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UPPER BOUNDS FOR SOLUTIONS OF AN EXPONENTIAL DIOPHANTINE EQUATION

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ABSTRACT. We consider the exponential Diophantine equation $a^x + b^y = c^z$ in positive integers x, y and z, where a, b and c are fixed pair-wise relatively prime positive integers greater than one. In this paper, we obtain several upper bounds for solutions x, y and z for which two of x, y and z are even. As their applications, we solve exponential Diophantine equations in which a, b and c are expressed as terms of linearly recurrence sequences.

1. Introduction. We consider the exponential Diophantine equation

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y and z, where a, b and c are fixed pair-wise relatively prime positive integers greater than one. It is not easy to solve (1.1), even if very particular values of a, b and c are given, or under the abc conjecture. In the study of equation (1.1), the method of coming to rational points of algebraic curves as used in the studies of the Fermat equation are not very effective because the exponents of (1.1) vary. By the theory of Diophantine approximation, we can obtain some general information on the solutions. Equation (1.1) can be regarded as a kind of unit equation. Schmidt's subspace theorem gives upper bounds for the number of general unit equations. In particular, equation (1.1) has only finitely many solutions. Also, by means of Baker's theory for linear forms in logarithms, we can obtain effectively computable upper bounds for the size of solutions of (1.1). We remark that these bounds

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are not very useful to determine the solutions of (1.1). Assuming the Tijdeman-Zagier conjecture on the generalized Fermat equation (cf., [**7,** Chapter 14], [**10,** Section B19]), we may conclude that equation (1.1) has no solutions with $\min\{x,y,z\} \geq 3$. The conjecture is also known as Beal's conjecture, and Beal has offered a prize of \$100,000 dollars (the money is held by the AMS and funds the Erdös lecture) for the solution (cf., [**13**]).

On the other hand, the study of determining solutions of (1.1) is more active than that of only estimating them. Originally this problem was considered for fixed values of a, b and c. Using various elementary methods in number theory (congruence, factorization in number fields, etc.), several authors determined complete solutions of (1.1) for small values of a, b and c. Almost all of the recent work concerns cases where a, b and c are expressed as terms of various recurrence sequences. The case of $a^p+b^q=c^r$ for some positive integers p, q and r has much interest. One of the famous unsolved problems is due to Jeśmanowicz ([**11**]). It states that equation (1.1) has the unique solution $x=y=z=2$ if $p=q=r=2$.

In this paper, we first obtain several upper bounds for solutions x, y and z of (1.1) for which two of x, y and z are even. For this, we use the theory of the ring of Gaussian integers and several p-adic calculations. One of our results (see (III-1) and (III-2) in Theorem 3.2 below) tells us that the estimate

$$\max\{x,y,z\} \leq \frac{4}{\log 2}\, \log\max\{a,b,c\}$$

holds for all solutions (x,y,z) of (1.1) for which x, y and z are all even, except for specific values of a, b, c, x, y, z. The above estimate is much smaller than any other known results. As applications of our results, we solve equations (1.1) in the case where a, b and c are Fibonacci numbers. Let $\{F_n\}_{n\geq 0}$ be Fibonacci numbers, defined by $F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$. Fibonacci numbers have the following elegant formulas:

$$F_n^2+F_{n+1}^2=F_{2n+1}, \quad F_n^2+F_{2n+2}=F_{n+2}^2$$

for $n \geq 0$ (cf., [**12,** page 79, Corollary 5.4]). It is worth stating that the formulas of these types seem not to be seen in other linearly recurrence sequences (see Remarks 5.4 and 5.10 below). In 2002, at a satellite meeting of the International Congress of Mathematics, Terai ([**17**]) proposed studying (1.1) in the case of $(a,b,c) = (F_n, F_{n+1}, F_{2n+1})$ with fixed $n \geq 3$, and he asked whether (1.1) has the unique solution $(x,y,z) = (2,2,1)$ or not. We solve his problem as follows.

Theorem F1. For each $n \geq 3$, the exponential Diophantine equation

$$F_n^x + F_{n+1}^y = F_{2n+1}^z$$

has the unique solution $(x,y,z) = (2,2,1)$ in positive integers x, y and z.

Also, we prove

Theorem F2. For each $n \geq 3$, the exponential Diophantine equation

$$F_n^x + F_{2n+2}^y = F_{n+2}^z$$

has the unique solution $(x,y,z) = (2,1,2)$ in positive integers x, y and z.

We can extend these results to more general linearly recurrence sequences (see Theorems P1 and P2 below).

In the next section, we quote preliminary results on Diophantine equations, the generalized Fermat equation and Catalan equation. In Section 3, we obtain several upper bounds for solutions x, y and z of (1.1) for which two of x, y and z are even. In Section 4, using the results proved in Section 3, we obtain several results related to the case of $a^2 + b^2 = c$ or $a^2 + b = c^2$. In the final section, we prove Theorems F1 and F2.

2. Lemmas. In this section, we quote several results on the generalized Fermat equation

$$S^p + T^q = U^r \tag{$*$}$$

in non-zero relatively prime integers S, T and U, where p, q and r are fixed positive integers greater than one. They are useful for reducing the divisibility properties of the solutions of (1.1).

Lemma 2.1. *Let $r \geq 2$ be a positive integer. If $(p,q,r) = (2,2,r)$, then all of the solutions of $(*)$ in relatively prime positive integers are given by*

$$S + T\sqrt{-1} = (k + l\sqrt{-1}\,)^r, \qquad U = k^2 + l^2,$$

where k and l are relatively prime positive integers of different parities with $k > l$.

Lemma 2.2. *If $(p,q,r) = (4,2,4)$, then $(*)$ has no solutions.*

Lemma 2.3 ([**5**] Theorem 3)**.** *Let $N \geq 2$ be a positive integer. If $(p,q,r) = (2N,2,4)$, then $(*)$ has no solutions with $Z \equiv 0 \pmod 2$.*

Lemma 2.4 ([**6**] Lemma 10)**.** *Let $N \geq 2$ be a positive integer. If $(p,q,r) = (2N,4,2)$, then $(*)$ has no solutions.*

Lemma 2.5 ([**7**] pages 489–490)**.** *Let $N \geq 6$ be a positive integer. If $(p,q,r) = (2,N,4)$, then $(*)$ has no solutions.*

Lemma 2.6 ([**1**])**.** *If $\{p,q,r\} = \{2,4,6\}$, then $(*)$ has no solutions.*

Lemma 2.7 ([**2**])**.** *Let N be a positive integer with $N \in \{4,5\}$. If $(p,q,r) = (3,3,N)$, then $(*)$ has no solutions.*

Lemma 2.8 ([**3**] Theorem 1.2)**.** *If $(p,q,r) = (2,5,4)$, then $(*)$ has no solutions other than $(S,T,U) = (\pm 122, -3, \pm 11)$, $(\pm 7, 2, \pm 3)$.*

Lemma 2.9 ([**16**])**.** *If $(p,q,r) = (3,4,5)$, then $(*)$ has no solutions.*

The following result is on the Catalan's equation.

Lemma 2.10 ([**14**])**.** *The equation*

$$X^U - Y^V = 1$$

has the unique solution $(X,Y,U,V) = (3,2,2,3)$ in positive integers $X,Y,U,V \geq 2$.

3. Estimates of solutions. We consider (1.1) in the case where a, b and c are relatively prime positive integers greater than one, one of which is even and the others are odd. In this section we obtain several upper bounds for solutions x, y and z of (1.1) in terms of a, b and c for the following cases:

(1) x, y and z are even.
(2) x, y are even and z is odd.
(3) x, z are even and y is odd.

For cases (1) and (2), we should consider the case where c is odd. Indeed, in the case where c is even, if (1.1) has a solution (x, y, z) for which x and y are even, then c^z is a sum of two squares of odd integers, so it is exactly divisible by 2, which implies that $z = 1$.

For a prime number p and a non-zero integer m, we denote $\mathrm{ord}_p(m)$ by the exact power of p in m. Also, we define the p-part of m by

$$m_{(p)} := p^{\mathrm{ord}_p(m)}.$$

For cases (1) and (3), we will use the following elementary fact on 2-adic calculations (cf., [**15,** page 11; P1.2]).

Lemma 3.1. *Let U and V be distinct relatively prime positive odd integers. Then, for any positive integer e, we have*

$$\mathrm{ord}_2\big(U^{2e} - V^{2e}\big) = \mathrm{ord}_2\big(U + (-1)^{(UV+1)/2} V\big) + \mathrm{ord}_2(e) + 1.$$

Theorem 3.2. *We consider the case where b is even. Let (x, y, z) be a solution of* (1.1). *Assume that x, y and z are even. We write $x = 2X$, $y = 2Y$ and $z = 2Z$, where X, Y and Z are positive integers. Then the following* (I), (II) *and* (III) *hold.*

(I) *If a is a prime power, then we have*

$$(X, Y, Z) \in \left\{ \left(\frac{\log(2b+1)}{2\log a}, 1, \frac{\log(b+1)}{\log c} \right), \right.$$
$$\left. \left(\frac{\log(2c-1)}{2\log a}, \frac{\log(c-1)}{\log b}, 1 \right) \right\}.$$

(II) *If b is a power of 2, then we have*

$$(a, c, X, Y, Z) = \left(\frac{b^{2i}}{4} - 1,\ \frac{b^{2i}}{4} + 1,\ 1,\ i,\ 1 \right)$$

for some $i \geq 1$.

(III) *We suppose that a is not a prime power, b is not a power of 2, and*

$$(X, Y, Z) \notin \left\{ \left(\frac{\log(2b+1)}{2\log a}, 1, \frac{\log(b+1)}{\log c} \right), \left(\frac{\log(2c-1)}{2\log a}, \frac{\log(c-1)}{\log b}, 1 \right) \right\},$$

$$(a, c, X, Y, Z) \neq \left(\frac{b^{2i}}{4} - 1,\ \frac{b^{2i}}{4} + 1,\ 1,\ i,\ 1 \right)$$

for any $i \geq 1$. Then the following (III-1) *and* (III-2) *hold.*

(III-1) *We have the upper estimates*

$$X < \frac{2Y\log(b/2) - \log 4}{\log a}, \quad Z < \frac{2Y\log(b/2) + \log 2}{\log c}.$$

(III-2) *Suppose that $Y > 1$. Then we have the upper estimate*

$$Y \leq \frac{\log\min\{a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a),\ 2\sqrt{c-1}\,\}}{\log b_{(2)}},$$

where $\mathrm{p}(a)$ is the least prime factor of a. Furthermore, if $c-1$ is not a square, $c-1$ has a prime factor not dividing b, or

$$Y \neq \frac{\mathrm{ord}_2(c-1) + 2}{2\,\mathrm{ord}_2(b)},$$

then we have

$$Y - \frac{\log Y}{\log b_{(p)}} < \frac{\log(2\log(b/2) + 1/2\log 2) + 1/2\log(c-4) - \log\log c}{\log b_{(p)}},$$

where p is the odd prime factor of b for which $b_{(p)}$ is minimum.

