d_{i_2} \cdots d_{i_k} (d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n) \right]$$

remains a τ-factorization, and [**9,** Lemma 1.3] ensures that this remains a U-factorization.

This yields

$$\begin{aligned}(a) &= (d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n)(d_{i_1} d_{i_2} \cdots d_{i_k}) \\ &= (d_1 \cdots d_n) = (c_1 \cdots c_k) \\ &= (c_1) \cdots (c_k) = (d_{i_1}) \cdots (d_{i_k}) = (d_{i_1} \cdots d_{i_k}).\end{aligned}$$

But then, $(d_1 \cdots \widehat{d_{i_1}} \widehat{d_{i_2}} \cdots \widehat{d_{i_k}} \cdots d_n)$ cannot be an essential divisor, a contradiction, unless $n = k$.

If $n = k$, then the sequences of ideals are identical, and we seek to prove this means the τ-U-factorizations are the same up to β. It is certainly true for $\beta =$ associate as demonstrated in [**9,** Theorem 2.9]. So we have a pairing of the c_i and d_i such that $c_i \sim b_j \sim d_i$ for one of the essential τ-U-divisors b_j. We know further that c_i and b_j (respectively, d_i and b_j) are β since R is by assumption a τ-U-β-WFFR.

It is well established that β is transitive, so we can conclude that this same pairing demonstrates that c_i and d_i are β, not just associate. Thus, the number of distinct τ-U-factorizations up to β is less than or equal to the number of non-comparable finite sequences of elements from the set $\{(b_1), (b_2), \ldots, (b_m)\}$.

From here, we direct the reader to the proof of the second claim in [**9,** Theorem 2.9] where it is shown that this set is finite. □

5. Direct products. For each i, $1 \le i \le N$, let R_i be commutative rings with τ_i a symmetric relation on $R_i^{\#}$. We define a relation $\tau_\times$ on $R = R_1 \times \cdots \times R_N$ which preserves many of the theorems about direct products from [**2**] for τ-factorizations. Let $(a_i), (b_i) \in R^{\#}$. Then $(a_i)\tau_\times(b_i)$ if and only if whenever a_i and b_i are both non-units in R_i, then $a_i \tau_i b_i$.

For convenience, we will adopt the following notation. Suppose $x \in R_i$. Then $x^{(i)} = (1_{R_1}, \ldots, 1_{R_{i-1}}, x, 1_{R_{i+1}}, \ldots 1_{R_N})$, so x appears in the ith coordinate, and all other entries are the identity. Thus, for any $(a_i) \in R$, we have $(a_i) = a_1^{(1)} a_2^{(2)} \cdots a_n^{(n)}$ is a $\tau_\times$-factorization. We will always move any $\tau_\times$-factors which may become units in this process to the front and collect them there.

Lemma 5.1. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$. Then $(a_i) \sim (b_i)$ (respectively, $(a_i) \approx (b_i)$) if and only if $a_i \sim b_i$ (respectively, $a_i \approx b_i$) for every i. Furthermore, $(a_i) \cong (b_i)$ implies $a_i \cong b_i$ for all i, and for a_i, b_i all non-zero, $a_i \cong b_i$ for all $i \Rightarrow (a_i) \cong (b_i)$.*

Proof. See [**8,** Theorem 2.15]. □

Example 5.2. If $a_{i_0} = 0$ for even one index $1 \leq i_0 \leq N$, then $a_i \cong b_i$ for all i need not imply $(a_i) \cong (b_i)$.

Consider the ring $R = \mathbb{Z} \times \mathbb{Z}$, with $\tau_i = \mathbb{Z}^\# \times \mathbb{Z}^\#$ for $i = 1, 2$, the usual factorization. We have $1 \cong 1$ and $0 \cong 0$ since $\mathbb{Z}$ is a domain; however, $(0,1) = (0,1)(0,1)$ shows $(0,1) \ncong (0,1)$.