Proof. By Lemma 2.1, we can write

$$a^X = k^2 - l^2, \qquad b^Y = 2kl, \qquad c^Z = k^2 + l^2,$$

where k and l are relatively prime positive integers of different parities with $k > l$. Since $(k+l)(k-l) = a^X$ and $\gcd(k+l, k-l) = 1$, we can write

$$k + l = u^X, \qquad k - l = v^X,$$

where u and v are relatively prime positive integers with $uv = a$.

(I) Assume that $v = 1$. Then $k - l = 1$, so $c^Z - b^Y = (k-l)^2 = 1$. It follows from Lemma 2.10 that $Y = 1$, $Z = 1$ or $(b, c, Y, Z) = (2, 3, 3, 2)$. We remark that $k - l = 1$ if a is a prime power.

If $Y = 1$, then $c^Z = b + 1$, that is, $Z = [\log(b+1)]/\log c$. Since $a^{2X} = c^{2Z} - b^2 = 2b + 1$, we have $X = [\log(2b+1)]/2\log a$.

If $Z = 1$, then $b^Y = c - 1$, that is, $Y = [\log(c-1)]/\log b$. Since $a^{2X} = c^2 - b^{2Y} = 2c - 1$, we have $X = [\log(2c-1)]/2\log a$.

If $(b, c, Y, Z) = (2, 3, 3, 2)$, then $a^{2X} = 3^4 - 2^6 = 17$, which is absurd.

(II) We assume that $l = 1$. Then

$$k^2 - a^X = 1, \qquad c^Z - k^2 = 1.$$

Since a is odd, it follows from Lemma 2.10 that $X = Z = 1$. Hence, we see that $a = k^2 - 1 = b^{2Y}/4 - 1$ and $c = k^2 + 1 = b^{2Y}/4 + 1$. We remark that $l = 1$ if b is a power of 2.

(III) In what follows, we assume that $v > 1$ and $l > 1$, in particular, a is not a prime power and b is not a power of 2. Then $v \geq \mathrm{p}(a)$ (since v is a divisor of a). We claim that $l \geq 2^{Y-1}$. If l is even, then $2l$ is a Y-th power of a positive even integer since $(2l)k = b^Y$ and $\gcd(k, 2l) = 1$, in particular, $l \geq 2^{Y-1}$. If l is odd, then l is a Y-th power of a positive odd integer since $(2k)l = b^Y$ and $\gcd(2k, l) = 1$, in particular, $l \geq 3^Y$. Hence, the claim is proved.

(III-1) Since $l \geq 2^{Y-1}$ and

$$a^X < k^2 = b^{2Y}/(4l^2), \qquad c^Z < 2k^2 = b^{2Y}/(2l^2),$$

we can obtain the desired upper bounds for X and Z.

(III-2) Suppose that $Y > 1$. From Lemmas 2.3 and 2.4, we see that both X and Z are odd. We will find two upper bounds for Y. Since

$$4kl = (k+l)^2 - (k-l)^2 = u^{2X} - v^{2X},$$

we see from Lemma 3.1 that

$$\begin{aligned}\operatorname{ord}_2(b)\,Y &= \operatorname{ord}_2(b^Y) = \operatorname{ord}_2(2kl)\\ &= \operatorname{ord}_2\!\left(\frac{u^{2X}-v^{2X}}{2}\right) = \operatorname{ord}_2\big(u^{2X}-v^{2X}\big) - 1\\ &= \operatorname{ord}_2\big(u + (-1)^{(a+1)/2}v\big) \le \frac{\log\big(u+(-1)^{(a+1)/2}v\big)}{\log 2}.\end{aligned}$$

Since a is not a prime power and $v \ge \mathrm{p}(a)$, we see that

$$u + (-1)^{(a+1)/2}v = a/v + (-1)^{(a+1)/2}v \le a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a);$$

hence, we obtain the first upper bound for Y:

$$Y \le \frac{\log\big(a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a)\big)}{\log b_{(2)}}.$$

On the other hand, we rewrite $k^2 + l^2 = c^Z$ as

$$(k + l\sqrt{-1})(k - l\sqrt{-1}) = c^Z.$$

Since c is odd, we see that two factors on the left-hand side of the above equality are relatively prime in the ring of Gaussian integers. Hence, we can write

$$k + l\sqrt{-1} = (a_1 + b_1\sqrt{-1}\,)^Z$$

for some integers a_1 and b_1 satisfying $a_1^2 + b_1^2 = c$. We remark that $a_1 \not\equiv b_1 \pmod 2$. Since Z is odd, we see that

$$(3.1)\qquad \begin{cases} k = a_1\left(a_1^{Z-1} - \dbinom{Z}{Z-2} a_1^{Z-3}\, b_1^2 + \cdots \right.\\ \qquad\qquad \left. \pm \dbinom{Z}{3} a_1^2\, b_1^{Z-3} \pm Z b_1^{Z-1}\right), \\ l = b_1\left(Z a_1^{Z-1} - \dbinom{Z}{Z-3} a_1^{Z-3}\, b_1^2 + \cdots \right.\\ \qquad\qquad \left. \pm \dbinom{Z}{2} a_1^2\, b_1^{Z-3} \pm b_1^{Z-1}\right). \end{cases}$$

It is clear that a_1 divides k and b_1 divides l. In particular, a_1 and b_1 are relatively prime non-zero integers. Since $a_1 \not\equiv b_1 \pmod 2$ and z is odd, we see from (3.1) that k/a_1 and l/b_1 are odd integers. Hence, we find that

$$\mathrm{ord}_2(b)\, Y = \mathrm{ord}_2(b^Y) = \mathrm{ord}_2(2kl) = \mathrm{ord}_2(2a_1b_1).$$

We only consider the case where a_1 is odd (the case where b_1 is odd is similar). Since b_1 is even and $|b_1| = \sqrt{c - a_1^2} \le \sqrt{c-1}$, we obtain the second upper bound for Y:

$$Y = \frac{\mathrm{ord}_2(2b_1)}{\mathrm{ord}_2(b)} \le \frac{\log(2|b_1|)}{\mathrm{ord}_2(b)\log 2} \le \frac{\log(2\sqrt{c-1}\,)}{\log b_{(2)}}.$$

In what follows, we only consider the case where a_1 is odd (the case where b_1 is odd is similar). If $a_1 = \pm 1$, then $c = a_1^2 + b_1^2 = 1 + b_1^2$; in particular, any prime factor of $c-1$ $(= b_1^2)$ divides b. Further, we see that

$$Y = \frac{\mathrm{ord}_2(2b_1)}{\mathrm{ord}_2(b)} = \frac{\mathrm{ord}_2(4b_1^2)}{2\,\mathrm{ord}_2(b)} = \frac{\mathrm{ord}_2(4(c-1))}{2\,\mathrm{ord}_2(b)} = \frac{\mathrm{ord}_2(c-1)+2}{2\,\mathrm{ord}_2(b)}.$$

Finally, we suppose that $c-1$ is not a square, or $c-1$ has a prime factor not dividing b, or

$$Y \ne \frac{\mathrm{ord}_2(c-1)+2}{2\,\mathrm{ord}_2(b)}.$$

Then, by the above remarks, we see that a_1 has an odd prime factor, say p. From the first equation in (3.1), we observe that p divides k, but p does not divide l since $\gcd(k,l) = 1$. We claim that

$$\mathrm{ord}_p(k)\, Y = \mathrm{ord}_p(a_1) + \mathrm{ord}_p(Z).$$

From the first equation in (3.1), it suffices to show that, if Z is divisible by p, then

$$\mathrm{ord}_p\left(\binom{Z}{i} a_1^{i-1}\right) > \mathrm{ord}_p(Z)$$

for $i = 3, 5, \ldots, Z$. Since $p \ge 3$, $i \ge 3$ and

$$\mathrm{ord}_p(i!) = \sum_{j=1}^{\infty} \left\lfloor \frac{i}{p^j} \right\rfloor < \sum_{j=1}^{\infty} \frac{i}{p^j} = \frac{i}{p-1},$$

where $\lfloor\cdot\rfloor$ is the floor function, we see that

$$\begin{aligned}
\operatorname{ord}_p\left(\binom{Z}{i} a_1^{i-1}\right) &= \operatorname{ord}_p\left(\frac{Z(Z-1)\cdots(Z-i+1)}{i!}\right) \\
&\quad + \operatorname{ord}_p\bigl(a_1^{i-1}\bigr) \\
&= \operatorname{ord}_p\bigl(Z(Z-1)\cdots(Z-i+1)\bigr) - \operatorname{ord}_p(i!) \\
&\quad + (i-1)\operatorname{ord}_p(a_1) \\
&> \operatorname{ord}_p(Z) - \frac{i}{p-1} + i - 1 \\
&= \operatorname{ord}_p(Z) + \left(\frac{p-2}{p-1}\right) i - 1 \\
&\geq \operatorname{ord}_p(Z) + \frac{1}{2} \\
&> \operatorname{ord}_p(Z).
\end{aligned}$$

Hence, the claim is proved. Since $b^Y = 2kl$ and $\gcd(2l, p) = 1$, we see that

$$\operatorname{ord}_p(b)\, Y = \operatorname{ord}_p(b^Y) = \operatorname{ord}_p(k) = \operatorname{ord}_p(a_1 Z).$$

Hence,

$$Y \leq \frac{\log(|a_1| Z)}{\log b_{(p)}}.$$

Since $|a_1| = \sqrt{c - b_1^2} \leq \sqrt{c-4}$, it follows from (III-1) that

$$\begin{aligned}
Y &\leq \frac{1/2\log(c-4) + \log Z}{\log b_{(p)}} \\
&< \frac{1/2\log(c-4) + \log\left(\frac{2Y\log(b/2) + \log 2}{\log c}\right)}{\log b_{(p)}} \\
&\leq \frac{\log Y}{\log b_{(p)}} + \frac{\log\bigl(2\log(b/2) + 1/2\log 2\bigr) + 1/2\log(c-4) - \log\log c}{\log b_{(p)}}.
\end{aligned}$$

This gives the desired conclusion. □

Remark 3.3. Under the assumption of Theorem 3.2, we cannot generally deduce upper bounds for Y such as $Y \leq \mathcal{C}\log b$, where $\mathcal{C}$

is an absolute constant. Indeed, the identity

$$\left(2^{2q-2} - 1\right)^2 + 2^{2q} = \left(2^{2q-2} + 1\right)^2$$

holds for $q \geq 2$.

Theorem 3.4. *We consider the case where b is even. Put*

$$M = \max\left\{ \frac{2\log a}{\log \alpha},\ \frac{\log b}{\log b_{(2)}} \right\},$$

$$m = \min\left\{ \frac{2\log a}{\log \alpha},\ \frac{\log b}{\log b_{(2)}} \right\},$$

where α is the minimum of $a_{(p)}$ when p runs over the prime factors of a. Let (x, y, z) be a solution of (1.1). Assume that x, y are even and z is odd. We write $x = 2X$ and $y = 2Y$, where X and Y are positive integers. Then we have the upper estimate

$$Y \leq \frac{\log(c-1)}{2\log b_{(2)}}.$$

Further, we suppose that $c - 1$ is not a square, $c - 1$ has a prime factor not dividing b, or the inequality

$$\frac{\log b}{\mathrm{ord}_2(b)} < \frac{\log(c-1)}{\mathrm{ord}_2(c-1)},$$

holds, or

$$Y \neq \frac{\mathrm{ord}_2(c-1)}{2\,\mathrm{ord}_2(b)}.$$

Then we may conclude that only one of the following estimates (I) *and* (II) *holds.*

(I)

$$X \leq \frac{\log(c-4)}{\log \alpha},\ z \leq \min\left\{ \frac{\log(c-1)}{\log c} M + \frac{1}{(c-1)^{M-m}\log c},\ \sqrt{c-4} \right\},$$

(II)

$$X \leq \frac{2\log z}{\log \alpha},\ \frac{z}{\log z} < \frac{2M}{\log c} + \frac{2}{(c-1)^{M-m}\log c\,\log(c-1)},\ z \geq \sqrt{c-1}.$$

Estimate (I) *is also valid if the inequality*

$$M < \frac{\sqrt{c-1}\,\log c}{\log(c-1)} - \frac{1}{(c-1)^{M-m}\log(c-1)}$$

holds.

Proof. Since $c^z = a^{2X} + b^{2Y}$ is a sum of two powers of relatively prime integers, we may assume that c does not have any prime factors congruent to 3 modulo 4, particularly, $c \equiv 1 \pmod 4$.

We rewrite (1.1) as

$$(a^X + b^Y\sqrt{-1}\,)(a^X - b^Y\sqrt{-1}\,) = c^z.$$

Since c is odd, we see that two factors on the left-hand side of the above equality are relatively prime in the ring of Gaussian integers. Hence, we can write

$$a^X + b^Y\sqrt{-1} = (a_2 + b_2\sqrt{-1}\,)^z$$

for some integers a_2 and b_2 satisfying $a_2^2 + b_2^2 = c$. We remark that $a_2 \not\equiv b_2 \pmod 2$. Since z is odd, we see that

$$\text{(3.2)}\qquad \left\{\begin{aligned} a^X &= a_2\left(a_2^{z-1} - \binom{z}{z-2}a_2^{z-3}b_2^2 + \cdots \right.\\ &\qquad \left. \pm \binom{z}{3}a_2^2 b_2^{z-3} \pm z b_2^{z-1}\right),\\ b^Y &= b_2\left(z a_2^{z-1} - \binom{z}{z-3}a_2^{z-3}b_2^2 + \cdots \right.\\ &\qquad \left. \pm \binom{z}{2}a_2^2 b_2^{z-3} \pm b_2^{z-1}\right). \end{aligned}\right.$$

It is clear that a_2 divides a^X and b_2 divides b^Y. In particular, a_2 and b_2 are relatively prime non-zero integers. Then a_2 is odd since a is odd, so b_2 is even. We can observe from the second equation in (3.2) that $b^Y/\,b_2$ is an odd integer, in particular,

$$Y = \frac{\mathrm{ord}_2(b_2)}{\mathrm{ord}_2(b)}.$$

Since $|b_2| = \sqrt{c - a_2^2} \le \sqrt{c-1}$, we obtain

$$Y \le \frac{\log(c-1)}{2\log b_{(2)}}.$$

We will consider the cases $|a_2| = 1$ and $|a_2| > 1$ separately. First, we assume that $|a_2| = 1$. Then $c = a_2^2 + b_2^2 = 1 + b_2^2$, in particular, any prime factor of $c - 1$ ($= b_2^2$) divides b. Further, we see that $b^{2Y}/(c-1) = (b^Y/b_2)^2$ is a positive odd integer, in particular,

$$\frac{\log(c-1)}{2\log b} \le Y = \frac{\mathrm{ord}_2(c-1)}{2\,\mathrm{ord}_2(b)}.$$

We suppose that $c - 1$ is not a square, $c - 1$ has a prime factor not dividing b, or the inequality

$$\frac{\log b}{\mathrm{ord}_2(b)} < \frac{\log(c-1)}{\mathrm{ord}_2(c-1)},$$

holds, or

$$Y \neq \frac{\mathrm{ord}_2(c-1)}{2\,\mathrm{ord}_2(b)}.$$

Then, by the preceding remarks, we see that $|a_2| > 1$. We remark that $c \neq 5$, so $c \ge 13$ (since c does not have any prime factors congruent to 3 modulo 4). Then a_2 has an odd prime factor, say p. From the first equation in (3.2), we see that p divides a, but p does not divide b_2 since $\gcd(a_2, b_2) = 1$. Similarly to an observation in the proof of Theorem 3.2, we see from the first equation in (3.2) that

$$\mathrm{ord}_p(a)\,X = \mathrm{ord}_p(a_2) + \mathrm{ord}_p(z).$$

Then, since $|a_2| \le \sqrt{c - b_2^2} \le \sqrt{c-4}$, we see that

$$\begin{aligned}\mathrm{ord}_p(a)\,X &\le \frac{\log|a_2|}{\log p} + \frac{\log z}{\log p}\\ &\le \frac{2\log\max\{|a_2|,\, z\}}{\log p}\\ &\le \frac{2\log\max\{z, \sqrt{c-4}\}}{\log p},\end{aligned}$$

so

$$X \le \frac{2\log\max\{z, \sqrt{c-4}\}}{\log a_{(p)}}.$$

We put

$$a' = a^{2/\log a_{(p)}}, \qquad b' = b^{1/\log b_{(2)}}.$$

We remark that $M = \log \max\{a', b'\}$ and $m = \log \min\{a', b'\}$.

We assume that $z \le \sqrt{c-4}$. Then we obtain

$$X \le \frac{\log(c-4)}{\log a_{(p)}}.$$

Since

$$\begin{aligned} c^z = a^{2X} + b^{2Y} &< a'^{\,\log(c-1)} + b'^{\,\log(c-1)} \\ &= \max\{a', b'\}^{\log(c-1)} \left(1 + \left(\frac{\min\{a', b'\}}{\max\{a', b'\}}\right)^{\log(c-1)}\right) \\ &= \max\{a', b'\}^{\log(c-1)} \left(1 + \frac{1}{(c-1)^{M-m}}\right), \end{aligned}$$

we see that

$$\begin{aligned} (\log c)\, z &< M \log(c-1) + \log\left(1 + \frac{1}{(c-1)^{M-m}}\right) \\ &< M \log(c-1) + \frac{1}{(c-1)^{M-m}}, \end{aligned}$$

so we obtain

$$z < \frac{\log(c-1)}{\log c} M + \frac{1}{(c-1)^{M-m} \log c}.$$

Hence, case (I) holds.

Finally, we assume that $z > \sqrt{c-4}$. We remark that $z \ge \sqrt{c-1}$. In fact, if $\sqrt{c-4} < z < \sqrt{c-1}$, then $1 < c - z^2 < 4$. But this does not hold since $c \equiv 1 \pmod 4$ and z is odd. Hence, we have

$$X \le \frac{2 \log z}{\log a_{(p)}}, \qquad Y \le \frac{\log(c-1)}{2 \log b_{(2)}} \le \frac{\log z}{\log b_{(2)}}.$$

Since

$$c^z = a^{2X} + b^{2Y} \le a'^{\,2\log z} + b'^{\,2\log z},$$

we can observe that the above yields

$$z \le \frac{2M}{\log c} \log z + \frac{1}{z^{2(M-m)} \log c}.$$

Since $z \geq \sqrt{c-1}$, we obtain

$$\begin{aligned}\frac{z}{\log z} &\leq \frac{2M}{\log c} + \frac{1}{(\log c)\, z^{2(M-m)} \log z} \\ &\leq \frac{2M}{\log c} + \frac{2}{(c-1)^{M-m} \log c \log(c-1)}.\end{aligned}$$

Hence, case (II) holds. Furthermore, since $z \geq \sqrt{c-1} \geq 2\sqrt{3} > \exp(1)$, it follows that

$$\frac{2\sqrt{c-1}}{\log(c-1)} = \frac{\sqrt{c-1}}{\log\sqrt{c-1}} \leq \frac{2M}{\log c} + \frac{2}{(c-1)^{M-m} \log c \log(c-1)},$$

so

$$M \geq \frac{\sqrt{c-1}\log c}{\log(c-1)} - \frac{1}{(c-1)^{M-m}\log(c-1)}.$$

□

Theorem 3.5. *We consider the case where b is odd and $b \geq 5$. Let (x, y, z) be a solution of (1.1). Assume that x and z are even and y is odd. We write $x = 2X$ and $z = 2Z$, where X and Z are positive integers. Suppose that $(a, b, c, X, y, Z) \neq (2, 17, 3, 3, 1, 2)$. Then the following* (I) *and* (II) *hold.*

(I) *If b is a prime power, then we have*

$$(X, y, Z) \in \left\{ \left(1, \frac{\log(2a+1)}{\log b}, \frac{\log(a+1)}{\log c}\right), \left(\frac{\log(c-1)}{\log a}, \frac{\log(2c-1)}{\log b}, 1\right) \right\}.$$

(II) *If b is not a prime power, and*

$$(X, y, Z) \notin \left\{ \left(1, \frac{\log(2a+1)}{\log b}, \frac{\log(a+1)}{\log c}\right), \left(\frac{\log(c-1)}{\log a}, \frac{\log(2c-1)}{\log b}, 1\right) \right\},$$

then we have

$$X < \frac{y \log(b/\mathrm{p}(b)) - \log 2}{\log a},$$

$$Z < \frac{\log(b/\mathrm{p}(b))}{\log c}\, y,$$
$$y \le \frac{\log a}{\log\bigl(\sqrt{b+1}-1\bigr)}\, X + 1,$$

where $\mathrm{p}(b)$ *is the least prime factor of* b*. Furthermore, we have*

$$X \le \frac{\log\bigl(b/\mathrm{p}(b) + (-1)^{(b+1)/2}\mathrm{p}(b)\bigr) - \log 2}{\log a_{(2)}} \quad \textit{if } a \textit{ is even},$$
$$Z \le \frac{\log\bigl(b/\mathrm{p}(b) + (-1)^{\frac{b+1}{2}}\mathrm{p}(b)\bigr) - \log 2}{\log c_{(2)}} \quad \textit{if } c \textit{ is even}.$$

Proof. From (1.1), we define positive odd integers D and E as follows:
$$b^y = DE,$$
where $D = c^Z + a^X$ and $E = c^Z - a^X$. It is easy to see that $\gcd(D,E)=1$. Hence, we can write
$$D = s^y, \qquad E = t^y,$$
where s and t are relatively prime positive integers with $st = b$. Then
$$s^y + t^y = 2c^Z, \qquad s^y - t^y = 2a^X.$$
We will consider the cases $t = 1$ and $t > 1$ separately.