Lemma 5.3. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^\#$ for each i. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$. If $(a_i) \in R$ is τ-α, then precisely one coordinate is not a unit.*

Proof. Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-α. Certainly not all coordinates can be units, or else $a \in U(R)$. Suppose for a moment there were at least two coordinates for which a_i is not a unit in R_i. After reordering, we may assume a_1 and a_2 are not units. Then $a = a_1^{(1)}(1_{R_1}, a_2, \ldots, a_N)$ is a $\tau_\times$-factorization. But a is not even associate to either $\tau_\times$-factor, a contradiction. □

Theorem 5.4. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^\#$ for each i.*

(1) *A non-unit $(a_i) \in R$ is $\tau_\times$-atomic (respectively, strongly atomic) if and only if a_{i_0} is τ_{i_0}-atomic (respectively, strongly atomic) for some $1 \leq i_0 \leq n$ and $a_i \in U(R_i)$ for all $i \neq i_0$.*
(2) *A non-unit $(a_i) \in R$ is $\tau_\times$-m-atomic if and only if a_{i_0} is τ_{i_0}-m-atomic for some $1 \leq i_0 \leq n$ and $a_i \in U(R_i)$ for all $i \neq i_0$.*

(3) *A non-unit* $(a_i) \in R$ *is* $\tau_\times$*-very strongly atomic if and only if* a_{i_0} *is* τ_{i_0}*-very strongly atomic and non-zero for some* $1 \leq i_0 \leq n$ *and* $a_i \in U(R_i)$ *for all* $i \neq i_0$*.*

Proof. (1) ($\Rightarrow$). Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-atomic (respectively, strongly atomic). By Lemma 5.3, there is only one non-unit coordinate. Suppose after reordering if necessary that a_1 is the non-unit. If a_1 were not τ_1-atomic (respectively, strongly atomic), then there is a τ_1-factorization, $\lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ for which $a_1 \not\sim a_{1_j}$ (respectively, $a_1 \not\approx a_{1_j}$) for any $1 \leq j \leq k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \ldots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization. Furthermore, by Lemma 5.1, $(a_i) \not\sim a_{1_j}^{(1)}$ (respectively, $(a_i) \not\sim a_{1_j}^{(1)}$) for all $1 \leq j \leq k$. This would contradict the assumption that a was $\tau_\times$-atomic (respectively, strongly atomic).

($\Leftarrow$). Let $a_1 \in R_1$ be a non-unit with a_1 being τ_1-atomic (respectively, strongly atomic). Let $\mu_i \in U(R_i)$ for $2 \leq i \leq N$. We show $a = (a_1, \mu_2, \cdots \mu_N)$ is $\tau_\times$-atomic (respectively, strongly atomic). Suppose $a = (\lambda_1, \ldots, \lambda_N)(a_{1_1}, \ldots, a_{1_N}) \cdots (a_{k_1}, \ldots, a_{k_N})$ is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \geq 2$. Furthermore, this means a_{i_1} is not a unit in R_1 for $1 \leq i \leq k$; otherwise, we would have units as factors in a $\tau_\times$ factorization. This means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-atomic (respectively, strongly atomic) element. Thus, we must have $a_1 \sim a_{j_1}$ (respectively, $a_1 \approx a_{j_1}$) for some $1 \leq j \leq k$. Hence, by Lemma 5.1, we have $a \sim (a_{j_1}, \ldots, a_{j_N})$ (respectively, $a \approx (a_{j_1}, \ldots, a_{j_N})$ for some $1 \leq j \leq k$ and a is $\tau_\times$ atomic (respectively, strongly atomic) as desired.

(2) ($\Rightarrow$). Let $a = (a_i) \in R$ be a non-unit which is $\tau_\times$-m-atomic. By Lemma 5.3, there is only one non-unit coordinate, say a_1 after reordering if necessary. Let $a_1 = \lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ be a τ_1 factorization for which $a_1 \not\sim a_{1_{j_0}}$ for at least one $1 \leq j_0 \leq k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \ldots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization of a for which (by Lemma 5.1) $a = (a_i) \not\sim a_{1_{j_0}}^{(1)}$. This contradicts the hypothesis that a is $\tau_\times$-m-atomic.

($\Leftarrow$). Let $a_1 \in R_1$ be a non-unit with a_1 being τ_1-m-atomic. Let $\mu_i \in U(R_i)$ for $2 \le i \le N$. We show $a = (a_1, \mu_2, \ldots, \mu_N)$ is $\tau_\times$-m-atomic. Suppose

$$a = (\lambda_1, \ldots, \lambda_N)(a_{1_1}, \ldots, a_{1_N}) \cdots (a_{k_1}, \ldots, a_{k_N})$$

is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \ge 2$. As before, this means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-m-atomic element. Hence, $a_1 \sim a_{j_1}$ for each $1 \le j \le k$. By Lemma 5.1, we have $a \sim (a_{j_1}, \ldots, a_{j_N})$ for all $1 \le j \le k$ and thus a is $\tau_\times$-m-atomic as desired.

(3) ($\Rightarrow$). Let $a = (a_1, \ldots, a_N)$ be a non-unit which is $\tau_\times$-very strongly atomic. By Lemma 5.3, we may assume a_1 is the non-unit, and a_j is a unit for $j \ge 2$. We suppose for a moment that $a_1 = 0_1$. But then $(0, a_2, \ldots, a_N) = (0, 1, \ldots, 1) \cdot (0, a_2, \ldots, a_N)$ shows that $a \ncong a$, a contradiction. Lemma 5.1 shows that, if $a \cong a$, then $a_i \cong a_i$ for each $1 \le i \le N$. Hence, if a_1 were not τ_1-very strongly atomic, then there is a τ_1-factorization, $\lambda_{1_1} a_{1_1} a_{1_2} \cdots a_{1_k}$ for which $a_1 \ncong a_{1_j}$ for any $1 \le j \le k$. But then

$$(a_i) = (\lambda_{1_1}, a_2, \ldots, a_n) a_{1_1}^{(1)} a_{1_2}^{(1)} \cdots a_{1_k}^{(1)}$$

is a $\tau_\times$-factorization. Furthermore, since every coordinate is non-zero, by Lemma 5.1, $(a_i) \ncong a_{1_j}^{(1)}$ for all $1 \le j \le k$. This would contradict the assumption that a was $\tau_\times$-very strongly atomic.

($\Leftarrow$). Let $a_1 \in R_1^{\#}$ be τ_1-very strongly atomic. Let $\mu_i \in U(R_i)$ for $2 \le i \le N$. We show $a = (a_1, \mu_2, \cdots \mu_N)$ is $\tau_\times$-very strongly atomic. We first check $a \cong a$. By the definition of τ_1-very strongly atomic, $a_1 \cong a_1$. Certainly as units, we have $\mu_i \cong \mu_i$ for each $i \ge 2$. Lastly, all of these are non-zero, so we may apply Lemma 5.1 to see that $a \cong a$. Suppose $a = (\lambda_1, \ldots, \lambda_N)(a_{1_1}, \ldots, a_{1_N}) \cdots (a_{k_1}, \ldots, a_{k_N})$ is a $\tau_\times$-factorization of a. We first note $a_{i_j} \in U(R_j)$ for all $j \ge 2$. As before, this means $a_1 = \lambda_1 a_{1_1} \cdots a_{k_1}$ is a τ_1 factorization of a τ_1-very strongly atomic element. Hence, $a_1 \cong a_{j_1}$ for some $1 \le j \le k$. By Lemma 5.1, we have $a \cong (a_{j_1}, \ldots, a_{j_N})$, and thus a is $\tau_\times$-very strongly atomic as desired. □

Lemma 5.5. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$. Then we have the following*:

(1) *If* $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ *is a* τ_i-U-α*-factorization of some non-unit* $a \in R_i$*, then* $a^{(i)} = \lambda^{(i)} a_1^{(i)} \cdots a_n^{(i)} \lceil b_1^{(i)} \cdots b_m^{(i)} \rceil$ *is a* $\tau_\times$*-U-*α*-factorization.*