(I) If $t = 1$, then $c^Z - a^X = E = 1$. Since $(a,b,c,X,y,Z) \ne (2,17,3,3,1,2)$, we see from Lemma 2.10 that $X = 1$ or $Z = 1$. Similarly to the proof of (I) in Theorem 3.2, we obtain the desired conclusion. We remark that $t = 1$ if b is a prime power.

(II) We assume that $t > 1$. Then b is not a prime power and $t \ge \mathrm{p}(b)$ (since t is a divisor of b). Since $a^X < D/2$, $c^Z < D$ and $D = b^y/E$, $E \ge \mathrm{p}(b)^y$, we can obtain the desired upper bounds for X and Z.

We claim that $s \ge \sqrt{b+1}+1$. Indeed, since $s \ge t+2$ and $st = b$, we find that $s^2 \ge b + 2s$, so $(s-1)^2 \ge b+1$. Hence,

$$\begin{aligned}2a^X &= s^y - t^y = s^y - \left(\frac{b}{s}\right)^y \\ &\ge \bigl(\sqrt{b+1}+1\bigr)^y - \bigl(\sqrt{b+1}-1\bigr)^y \ge 2y\bigl(\sqrt{b+1}-1\bigr)^{y-1},\end{aligned}$$

so

$$\left(\sqrt{b+1}-1\right)^{y-1} \le a^X.$$

Since $b \ge 5$, we have

$$y \le \frac{\log a}{\log\left(\sqrt{b+1}-1\right)}\, X + 1.$$

Since $4a^X c^Z = s^{2y} - t^{2y}$ and

$$s + (-1)^{(b+1)/2} t = b/t + (-1)^{(b+1)/2} t \le b/\mathrm{p}(b) + (-1)^{(b+1)/2}\mathrm{p}(b),$$

we see from Lemma 3.1 that

$$\begin{aligned}
\mathrm{ord}_2(a)\, X + \mathrm{ord}_2(c)\, Z &= \mathrm{ord}_2(a^X c^Z) \\
&= \mathrm{ord}_2(s^{2y} - t^{2y}) - 2 \\
&= \mathrm{ord}_2\left(s + (-1)^{(b+1)/2} t\right) - 1 \\
&\le \frac{\log\left(s + (-1)^{(b+1)/2} t\right)}{\log 2} - 1 \\
&\le \frac{\log\left(b/\mathrm{p}(b) + (-1)^{(b+1)/2}\mathrm{p}(b)\right) - \log 2}{\log 2}.
\end{aligned}$$

This gives the desired conclusions. □

In a similar manner to Theorems 3.2-3.5, we can also prove the following result. Actually, we can further conclude that $y = 1$ if we use the result in [**8**].

Proposition 3.6. *We consider the case where b is even. Let (x, y, z) be a solution of* (1.1)*. Assume that x and z are divisible by* 4*. We write $x = 4X$ and $z = 4Z$, where X and Z are positive integers. Then we have*

$$y < \frac{\log b}{\log b_{(2)}}, \qquad X < \frac{\log(b/b_{(2)})}{2\log a}\, y, \qquad Z < \frac{\log(2b/b_{(2)})}{2\log c}\, y.$$

Proof. By Lemma 2.2, we may assume that y is odd. From (1.1), we define positive even integers D and E as follows:

$$b^y = DE,$$

where $D = c^{2Z} + a^{2X}$ and $E = c^{2Z} - a^{2X}$. It is easy to see that $\gcd(D, E) = 2$, and that D is exactly divisible by 2 since it is a sum of two squares of odd integers. Hence, we can write

$$D = 2s^y, \qquad E = 2^{\beta y-1}t^y,$$

for some relatively prime positive odd integers s and t satisfying $2^\beta st = b$, where $\beta = \mathrm{ord}_2(b) \geq 1$. We can easily show that $s \geq 5$.

Since $a^{2X} < D/2$, $c^{2Z} < D$ and $D \leq b^y/2^{\beta y-1}$, we can obtain the desired upper bounds for X and Z.

We claim that $\beta \geq 2$ or $t \geq 3$. Indeed, if $\beta = 1$ and $t = 1$, then $2^{y-1} = (c^Z + a^X)(c^Z - a^X)$, so $c^Z + a^X = 2^{y-2}$. This implies that $2s^y = D < (c^Z + a^X)^2 = 2^{2y-4}$, so $s < 4$. This is absurd.

We rewrite $c^{2Z} + a^{2X} = 2s^y$ as

$$(c^Z + a^X\sqrt{-1}\,)(c^Z - a^X\sqrt{-1}\,) = (1+\sqrt{-1}\,)(1-\sqrt{-1}\,)s^y.$$

It is easy to see that two factors on the left-hand side of the above equality are relatively prime in the ring of Gaussian integers. Hence, we can write

$$c^Z + a^X\sqrt{-1} = (1+\varepsilon\sqrt{-1}\,)(d_1 + e_1\sqrt{-1}\,)^y$$

for some integers d_1 and e_1 satisfying $d_1^2 + e_1^2 = s$, where $\varepsilon = \pm 1$. Note that $d_1 \not\equiv e_1 \pmod 2$. Let I and J be the real part and the imaginary part of $(d_1 + e_1\sqrt{-1}\,)^y$, respectively, that is,

$$\begin{aligned} I &= d_1\left(d_1^{y-1} - \binom{y}{2}d_1^{y-3}e_1^2 + \cdots \pm y e_1^{y-1}\right), \\ J &= e_1\left(y d_1^{y-1} - \binom{y}{3}d_1^{y-3}e_1^2 + \cdots \pm e_1^{y-1}\right). \end{aligned}$$

Then $a^X = \varepsilon I + J$ and $c^Z = I - \varepsilon J$. Hence,

$$2^{\beta y-1}\,t^y = E = (c^Z + \varepsilon a^X)(c^Z - \varepsilon a^X) = -4\varepsilon IJ.$$

Since y is odd and $d_1 \not\equiv e_1 \pmod 2$, we see that I/d_1 and J/e_1 are odd integers. It follows that

$$\begin{aligned} \beta y &= \mathrm{ord}_2(4\varepsilon IJ) + 1 = \mathrm{ord}_2(4d_1e_1) + 1 \\ &= \mathrm{ord}_2(d_1e_1) + 3 \leq \frac{\log|d_1e_1|}{\log 2} + 3 \end{aligned}$$

$$
\begin{aligned}
&< \frac{\log\bigl(b/(\,2^{\beta+1}\,t)\bigr)}{\log 2} + 3 = \frac{\log b}{\log 2} - \left(\beta + \frac{\log t}{\log 2}\right) + 2 \\
&\le \frac{\log b}{\log 2},
\end{aligned}
$$

where we used the fact that $2|d_1 e_1| < d_1^2 + e_1^2 = s = b/(2^\beta t)$, and that $\beta \ge 2$ or $t \ge 3$. $\square$

4. Applications. In this section, we apply Theorems 3.2–3.5 to the case of $a^2+b^2 = c$ or $a^2+b = c^2$. As a result, we obtain several sufficient conditions for which equation (1.1), with divisibility properties, has a unique solution. We begin by showing a basic, actually important, lemma.

Lemma 4.1. *Let A, B and C be positive integers greater than one such that $A + B = C$. Let x, y and z be positive integers satisfying $A^x + B^y = C^z$. If $z = 1$ or $\max\{x, y\} \le z$, then $x = y = z = 1$.*

Proof. It is clear that $x = y = 1$ if $z = 1$. If $\max\{x, y\} \le z$, then

$$A^z + B^z \le (A + B)^z = C^z = A^x + B^y \le A^z + B^z.$$

This gives that $A^z + B^z = (A + B)^z$, so $z = 1$; hence, $x = y = 1$.

First, we consider the case of $a^2 + b^2 = c$. It is easy to see that the inequality

$$M < \frac{\sqrt{c-1}\,\log c}{\log(c-1)} - \frac{1}{(c-1)^{M-m}\,\log(c-1)},$$

which appears at the end of the statement of Theorem 3.4, holds if $a^2 + b^2 = c$. $\square$

In what follows, we denote the ceiling function by $\lceil\cdot\rceil$. Using Theorems 3.2 and 3.4, we prove the following result.

Proposition 4.2. *Let a, b and c be pair-wise relatively prime positive integers greater than one such that $a^2 + b^2 = c$ and b is even. Assume that $c - 1$ is not a square, or $c - 1$ has a prime factor not dividing b, or the inequality*

$$\frac{\log b}{\mathrm{ord}_2(b)} < \frac{\log(c-1)}{\mathrm{ord}_2(c-1)}$$

holds. We further assume at least one of the following conditions (i)–(iv) :

(i) $$a > b, \quad \left\lceil \frac{\log(7a^2)}{2\log b} \right\rceil \geq \frac{\log\bigl(1+(a/b)^2\bigr)}{2\log b_{(2)}}.$$

(ii) *a is a prime power, and the inequalities*

$$a < b, \quad \left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil \geq \frac{\log\bigl(1+(b/a)^2\bigr)}{\log a}$$

hold.

(iii) *a is not a prime power and the inequalities*

$$a < b,$$

$$\left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil$$
$$\geq \max\left\{ \frac{2\log\bigl(a/\mathrm{p}(a)+(-1)^{(a+1)/2}\mathrm{p}(a)\bigr)\log(b/2)}{\log a\ \log b_{(2)}\log c}, \frac{1}{\log \alpha} \right\} \log\bigl(1+(b/a)^2\bigr)$$

hold, where $\mathrm{p}(a)$ *is the least prime factor of a and α is the minimum of $a_{(p)}$ when p runs over the prime factors of a.*

(iv) $$b \geq \sqrt{2}a^3.$$

Then the exponential Diophantine equation

$$a^{2X} + b^{2Y} = c^z$$

has the unique solution $X = Y = z = 1$ in positive integers X, Y and z.

Proof. Let (X, Y, z) be a solution of the equation

$$a^{2X} + b^{2Y} = c^z \tag{4.1}$$

where X, Y, z are positive integers. Since $\max\{a^{2X}, b^{2Y}\} < c^z$, we see that

$$X - z < X - \frac{2\log a}{\log c} X = \frac{\log(c/a^2)}{\log c} X = \frac{\log\bigl(1+(b/a)^2\bigr)}{\log c} X,$$

$$Y - z < Y - \frac{2\log b}{\log c}\,Y = \frac{\log(c/b^2)}{\log c}\,Y = \frac{\log\bigl(1+(a/b)^2\bigr)}{\log c}\,Y.$$

We will obtain upper bounds for X and Y by using Theorems 3.2 and 3.4.