(2) *Conversely, let* $a_{i_0} \in R_{i_0}$ *be a non-unit and* $\mu_i \in U(R_i)$ *for all* $i \neq i_0$*. Let*

$$(\mu_1, \mu_2, \ldots, \mu_{i_0-1}, a_{i_0}, \mu_{i_0+1}, \ldots, \mu_N) = (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \rceil$$

be a $\tau_\times$-U-α*-factorization. Then*

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} \cdots b_{_{i_0}} \right\rceil$$

is a τ_{i_0}-U-α*-factorization.*

Proof. (1) Let $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ be a τ_i-U-α-factorization of some non-unit $a \in R_i$. It is easy to see that

$$a^{(i)} = \lambda^{(i)} a_1^{(i)} \cdots a_n^{(i)} \left\lceil b_1^{(i)} \cdots b_m^{(i)} \right\rceil$$

is a $\tau_\times$-factorization. Furthermore, $b_j \neq 0$ for all $1 \leq j \leq m$ or else it would not be a τ_i-factorization. Hence, by Theorem 5.4, $b_j^{(i)}$ is $\tau_\times$-α for each $1 \leq j \leq m$. Thus, it suffices to show that we actually have a U-factorization.

Since $a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ is a U-factorization, we know $a_k(b_1 \cdots b_m) = (b_1 \cdots b_m)$ for all $1 \leq k \leq n$. In the other coordinates, we have $(1_{R_j}) = (1_{R_j})$ for all $j \neq i$. Hence, we apply Lemma 5.1 and see that this implies that $a_k^{(i)}(b_1^{(i)} \cdots b_m^{(i)}) = (b_1^{(i)} \cdots b_m^{(i)})$ for all $1 \leq k \leq n$. Similarly, we have $b_j(b_1 \cdots \widehat{b_j} \cdots b_m) \neq (b_1 \cdots \widehat{b_j} \cdots b_m)$ which implies $b_j^{(i)}(b_1^{(i)} \cdots \widehat{b_j^{(i)}} \cdots b_m^{(i)}) \neq (b_1^{(i)} \cdots \widehat{b_j^{(i)}} \cdots b_m^{(i)})$, so this is indeed a U-factorization.

(2) Let

$$(\mu_1, \mu_2, \ldots, \mu_{i_0-1}, a_{i_0}, \mu_{i_0+1} \ldots, \mu_N) = (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \rceil$$

be a $\tau_\times$-U-α-factorization. We note that $a_{j_i} \in U(R_i)$ for all $i \neq i_0$ and all $1 \leq j \leq n$ and $b_{j_i} \in U(R_i)$ for all $i \neq i_0$ and all $1 \leq j \leq m$ since they divide the unit μ_i. Next, every coordinate in the i_0 place must be a non-unit in R_{i_0} or else this factor would be a unit in R and therefore could not occur as a factor in a $\tau_\times$-factorization. This tells us that

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} \cdots b_{_{i_0}} \right\rceil$$

is a τ_{i_0}-factorization. Furthermore, (b_{k_i}) is assumed to be $\tau_\times$-α for all $1 \leq k \leq m$, and the other coordinates are units, so $b_{k_{i_0}}$ is τ_{i_0}-α for all $1 \leq k \leq m$ by Theorem 5.4. Again, we need only show that

$$a_{i_0} = \lambda_{i_0} a_{1_{i_0}} a_{2_{i_0}} \cdots a_{n_{i_0}} \left\lceil b_{1_{i_0}} b_{2_{i_0}} \cdots b_{m_{i_0}} \right\rceil$$

is a U-factorization. Since all the coordinates other than i_0 are units, we simply apply Lemma 5.1 and see that we indeed maintain a U-factorization. □

Theorem 5.6. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^\#$. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$. Then R is $\tau_\times$-U-α if and only if R_i is a τ_i-U-α for each $1 \leq i \leq N$.*

Proof. ($\Rightarrow$). Let $a \in R_{i_0}$ be a non-unit. Then $a^{(i_0)}$ is a non-unit in R and therefore has a $\tau_\times$-U-α-factorization. Furthermore, the only possible non-unit factors in this factorization must occur in the i_0th coordinate. Thus, as in Lemma 5.5 (2), we have found a τ_{i_0}-U-α-factorization of a by taking the product of the i_0th entries. This shows R_{i_0} is τ_{i_0}-U-α as desired.

($\Leftarrow$). Let $a = (a_i) \in R$ be a non-unit. For each non-unit $a_i \in R_i$, there is a τ_i-U-α-factorization of a_i, say

$$a_i = \lambda_i a_{i_1} \cdots a_{i_{n_i}} \left\lceil b_{i_1} \cdots b_{i_{m_i}} \right\rceil.$$

If $a_i \in U(R_i)$, then $a_i^{(i)} \in U(R)$, and we can simply collect these unit factors in the front, so we need not worry about these factors. This yields a $\tau_\times$-U-α-factorization

$$a = (a_i) = \prod_{i=1}^{n} \lambda_i^{(i)} a_{i_1}^{(i)} \cdots a_{i_{n_i}}^{(i)} \left\lceil \prod_{i=0}^{m} b_{i_1}^{(i)} \cdots b_{i_{m_i}}^{(i)} \right\rceil.$$

It is certainly a $\tau_\times$-factorization. Furthermore, $b_{j_k} \neq 0_j$ for $1 \leq j \leq m$ and $1 \leq k \leq m_j$, so $b_{j_k}^{(j)}$ is $\tau_\times$-α by Theorem 5.4. It is also clear from

Lemma 5.5 that this is a U-factorization, showing every non-unit in R has a $\tau_\times$-U-α-factorization. □

Theorem 5.7. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Let $\alpha \in \{$irreducible, strongly irreducible, m-irreducible, very strongly irreducible$\}$, and let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Then R is a $\tau_\times$-U-α-β-*df *ring if and only if R_i is τ_i-U-α-*df *ring for each $1 \leq i \leq N$.*

Proof. ($\Rightarrow$). Let $a \in R_{i_0}$ be a non-unit. Suppose there were an infinite number of τ_{i_0}-U-α essential divisors of a, say $\{b_j\}_{j=1}^{\infty}$, none of which are β. But then $\{b_j^{(i_0)}\}_{j=1}^{\infty}$ yields an infinite set of $\tau_\times$-U-α-divisors of $a^{(i_0)}$ by Lemma 5.5. Furthermore, none of them are β by Lemma 5.1.

($\Leftarrow$). Let $(a_i) \in R$ be a non-unit. We look at the collection of $\tau_\times$-U-α essential divisors of (a_i). Each must be of the form $(\lambda_1, \ldots, b_{i_0}, \ldots, \lambda_N)$ with $\lambda_i \in U(R_i)$ for each i, and with b_{i_0} τ_{i_0}-α for some $1 \leq i_0 \leq N$. But, then b_{i_0} is a τ_{i_0}-α essential divisor of a_{i_0}. For each i between 1 and N, R_i is a τ_i-U-α-β-df ring, so there can only be finitely many τ_i-α essential divisors of a_i up to β, say $N(a_i)$. If $a_i \in R_i$, then we can simply set $N(a_i) = 0$ since it is a unit and has no non-trivial τ_i-U-factorizations. Hence, there can be only

$$N((a_i)) := N(a_1) + N(a_2) + \cdots + N(a_N) = \sum_{i=1}^{N} N(a_i)$$

$\tau_\times$-α essential divisors of (a_i) up to β. This proves the claim. □

Corollary 5.8. *Let α and β be as in the theorem. Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Then R is a $\tau_\times$-U-α $\tau_\times$-U-α-β-*df *ring if and only if R_i is a τ_i-U-α τ_i-U-α-β-*df *ring for each $1 \leq i \leq N$.*

Proof. This is immediate from Theorems 5.6 and 5.7. □

Theorem 5.9. *Let $R = R_1 \times \cdots \times R_N$ for $N \in \mathbb{N}$ with τ_i a symmetric relation on $R_i^{\#}$. Then R is a $\tau_\times$-*U-BFR *if and only if R_i is a τ_i-*U-BFR *for every i.*