We claim that

$$(X, Y, z) \neq \left(\frac{\log(2c-1)}{2\log a}, \frac{\log(c-1)}{\log b}, 2\right).$$

Suppose that $a^{2X} = 2c-1$, $b^Y = c-1$ and $z = 2$. Then

$$a^2 + b^2 = c = \frac{a^{2X}+1}{2} = b^Y + 1.$$

From this, we easily observe that $X > 1$ and $Y > 2$. If $a \geq b$, then the above implies that $4a^2 - 1 \geq a^{2X}$, which implies $X = 1$. If $b \geq a$, then the above implies that $2b^2 - 1 \geq b^Y$, which implies $Y \leq 2$. This is a contradiction. The claim is proved.

If

$$(X, Y, z) = \left(\frac{\log(2b+1)}{2\log a}, 1, \frac{2\log(b+1)}{\log c}\right),$$

then $a^{2X} = 2b+1$, $Y = 1$ and $c^z = b^2 + 2b + 1$. Then

$$(a^2 + b^2)^z = c^z = b^2 + 2b + 1.$$

From this, we easily observe that $z = 1$. Hence, $a^2 = 2b+1$, so $X = 1$.

First we assume that a is a prime power. By the above remarks, we see from (I) in Theorem 3.2 that z has to be odd. Then, by Theorem 3.4, we have

$$X \leq \frac{\log(c-4)}{\log a}, \quad Y \leq \frac{\log(c-1)}{2\log b_{(2)}}.$$

Next we assume that a is not a prime power. If z is odd, then we see from Theorem 3.4 that

$$X \leq \frac{\log(c-4)}{\log \alpha}, \qquad Y \leq \frac{\log(c-1)}{2\log b_{(2)}}.$$

Since

$$a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a) \leq a/3 + 3 < a < \sqrt{c-1},$$

we see from (III-1) in Theorem 3.2 that, if z is even, then we have

$$X < \frac{2\log(b/2)}{\log a}\left(\leq \frac{\log(c-4)}{\log \alpha}\right), \quad Y = 1,$$

or

$$X < \frac{2\log\bigl(a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a)\bigr)\log(b/2)}{\log a \, \log b_{(2)}},$$
$$Y \leq \frac{\log\bigl(a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a)\bigr)}{\log b_{(2)}}.$$

Therefore, we may conclude that

$$X \leq \max\left\{\frac{2\log\bigl(a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a)\bigr)\log(b/2)}{\log a \, \log b_{(2)}}, \ \frac{\log(c-4)}{\log \alpha}\right\},$$
$$Y \leq \frac{\log(c-1)}{2\log b_{(2)}}.$$

On the other hand, taking equation (4.1) modulo b^2, we find that

$$a^{2X} \equiv a^{2z} \pmod{b^2}.$$

Since $\gcd(a,b) = 1$, it follows that

$$a^{2|X-z|} \equiv 1 \pmod{b^2}.$$

Suppose that $X - z > 0$. Then $a^{2(X-z)} \geq 1 + 2b^2$, so

$$X - z \geq \left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil.$$

Therefore, we see that if a is a prime power. Then

$$\left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil < \frac{\log\bigl(1 + (b/a)^2\bigr)}{\log a},$$

and that if a is not a prime power, then

$$\left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil < \max\left\{\frac{2\log\bigl(a/\mathrm{p}(a) + (-1)^{\frac{a+1}{2}}\mathrm{p}(a)\bigr)\log(b/2)}{\log a \, \log b_{(2)} \log c}, \right.$$
$$\left. \frac{1}{\log \alpha}\right\} \log\bigl(1 + (b/a)^2\bigr).$$

We can easily observe that both the above two inequalities do not hold if $a > b$ since the values of their right-hand side are less than 1. We remark that both the values of their right-hand sides are greater than or equal to $\rho := \log(1 + (b/a)^2)/\log a$. Furthermore, if $b \geq \sqrt{2}a^3$, then

$$\rho > \frac{2\log(b/a)}{\log a} = \frac{\log(b^4/a^4)}{2\log a} \geq \frac{\log(2a^2b^2)}{2\log a}$$
$$= \frac{\log(2b^2)}{2\log a} + 1 > \left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil.$$

Therefore, we may conclude that $X \leq z$ if $a > b$ or $b \geq \sqrt{2}a^3$. Also, taking equation (4.1) modulo a^2, we find that

$$b^{2|Y-z|} \equiv 1 \pmod{a^2}.$$

Suppose that $Y - z > 0$. Then $b^{2(Y-z)} \geq 1 + 7a^2$ (since $a^2 \equiv 1 \pmod 8$ and $b^{2(Y-z)} \equiv 0 \pmod 8$), so

$$Y - z \geq \left\lceil \frac{\log(7a^2)}{2\log b} \right\rceil.$$

This implies that

$$\left\lceil \frac{\log(7a^2)}{2\log b} \right\rceil < \frac{\log\bigl(1 + (a/b)^2\bigr)}{2\log b_{(2)}}.$$

We remark that the above inequality does not hold if $a < b$ since the value of the right-hand side is less than 1. Therefore, we may conclude that $Y \leq z$ if $a < b$. To sum up, under each of the assumptions (i), (ii), (iii) and (iv), we obtain $X = Y = z = 1$ by Lemma 4.1. □

From Proposition 4.2, we may obtain the following corollary.

Corollary 4.3. *Let a, b and c be pair-wise relatively prime positive integers greater than one such that $a^2 + b^2 = c$ and b is even. Assume that $c - 1$ is not a square, $c - 1$ has a prime factor not dividing b, or the inequality*

$$\frac{\log b}{\mathrm{ord}_2(b)} < \frac{\log(c-1)}{\mathrm{ord}_2(c-1)}$$

holds. We further assume at least one of the following conditions:
(i)

$$\frac{1}{\sqrt{b_{(2)}^4 - 1}} \le b/a \le \begin{cases} \sqrt{a^2-1} & \text{if } a \text{ is a prime power,} \\ \sqrt{\min\{\alpha, b_{(2)}\}^2 - 1} & \text{if } a \text{ is not a prime power,} \end{cases}$$

where α is the minimum of $a_{(p)}$ when p runs over the prime factors of a.

(ii)
$$\frac{1}{\sqrt{15}} \le b/a \le \sqrt{3}.$$

(iii)
$$\sqrt{2}a^3 \le b.$$

Then, the exponential Diophantine equation

$$a^{2X} + b^{2Y} = c^z$$

has the unique solution $X = Y = z = 1$ in positive integers X, Y and z.

Proof. Case (ii) is an immediate consequence of case (i). Further, case (iii) is just (iv) in Proposition 4.2, so we only consider case (i).

Assume that $s := a/b > 1$. If $s \le \sqrt{b_{(2)}^4 - 1}$, then we see that

$$\frac{\log(1+s^2)}{2\log b_{(2)}} \le 2 \le \left\lceil \frac{\log(7a^2)}{2\log b} \right\rceil,$$

so the desired conclusion follows from Proposition 4.2 (i).

Assume that $t := b/a > 1$. Suppose that a is a prime power. If $t \le \sqrt{a^2-1}$, then we see that

$$\frac{\log\big(1+t^2\big)}{\log a} \le 2 \le \left\lceil \frac{\log(2b^2)}{2\log a} \right\rceil,$$

so the desired conclusion follows from Proposition 4.2 (ii).

Suppose that a is not a prime power. Since $a \ge 15$, $\mathrm{p}(a) \ge 3$, $b = at$

and $c = a^2(t^2+1)$, we see that

$$\begin{aligned}
&\frac{2\log\bigl(a/\mathrm{p}(a) + (-1)^{(a+1)/2}\mathrm{p}(a)\bigr)\log(b/2)}{\log a\,\log b_{(2)}\log c}\\
&\le \frac{\log(a/3+3)}{\log a}\,\frac{\log(a^2t^2/4)}{\log\bigl(a^2(t^2+1)\bigr)}\,\frac{1}{\log b_{(2)}}\\
&< \frac{1}{\log b_{(2)}}.
\end{aligned}$$

Hence, if $t \le \sqrt{\min\{\alpha, b_{(2)}\}^2 - 1}$, then we find that

$$\max\left\{\frac{1}{\log\alpha}, \frac{1}{\log b_{(2)}}\right\}\log(1+t^2) = \frac{\log(1+t^2)}{\log\min\{\alpha, b_{(2)}\}} \le 2 \le \left\lceil\frac{\log(2b^2)}{2\log a}\right\rceil,$$

so the desired conclusion follows from Proposition 4.2 (iii). $\square$

Next, we consider the case of $a^2 + b = c^2$. Using Theorems 3.2 and 3.5, we prove the following result.

Proposition 4.4. *Let a, b and c be pair-wise relatively prime positive integers greater than one such that $a^2+b=c^2$ and b is odd. We assume at least one of the following conditions* (i)–(iv) :

(i) $c \equiv 2 \pmod 4$.

(ii) *b is a prime power.*

(iii) *b is not a prime power, a is even and the inequalities*

$$\max\left\{\left\lceil\frac{\log(3b)}{2\log a}\right\rceil, 2\right\} \ge \frac{\log\bigl(b/\mathrm{p}(b) - \mathrm{p}(b)\bigr)}{\log a_{(2)}\log c}\log(c/a),$$

$$\left\lceil\frac{\log(2a^2)}{2\log c}\right\rceil \ge \frac{2\log(a/2)\,\log\bigl(b/\mathrm{p}(b) - \mathrm{p}(b)\bigr)}{\log a_{(2)}\log b\,\log c}\log\left(1+\frac{1}{(c/a)^2-1}\right)$$

hold, where $\mathrm{p}(b)$ is the least prime factor of b.

(iv) *b is not a prime power, c is even and the inequalities*

$$\max\left\{\left\lceil\frac{\log(8b)}{2\log a}\right\rceil, 2\right\} \ge \frac{\log\bigl(b/\mathrm{p}(b) + \mathrm{p}(b)\bigr) - \log 2}{\log a\,\log c_{(2)}}\log(c/a),$$

$$\max\left\{\left\lceil \frac{\log(7a^2)}{2\log c}\right\rceil,\ 2\right\}$$
$$\geq \frac{\log\big(b/\mathrm{p}(b)+\mathrm{p}(b)\big)-\log 2}{\log b\ \log c_{(2)}}\ \log\left(1+\frac{1}{(c/a)^2-1}\right)$$

hold, where $\mathrm{p}(b)$ *is the least prime factor of* b*. Then the exponential Diophantine equation*

$$a^{2X}+b^y=c^{2Z}$$

has the unique solution $X=y=Z=1$ *in positive integers* X*,* y *and* Z*.*

Proof. Let (X,y,Z) be a solution of the equation

$$a^{2X}+b^y=c^{2Z} \tag{4.2}$$

where X,y,Z are positive integers. We remark that $Z<2y$. This follows from the relation

$$c^{2y}>(c^2-a^2)^y=b^y=(c^Z+a^X)(c^Z-a^X)\geq c^Z+a^X>c^Z.$$

In particular, we find that $X=y=Z=1$ if $y=1$.