Proof. (⇒). Let $a \in R_{i_0}$ be a non-unit. Then $a^{(i_0)}$ is a non-unit in R and hence has a bound on the number of essential divisors in any $\tau_\times$-U-factorization, say $N_e(a^{(i_0)})$. We claim this also bounds the number of essential divisors in any τ_{i_0}-U-factorization of a. Suppose for a moment $a = a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil$ were a τ_{i_0}-U-factorization with $m > N_e(a^{(i)})$. But then

$$a = \lambda^{(i_0)} a_1^{(i_0)} \cdots a_n^{(i_0)} \left\lceil b_1^{(i_0)} \cdots b_m^{(i_0)} \right\rceil$$

is a $\tau_\times$-U-factorization with more essential divisors than is allowed, a contradiction.

(⇐). Let $a = (a_i) \in R$ be a non-unit. Let $B(a) = \max\{N_e(a_i)\}_{i=1}^N$, where $N_e(a_i)$ is the number of essential divisors in any τ_i-U-factorization of a_i, and for $a_i \in U(R_i)$, $N_e(a_i) = 0$. We claim that $B(a)N$ is a bound on the number of essential divisors in any $\tau_\times$-U-factorization of a. Let

$$(a_i) = (\lambda_i)(a_{1_i}) \cdots (a_{n_i}) \lceil (b_{1_i}) \cdots (b_{m_i}) \rceil$$

be a $\tau_\times$-U-factorization. We can decompose this factorization so that each factor has at most one non-unit entry as follows:

$$(a_i) = \prod_{i=1}^{N} \lambda_i^{(i)} a_{1_i}^{(i)} \cdots \prod_{i=1}^{N} a_{n_i}^{(i)} \prod_{i=1}^{N} b_{1_i}^{(i)} \cdots \prod_{i=1}^{N} b_{m_i}^{(i)}.$$

Some of these factors may indeed be units; however, by allowing a unit factor in the front of every τ-U-factorization, we simply combine all the units into one at the front and maintain a $\tau_\times$-factorization. We can always rearrange this to be a $\tau_\times$-U-factorization. Furthermore, since a_{j_i} is inessential, by Lemma 5.1, $a_{j_i}^{(i)}$ is inessential. Only some of the components of the essential divisors could become inessential; for instance, if one coordinate were a unit. At worst, when we decompose, $b_{j_i}^{(i)}$ remains an essential divisor for all $1 \leq j \leq m$ and for all $1 \leq i \leq N$. But then, the product of each of the ith coordinates gives a τ_i-U-factorization of a_i and thus is bounded by $N_e(a_i)$, so we have $m \leq N_e(a_i) \leq B(a)$, and therefore there are no more than $B(a)N$ essential divisors. Certainly the original factorization is no longer than the one we constructed through the decomposition, proving the claim and completing the proof. □

Theorem 5.10. *Let* $R = R_1 \times \cdots \times R_N$ *for* $N \in \mathbb{N}$ *with* τ_i *a symmetric relation on* $R_i^{\#}$. *Let* $\alpha \in \{$*irreducible, strongly irreducible,* m-*irreducible, very strongly irreducible*$\}$. *Then* R *is* $\tau_\times$-U-α-HFR *if and only if* R_i *is a* τ_i-U-α-HFR *for each* i.

Proof. ($\Rightarrow$). Let $a \in R_{i_0}$ be a non-unit. We know by Theorem 5.6 that $a^{(i_0)}$ is a non-unit in R and has a $\tau_\times$-U-α-factorization. Suppose there were τ_{i_0}-U-α-factorizations of a with different numbers of essential divisors, say:

$$a = \lambda a_1 \cdots a_n \lceil b_1 \cdots b_m \rceil = \mu c_1 \cdots c_{n'} \lceil d_1 \cdots d_{m'} \rceil$$

where $m \neq m'$. By Lemma 5.5, this yields two $\tau_\times$-U-α-factorizations:

$$\begin{aligned} a^{(i_0)} &= \lambda^{(i_0)} a_1^{(i_0)} \cdots a_n^{(i_0)} \left\lceil b_1^{(i_0)} \cdots b_n^{(i_0)} \right\rceil \\ &= \mu^{(i_0)} c_1^{(i_0)} \cdots c_{n'}^{(i_0)} \left\lceil d_1^{(i_0)} \cdots d_{n'}^{(i_0)} \right\rceil. \end{aligned}$$

This contradicts the hypothesis that R is a $\tau_\times$-U-α-HFR.