(i) We assume that $c\equiv 2 \pmod 4$. Then we see that $b=c^2-a^2\equiv 4-1\equiv 3 \pmod 8$. Taking equation (4.2) modulo 8, we find that $1+3^y\equiv 4^Z \pmod 8$. If $Z>1$, then $3^y\equiv -1 \pmod 8$, which does not hold. Hence $Z=1$, so $X=y=1$ by Lemma 4.1.

(ii) We assume that b is a prime power. From Lemma 2.1, we see that y has to be odd if c is even. Then, by (I) in Theorem 3.2 and (I) in Theorem 3.5, we have

$$(X,y,Z)\in\left\{\left(1,\ \frac{\log(2a+1)}{\log b},\ \frac{\log(a+1)}{\log c}\right),\right.$$
$$\left.\left(\frac{\log(c-1)}{\log a},\ \frac{\log(2c-1)}{\log b},\ 1\right)\right\}.$$

Since

$$b=c^2-a^2\geq c^2-(c-1)^2=2c-1\geq 2a+1,$$

we conclude that $c=a+1$ and $X=y=Z=1$.

In what follows, we assume that b is not a prime power. Also, by the above remarks, we may assume that

$$(X, y, Z) \neq \left(1, \frac{\log(2a+1)}{\log b}, \frac{\log(a+1)}{\log c}\right), \left(\frac{\log(c-1)}{\log a}, \frac{\log(2c-1)}{\log b}, 1\right).$$

(iii) We assume condition (iii). Next we remark that $b \equiv 1 \pmod 4$. Since $\max\{a^{2X}, b^y\} < c^{2Z}$, we see that

$$\begin{aligned} X - Z &< X - \frac{\log a}{\log c} X = \frac{X}{\log c} \log(c/a), \\ y - Z &< y - \frac{\log b}{2\log c} y = \frac{\log(c^2/b)}{2\log c} y \\ &= \frac{y}{2\log c} \log\left(1 + \frac{1}{(c/a)^2 - 1}\right). \end{aligned}$$

We will obtain upper bounds for X and y by using Theorems 3.2 and 3.5.

Suppose that y is odd. By Theorem 3.5 (II), we have

$$X \leq \frac{\log\big(b/\mathrm{p}(b) - \mathrm{p}(b)\big)}{\log a_{(2)}}, \qquad y \leq \frac{\log a \, \log\big(b/\mathrm{p}(b) - \mathrm{p}(b)\big)}{\log a_{(2)} \log\big(\sqrt{b+1} - 1\big)} + 1.$$

Suppose that y is even. From Lemma 2.1, we see that c has to be odd. By Theorem 3.2 (III-1), we have

$$X = 1, \quad y < \frac{4\log(a/2)}{\log b},$$

or

$$\begin{aligned} X &\leq \frac{\log \min\big\{b/\mathrm{p}(b) - \mathrm{p}(b),\ 2\sqrt{c-1}\big\}}{\log a_{(2)}}, \\ y &< \frac{4\log(a/2) \log \min\big\{b/\mathrm{p}(b) - \mathrm{p}(b),\ 2\sqrt{c-1}\big\}}{\log a_{(2)} \log b}. \end{aligned}$$

If the former case holds, then since $b \geq 2a + 1$, we see that

$$y < \frac{4\log(a/2)}{\log b} \leq \frac{4\log(a/2)}{\log(2a+1)} < 4,$$

so $y = 2$. This implies that $a^2 + b^2 = (a^2 + b)^Z$. But this clearly does not hold. Hence, the latter case holds. To sum up, we may conclude that

$$X \le \frac{\log\bigl(b/\mathrm{p}(b)-\mathrm{p}(b)\bigr)}{\log a_{(2)}}, \qquad y < \frac{4\log(a/2)\log\bigl(b/\mathrm{p}(b)-\mathrm{p}(b)\bigr)}{\log a_{(2)}\log b}.$$

Therefore, we see that

$$\begin{aligned} X-Z &< \frac{\log\bigl(b/\mathrm{p}(b)-\mathrm{p}(b)\bigr)}{\log a_{(2)}\log c}\,\log(c/a), \\ y-Z &< \frac{2\log(a/2)\,\log\bigl(b/\mathrm{p}(b)-\mathrm{p}(b)\bigr)}{\log a_{(2)}\log b\,\log c}\,\log\Bigl(1+\frac{1}{(c/a)^2-1}\Bigr). \end{aligned}$$

On the other hand, taking equation (4.2) modulo b, we find that

$$a^{2X} \equiv a^{2Z} \pmod{b}.$$

Since $\gcd(a,b)=1$, it follows that

$$a^{2|X-Z|} \equiv 1 \pmod{b}.$$

Suppose that $X-Z>0$. We will observe that this leads to a contradiction. Then $a^{2(X-Z)} \ge 1+3b$ (since a is even and $b \equiv 1 \pmod 4$). We will show that $X-Z \ge 2$. Suppose that $X-Z=1$. We remark that $y \ge 2$ and $Z \ge 2$ (since $(X,y,Z)=(1,1,1)$ if $y=1$ or $Z=1$). It is clear that X or Z is even. Then $y \le 3$ by Lemmas 2.2, 2.5 and 2.8. If $y=2$, then, and since $Z<2y\le 4$, we find that $(X,Z)\in\{(3,2),(4,3)\}$. This contradicts Lemma 2.6. If $y=3$, then $(X,Z)\in\{(3,2),(4,3),(5,4),(6,5)\}$. This contradicts Lemmas 2.7 and 2.9. It follows that $X-Z\ge 2$. Hence,

$$X-Z \ge \max\Bigl\{\Bigl\lceil\frac{\log(3b)}{2\log a}\Bigr\rceil,\ 2\Bigr\}.$$

This implies that

$$\max\Bigl\{\Bigl\lceil\frac{\log(3b)}{2\log a}\Bigr\rceil,\ 2\Bigr\} < \frac{\log\bigl(b/\mathrm{p}(b)-\mathrm{p}(b)\bigr)}{\log a_{(2)}\log c}\,\log(c/a).$$

But this contradicts our assumptions. Hence, $X\le Z$. Also, taking equation (4.2) modulo a^2, we find that

$$c^{2|y-Z|} \equiv 1 \pmod{a^2}.$$

Suppose that $y - Z > 0$. Then $c^{2(y-Z)} \geq 1 + 2a^2$. It follows that

$$y - Z \geq \left\lceil \frac{\log(2a^2)}{2\log c} \right\rceil.$$

This implies that

$$\left\lceil \frac{\log(2a^2)}{2\log c} \right\rceil < \frac{2\log(a/2)\,\log\big(b/\mathrm{p}(b) - \mathrm{p}(b)\big)}{\log a_{(2)}\log b\,\log c}\,\log\left(1 + \frac{1}{(c/a)^2 - 1}\right).$$

But this contradicts our assumptions. Hence, $y \leq Z$. Therefore, we obtain $X = y = Z = 1$ by Lemma 4.1.

(iv) We assume condition (iv). We remark that $b \equiv 3 \pmod 4$. From Lemma 2.1, we see that y has to be odd (since c is even). Then, by (II) in Theorem 3.5, we have

$$Z \leq \frac{\log\big(b/\mathrm{p}(b) + \mathrm{p}(b)\big) - \log 2}{\log c_{(2)}}.$$

Since $\max\{a^{2X}, b^y\} < c^{2Z}$, we see that

$$\begin{aligned}
-Z &< \frac{\log c}{\log a}\,Z - \,Z = \frac{Z}{\log a}\,\log(c/a)\\
&\leq \frac{\log\big(b/\mathrm{p}(b) + \mathrm{p}(b)\big) - \log 2}{\log a\,\log c_{(2)}}\,\log(c/a),\\
y - Z &< \frac{2\log c}{\log b}\,Z - Z = \frac{\log(c^2/b)}{\log b}\,Z = \frac{Z}{\log b}\,\log\left(1 + \frac{1}{(c/a)^2 - 1}\right)\\
&\leq \frac{\log\big(b/\mathrm{p}(b) + \mathrm{p}(b)\big) - \log 2}{\log b\,\log c_{(2)}}\,\log\left(1 + \frac{1}{(c/a)^2 - 1}\right).
\end{aligned}$$

Suppose that $X - Z > 0$. Similarly to (iii), we can observe that $a^{2(X-Z)} \geq 1 + 8b$ and $X - Z \geq 2$. But this contradicts our assumptions. Hence, $X \leq Z$.

Suppose that $y - Z > 0$. We will observe that this leads to a contradiction. Similarly to (iii), we can observe that $c^{2(y-Z)} \geq 1 + 7a^2$. We will show that $y - Z \geq 2$. Suppose that $y - Z = 1$. Since y is odd, we see that Z is even. Then $y \leq 3$ by Lemmas 2.2, 2.5 and 2.8. Hence, $y = 3$ and $Z = 2$. Since $a^2 + b = c^2$ and $a^{2X} + b^3 = c^4$, we see that

$$(4b^2 <)\ a^{2X} + b^3 = (a^2 + b)^2 \leq 4\max\{a^2, b\}^2.$$

This implies that $a^2 > b$. Taking the above modulo a^2, we see that $b^3 \equiv b^2 \pmod{a^2}$, so $b \equiv 1 \pmod{a^2}$ since $\gcd(a,b) = 1$. This is a contradiction since $a^2 > b$ and $b > 1$. It follows that

$$y - Z \geq \max\left\{ \left\lceil \frac{\log(7a^2)}{2\log c} \right\rceil, \ 2 \right\}.$$

This implies that

$$\max\left\{ \left\lceil \frac{\log(7a^2)}{2\log c} \right\rceil, \ 2 \right\} < \frac{2\log(a/2)\,\log\bigl(b/\mathrm{p}(b) - \mathrm{p}(b)\bigr)}{\log a_{(2)} \log b \, \log c} \log\left(1 + \frac{1}{(c/a)^2 - 1}\right).$$

But this contradicts our assumptions. Hence, $y \leq Z$. Therefore, we obtain $X = y = Z = 1$ by Lemma 4.1. □

From Proposition 4.4, we may obtain the following corollary.

Corollary 4.5. *Let a, b and c be pair-wise relatively prime positive integers greater than one such that $a^2 + b = c^2$ and b is odd. We assume at least one of the following conditions* (i), (ii) *and* (iii):

(i) $$a \equiv 0 \pmod 2, \quad \sqrt{1 + \frac{1}{\sqrt{a_{(2)}} - 1}} \leq c/a \leq a_{(2)}.$$

(ii) $$c \equiv 0 \pmod 2, \quad \sqrt{1 + \frac{1}{c_{(2)}^2 - 1}} \leq c/a \leq \sqrt{c_{(2)}}.$$

(iii) $$\sqrt[4]{2} \leq c/a \leq 2.$$

Then the exponential Diophantine equation

$$a^{2X} + b^y = c^{2Z}$$

has the unique solution $X = y = Z = 1$ in positive integers X, y and Z.

Proof. By Proposition 4.4 (i), we observe that case (iii) is an immediate consequence of cases (i) and (ii). So we only consider cases (i) and (ii). By Proposition 4.4 (ii), we may assume that b is not a prime power.

We assume that a is even. If $u := c/a \leq a_{(2)}$, then, since $b/\mathrm{p}(b) - \mathrm{p}(b) \leq b/3 - 3 < b < c^2$, we see that

$$\frac{\log\bigl(b/\mathrm{p}(b) - \mathrm{p}(b)\bigr)}{\log a_{(2)} \log c} \log u < \frac{2\log u}{\log a_{(2)}} \leq 2.$$

Also, if $u \geq \sqrt{1 + (1/\sqrt{a_{(2)}} - 1)}$, then, since $b/\mathrm{p}(b) - \mathrm{p}(b) < b$, we see that

$$\frac{2\log(a/2)\,\log\bigl(b/\mathrm{p}(b) - \mathrm{p}(b)\bigr)}{\log a_{(2)} \log b\, \log c} \log\left(1 + \frac{1}{u^2 - 1}\right) < \frac{\log(a^2/4)}{\log a_{(2)} \log c} \log \sqrt{a_{(2)}}$$
$$= \frac{\log(a^2/4)}{2\log c} < \left\lceil \frac{\log(2a^2)}{2\log c} \right\rceil,$$

so the desired conclusion follows from (iii) in Proposition 4.4.

We assume that c is even. If $u := c/a \leq \sqrt{c_{(2)}}$, then, since $b/\mathrm{p}(b) + \mathrm{p}(b) \leq b/3 + 3 < b$, we see that

$$\frac{\log\bigl(b/\mathrm{p}(b) + \mathrm{p}(b)\bigr) - \log 2}{\log a\, \log c_{(2)}} \log u < \frac{\log b}{\log a\, \log c_{(2)}} \log \sqrt{c_{(2)}}$$
$$= \frac{\log b}{2\log a} < \left\lceil \frac{\log(8b)}{2\log a} \right\rceil.$$

Also, if $u \geq \sqrt{1 + (1/c_{(2)}^2 - 1)}$, then we see that

$$\frac{\log\bigl(b/\mathrm{p}(b) + \mathrm{p}(b)\bigr) - \log 2}{\log b\, \log c_{(2)}} \log\left(1 + \frac{1}{u^2 - 1}\right) < \frac{1}{\log c_{(2)}} \log c_{(2)}^2 = 2,$$

so the desired conclusion follows from Proposition 4.4 (iv). □

5. Proofs of Theorems F1 and F2. In this section, we prove Theorems F1 and F2. Let $\{F_n\}_{n\geq 0}$ be Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. In the proofs of Theorems F1 and F2, we will use *Cassini's identity*:

$$F_n^2 = (-1)^{n+1} + F_{n-1}F_{n+1}$$

for $n \geq 1$ (cf., [**12,** page 74, Theorem 5.3]). This will play a crucially important role in the proofs. In the study of Fibonacci numbers we often observe that Lucas numbers $\{L_n\}_{n\geq 0}$ work well, as well as in the proofs of Theorems F1 and F2. They are defined by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$.

5.1. Proof of Theorem F1. Let $n \geq 3$. We first note that F_n, F_{n+1} and F_{2n+1} are pair-wise relatively prime positive integers greater than one. Let (x, y, z) be a solution of the equation

$$F_n^x + F_{n+1}^y = F_{2n+1}^z \tag{5.1}$$

where x, y and z are positive integers. First, we determine the parities of x and y by using congruence reductions. Further, we obtain congruence relations among x, y and z. Actually, we do not use statement (ii) in the following lemma (see Remark 5.4).

Lemma 5.1. *The following hold*:

(i) *x and y are even.*
(ii) *$X \equiv z \pmod{F_{n+1}}$ and $Y \equiv z \pmod{F_n}$, where $X = x/2$ and $Y = y/2$.*

Proof. We first consider the case of $n = 3$. In this case, we rewrite (5.1) as

$$2^x + 3^y = 13^z. \tag{5.2}$$

Taking (5.2) modulo 3, we have $(-1)^x \equiv 1 \pmod 3$, so x is even. Then, taking (5.2) modulo 4, we have $(-1)^y \equiv 1 \pmod 4$, so y is even. By Corollary 4.3, we have $X = Y = z = 1$; in particular, (ii) also holds. Hence the lemma holds for $n = 3$. Similarly, we can prove the lemma for $n = 4$. Hence, it suffices to consider the case of $n \geq 5$.

For any $m \geq 4$, we see that $F_{m+1} - F_m = F_{m-1} \geq F_3 = 2 > 1$ and $F_m > 1$. Hence, $F_m \not\equiv \pm 1 \pmod{F_{m+1}}$. In particular, we find that

$$F_n \not\equiv \pm 1 \pmod{F_{n+1}}, \qquad F_{n+1} \equiv F_{n-1} \not\equiv \pm 1 \pmod{F_n}.$$

We write $x = 2X + x_1$, where X is a non-negative integer and $x_1 \in \{0, 1\}$. By Cassini's identity, we see that

$$F_n^2 = \delta + F_{n-1}F_{n+1} \equiv \delta - F_nF_{n+1} \pmod{F_{n+1}^2},$$

where $\delta = (-1)^{n+1}$. Hence, we observe that

$$F_n^{2X} = F_n^{2X} F_n^{x_1}$$

$$
\begin{aligned}
&\equiv (\delta - F_nF_{n+1})^X F_n^{x_1} \\
&\equiv (\delta^X - \delta^{X-1}F_nF_{n+1})F_n^{x_1} \pmod{F_{n+1}^2}, F_{2n+1}^z \ = \big(F_n^2 + F_{n+1}^2\big)^z \\
&\equiv F_n^{2z} \\
&\equiv (\delta - F_nF_{n+1})^z \\
&\equiv \delta^z - \delta^{z-1}F_nF_{n+1}z \pmod{F_{n+1}^2}.
\end{aligned}
$$

It follows from (5.1) that

$$(\delta^X - \delta^{X-1}F_nF_{n+1})F_n^{x_1} + F_{n+1}^y \equiv \delta^z - \delta^{z-1}F_nF_{n+1}z \pmod{F_{n+1}^2}.$$

Reducing this modulo F_{n+1}, we have

$$\delta^X F_n^{x_1} \equiv \delta^z \pmod{F_{n+1}}.$$

If $x_1 = 1$, then $F_n \equiv \pm 1 \pmod{F_{n+1}}$, which is absurd. Hence, $x_1 = 0$, that is, $x = 2X$. Then, $\delta^X \equiv \delta^z \pmod{F_{n+1}}$. This implies that $\delta^X = \delta^z$ since $\delta = \pm 1$ and $F_{n+1} \geq 3$. Hence, we find that

$$-\delta^{X-1}F_nX + F_{n+1}^{y-1} \equiv -\delta^{X-1}F_nz \pmod{F_{n+1}}.$$

Similarly, we can prove that y is even by taking (5.1) modulo F_n^2 (for this, we use the congruences $F_{n+1}^2 \equiv -\delta + F_nF_{n+1} \pmod{F_n^2}$ and $F_{n+1} \not\equiv \pm 1 \pmod{F_n}$), and that

$$F_n^{x-1} + (-\delta)^{Y-1}F_{n+1}Y \equiv (-\delta)^{Y-1}F_{n+1}z \pmod{F_n},$$

where $Y = y/2$. Since $x \geq 2$ and $y \geq 2$, it follows from the above two congruences that $X \equiv z \pmod{F_{n+1}}$ and $Y \equiv z \pmod{F_n}$.

By Lemma 5.1, we can write $x = 2X$ and $y = 2Y$, where X and Y are positive integers.

It suffices to consider the case where F_{2n+1} is odd. Indeed, if F_{2n+1} is even, then F_n and F_{n+1} are odd, so $F_{2n+1}^z = F_n^{2X} + F_{n+1}^{2Y} \equiv 2 \pmod 4$. This gives that $z = 1$; hence, $X = Y = 1$ (by Lemma 4.1). □

In what follows, we consider the case where F_{2n+1} is odd. In order to use Corollary 4.3, we show the following lemma.

Lemma 5.2. $F_{2n+1} - 1$ *has prime factors not dividing* F_n *and* F_{n+1}*, respectively.*

Proof. We first remark that both F_n and L_n are prime to F_{n+1}, and that both F_{n+1} and L_{n+1} are prime to F_n. Since $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $L_m = F_{m+1} + F_{m-1}$ for all $m \geq 1$, we see from Cassini's identity that

$$F_{2n+1} - 1 = \begin{cases} F_n L_{n+1} & \text{if } n \text{ is even,} \\ L_n F_{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

This proves the lemma. □

Since $F_{n+1}/F_n \leq 3/2$, combining Corollary 4.3 (ii) with Lemma 5.1 (i) and Lemma 5.2, we obtain $X = Y = z = 1$. This completes the proof of Theorem F1. □

Remark 5.3. Although we show in Lemma 5.2 that a condition of Corollary 4.3 (essentially, of Theorem 3.4) holds, we can actually verify the other conditions. Using the result in **[9]** (see also **[4]**), which states that all of Fibonacci numbers being a square increased by 1 are given by $F_1 = F_2 = 1$, $F_3 = 2$ and $F_5 = 5$, we may conclude that $F_{2n+1} - 1$ is not a square. Further, by easy calculations of 2-adic valuations of Fibonacci and Lucas numbers, for example, in the case where F_{n+1} is even, we can prove the inequality

$$\frac{\operatorname{ord}_2(F_{2n+1} - 1)}{\operatorname{ord}_2(F_{n+1})} \leq 2\left(< \frac{\log(F_{2n+1} - 1)}{\log F_{n+1}} \right).$$

Also, if the equality $Y = (\operatorname{ord}_2(F_{2n+1} - 1))/(2 \operatorname{ord}_2(F_{n+1}))$ holds, then we have $Y = 1$ by the above inequality. Then, by means of Baker's theory of liner forms in (two) logarithms, we can estimate the value of X as $X = O(\log F_n)$. But the implied constant is much larger than the one obtained from Theorems 3.2 and 3.4.

Remark 5.4. For a positive integer t, let $\{f_n(t)\}_{n \geq 0}$ be the linearly recurrence sequence defined by $f_0(t) = 0$, $f_1(t) = 1$, $f_{n+2}(t) = t\, f_{n+1}(t) + f_n(t)$. It is clear that $f_n(1) = F_n$. According to **[12,** Chapters 37, 38], the formula

$$f_n(t)^2 + f_{n+1}(t)^2 = f_{2n+1}(t)$$

holds for $n \geq 0$. Also $\{f_n(t)\}_{n \geq 0}$ has a formula similar to Cassini's identity. Similarly to the proof of Theorem F1 (with congruence relations similar to Lemma 5.1) (ii), we can prove the following result.