($\Leftarrow$). Let $(a_i) \in R$ be a non-unit. Suppose we have two $\tau_\times$-U-α factorizations

$$\begin{aligned} (a_i) &= (\lambda_i)(a_{1_i})(a_{2_i}) \cdots (a_{n_i}) \lceil (b_{1_i})(b_{2_i}) \cdots (b_{m_i}) \rceil \\ &= (\mu_i)(a'_{1_i})(a'_{2_i}) \cdots (a'_{n'_i}) \left\lceil (b'_{1_i})(b'_{2_i}) \cdots (b'_{m'_i}) \right\rceil. \end{aligned}$$

For each i_0, if a_{i_0} is a non-unit in R_{i_0}, then since each $\tau_\times$-α element can only have one coordinate which is not a unit, we can simply collect all the $\tau_\times$-divisors which have the i_0 coordinate a non-unit. This product forms a τ_{i_0}-U-α-factorization of a_{i_0}, and therefore the number of essential $\tau_\times$-factors with coordinate i_0 a non-unit must be the same in the two factorizations. This is true for each coordinate i_0, hence $m = m'$, as desired. $\square$

Theorem 5.11. *Let* $R = R_1 \times \cdots \times R_N$ *for* $N \in \mathbb{N}$ *with* τ_i *a symmetric relation on* $R_i^{\#}$. *Let* $\alpha \in \{$*irreducible, strongly irreducible,* m-*irreducible, very strongly irreducible*$\}$, *and let* $\beta \in \{$*associate, strongly associate*$\}$. *Then* R *is* $\tau_\times$-*U*-α-β-UFR *if and only if* R_i *is a* τ_i-U-α-β-UFR *for each* i.

Proof. We simply apply Lemma 5.1 to the proof of Theorem 5.10 to see that the factors can always be rearranged to match associates of the correct type. □

REFERENCES

1. A.G. Ağargün, D.D. Anderson and S. Valdez-Leon, *Unique factorization in commutative rings with zero divisors*, Comm. Alg. **27** (1999), 1967–1974.

2. ———, *Factorization in commutative rings with zero divisors*, III, Rocky Mountain J. Math. **31** (2001), 1–21.

3. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Alg. **69** (1990), 1–19.

4. D.D. Anderson, M. Axtell, S.J. Forman and J. Stickles, *When are associates unit multiples?*, Rocky Mountain J. Math. **34** (2004), 811–828.

5. D.D. Anderson and S. Chun, *Irreducible elements in commutative rings with zero-divisors*, Rocky Mountain J. Math. **37** (2011), 741–744.

6. D.D. Anderson and Andrea M. Frazier, *On a general theory of factorization in integral domains*, Rocky Mountain J. Math. **41** (2011), 663–705.

7. D.D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Alg. **159** (1993), 500–514.

8. D.D. Anderson and S. Valdes-Leon, *Factorization in commutative rings with zero divisors*, Rocky Mountain J. Math. **26** (1996), 439–480.

9. M. Axtell, *U-factorizations in commutative rings with zero-divisors*, Comm. Alg. **30** (2002), 1241–1255.

10. M. Axtell, S. Forman, N. Roersma and J. Stickles, *Properties of U-factorizations*, Inter. J. Comm. Rings **2** (2003), 83–99.

11. C.R. Fletcher, *Unique factorization rings*, Proc. Cambr. Philos. Soc. **65** (1969), 579–583.

12. ———, *The structure of unique factorization rings*, Proc. Cambr. Philos. Soc. **67** (1970), 535–540.

13. C.P. Mooney, *Generalized factorization in commutative rings with zero-divisors*, Houston J. Math., to appear.

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