Theorem P1. For each $n \geq 3$ and positive integer t, the exponential Diophantine equation

$$f_n(t)^x + f_{n+1}(t)^y = f_{2n+1}(t)^z$$

has the unique solution $(x, y, z) = (2, 2, 1)$ in positive integers x, y and z.

This is a generalization of Theorem F1. We omit the proof.

5.2. Proof of Theorem F2. Let $n \geq 3$. We first note that F_n, F_{n+2} and F_{2n+2} are pair-wise relatively prime positive integers greater than one. We remark that $F_{2n+2} = F_{n+1}L_{n+1}$. Let (x, y, z) be a solution of the equation

$$F_n^x + F_{2n+2}^y = F_{n+2}^z \tag{5.3}$$

where x, y, z are positive integers. We prepare several lemmas. First, we determine the parities of x and z by using congruence reductions. Further, we obtain congruence relations among x, y and z.

Lemma 5.5. *The following hold*:

(i) *x and z are even.*
(ii) *$3(X - y) \equiv 0 \pmod{F_{n+2}}$ and $3(Z - y) \equiv 0 \pmod{F_n}$, where $X = x/2$ and $Z = z/2$.*

Proof. We first consider the case of $n = 3$. In this case, we rewrite (5.3) as

$$2^x + 21^y = 5^z. \tag{5.4}$$

Taking (5.4) modulo 3, we have $(-1)^x \equiv (-1)^z \pmod 3$, so $x \equiv z \pmod 2$. Also taking (5.4) modulo 5, we have $2^x \equiv -1 \pmod 5$, so x is even; hence, z is even. By Proposition 4.4 (iii), we have $X = y = Z = 1$; in particular, (ii) also holds. Hence, the lemma holds for $n = 3$. Similarly, we can prove the lemma for $n = 4$. Hence, it suffices to consider the case of $n \geq 5$. Then, as observed in the proof of Lemma 5.1, we see that F_n $(\equiv -F_{n+1}) \not\equiv \pm 1 \pmod{F_{n+2}}$ and F_{n+2} $(\equiv F_{n+1}) \not\equiv \pm 1 \pmod{F_n}$.

We write $x = 2X + x_2$, where X is a non-negative integer and $x_2 \in \{0,1\}$. By Cassini's identity, we easily see that

$$F_n^2 \equiv -\delta + 3F_nF_{n+2} \pmod{F_{n+2}^2},$$

where $\delta = (-1)^{n+1}$. Hence, we observe that

$$\begin{aligned}
F_n^x &= F_n^{2X}F_n^{x_2} \\
&\equiv (-\delta + 3F_nF_{n+2})^X F_n^{x_2} \\
&\equiv \big((-\delta)^X + (-\delta)^{X-1}3F_nF_{n+2}X\big)F_n^{x_2} \pmod{F_{n+2}^2}, \\
F_{2n+2}^y &= \big(F_{n+2}^2 - F_n^2\big)^y \\
&\equiv (-F_n^2)^y \\
&\equiv (\delta - 3F_nF_{n+2})^y \\
&\equiv \delta^y - \delta^{y-1}3F_nF_{n+2}\,y \pmod{F_{n+2}^2}.
\end{aligned}$$

It follows from (5.3) that

$$\begin{aligned}
&\big((-\delta)^X + (-\delta)^{X-1}3F_nF_{n+2}\,X\big)F_n^{x_2} \\
&\qquad\qquad \equiv -\delta^y + \delta^{y-1}3F_nF_{n+2}\,y + F_{n+2}^z \pmod{F_{n+2}^2}.
\end{aligned}$$

Reducing this modulo F_{n+2}, we have

$$(-\delta)^X F_n^{x_2} \equiv -\delta^y \pmod{F_{n+2}}.$$

If $x_2 = 1$, then the above implies that $F_n \equiv \pm 1 \pmod{F_{n+2}}$, which is absurd. Hence, $x_2 = 0$, that is, x is even. Then $(-\delta)^X \equiv -\delta^y \pmod{F_{n+2}}$. This implies that $(-\delta)^X = -\delta^y$ since $\delta = \pm 1$ and $F_{n+2} \geq 3$. Hence, we find that

$$(-\delta)^{X-1}3F_nF_{n+2}\,X \equiv (-\delta)^{X-1}3F_nF_{n+2}\,y + F_{n+2}^z \pmod{F_{n+2}^2},$$

so

$$(-\delta)^{X-1}3F_n(X-y) \equiv F_{n+2}^{z-1} \pmod{F_{n+2}}.$$

Similarly, we can prove that z is even by taking (5.3) modulo F_n^2 (for this, we use the congruences $F_{n+2}^2 \equiv -\delta + 3F_nF_{n+1} \pmod{F_n^2}$ and $F_{n+2} \not\equiv \pm 1 \pmod{F_n}$), and that

$$(-\delta)^{X-1}3F_{n+2}(Z-y) \equiv F_n^{x-1} \pmod{F_n},$$

where $Z = z/2$. Since $x \geq 2$ and $z \geq 2$, it follows from the above two congruences that $3(X - y) \equiv 0 \pmod{F_{n+2}}$ and $3(Z - y) \equiv 0 \pmod{F_n}$. □

By Lemma 5.5 (i), we can write $x = 2X$ and $z = 2Z$, where X and Z are positive integers. We recall that, if $y = 1$, then $X = Z = 1$ (which has already been shown at the beginning of the proof of Proposition 4.4).

We first consider the case where F_{n+1} is odd. Then F_{2n+2} is also odd. In this case, we can use Corollary 4.5.

Lemma 5.6. *If F_{n+1} is odd, then $X = y = Z = 1$.*

Proof. Assume that F_{n+1} is odd. First we assume that $F_{n+1} \equiv 1 \pmod 4$. Then, F_n or F_{n+2} is divisible by 8. Since $F_{n+2}/F_n \leq 5/2$, by Corollary 4.5 (i), (ii) , we have $X = y = Z = 1$.

Next we assume that $F_{n+1} \equiv 3 \pmod 4$. Then $F_n \equiv 2 \pmod 4$ or $F_{n+2} \equiv 2 \pmod 4$. By Proposition 4.4 (i), we may assume that $F_n \equiv 2 \pmod 4$. We claim y is odd. Suppose that y is even. We will observe that this leads to a contradiction. We can write $y = 2Y$, where Y is a positive integer. We remark that F_{n+2} has to be odd, so $F_n \equiv 2 \pmod 4$. Then, we observe from Lemma 5.5 (ii) that Z is even. Hence, Y is odd by Lemma 2.2. By Lemma 2.1, we can write

$$F_{2n+2}^Y = k^2 - l^2, \qquad F_{n+2}^Z = k^2 + l^2$$

for some relatively prime integers k and l. Since $F_{2n+2} = F_{n+2}^2 - F_n^2 \equiv 5 \pmod 8$, and Z is even, we see from the above two equations that

$$2l^2 = (F_{n+2}^2)^{Z/2} - F_{2n+2}^Y \equiv 1^{Z/2} - 5^Y \equiv 4 \pmod 8,$$

so $l^2 \equiv 2 \pmod 4$. But this does not hold. Hence, the claim is proved. Since $F_{2n+2} \equiv 5 \pmod 8$ and F_{n+2} is odd, it follows from (5.3) that $4^X \equiv 4 \pmod 8$. This implies that $X = 1$. By Theorem 3.5 (II), we find that

$$y \leq \frac{\log F_n}{\log(\sqrt{F_{2n+2} + 1} - 1)} + 1 < 2,$$

so $y = 1$; hence, $Z = 1$. □

Finally, we consider the case where F_{n+1} is even. We shall first show the following lemma.

Lemma 5.7. *If F_{n+1} is even, then X and Z are odd.*

Proof. We assume that F_{n+1} is even. Suppose that X or Z is even. We will observe that this leads to a contradiction. We remark that $y > 1$. By Cassini's identity, we see that

$$F_n^2 = \delta + F_{n-1}F_{n+1} \equiv \delta - F_nF_{n+1} \pmod{F_{n+1}^2},$$
$$F_{n+2}^2 = \delta + F_{n+1}F_{n+3} \equiv \delta + F_nF_{n+1} \pmod{F_{n+1}^2},$$

where $\delta = (-1)^{n+1}$. Hence, we observe that

$$F_n^{2X} \equiv (\delta - F_nF_{n+1})^X \equiv \delta^X - \delta^{X-1}F_nF_{n+1}X \pmod{F_{n+1}^2},$$
$$F_{n+2}^{2Z} \equiv (\delta + F_nF_{n+1})^Z \equiv \delta^Z + \delta^{Z-1}F_nF_{n+1}Z \pmod{F_{n+1}^2}.$$

Since $F_{2n+2} = F_{n+1}L_{n+1}$ and $y > 1$, it follows from (5.3) that

$$\delta^X - \delta^{X-1}F_nF_{n+1}X \equiv \delta^Z + \delta^{Z-1}F_nF_{n+1}Z \pmod{F_{n+1}^2}.$$

Reducing this modulo F_{n+1}, we have $\delta^X \equiv \delta^Z \pmod{F_{n+1}}$. This implies that $\delta^X = \delta^Z$ since $\delta = \pm 1$ and $F_{n+1} \geq 3$. It follows from the above congruence that

$$\delta^{X-1}F_n(X+Z) \equiv 0 \pmod{F_{n+1}}.$$

This implies that $X + Z \equiv 0 \pmod{F_{n+1}}$. In particular, we see that $X + Z \geq F_{n+1}$ and $X \equiv Z \pmod 2$. Hence, both X and Z are even. Applying Proposition 3.6 to the case of $(a,b,c) = (F_n, F_{2n+2}, F_{n+2})$, we have

$$y < \frac{\log F_{2n+2}}{3\log 2},$$
$$X < \frac{\log(F_{2n+2}/8)}{\log F_n}\, y < 2y,$$
$$Z < \frac{\log(F_{2n+2}/4)}{\log F_{n+2}}\, y < 2y.$$

This yields

$$X + Z \leq 4y - 4 < \frac{4\log F_{2n+2}}{3\log 2} - 4 < F_{n+1}.$$

This is a contradiction. We conclude that X and Z are odd.

From (5.3), we define positive even integers D and E as follows:

$$F_{2n+2}^{y} = DE, \tag{5.5}$$

where $D = F_{n+2}^{Z} + F_{n}^{X}$ and $E = F_{n+2}^{Z} - F_{n}^{X}$. It is easy to see that $\gcd(D, E) = 2$. □

We will consider the cases $F_{n+1} \equiv 2 \pmod 4$ and $F_{n+1} \equiv 0 \pmod 4$ separately. Those cases can be handled by use of Cassini's identity.

Lemma 5.8. *If $F_{n+1} \equiv 2 \pmod 4$, then $X = y = Z = 1$.*

Proof. We assume that $F_{n+1} \equiv 2 \pmod 4$. We remark that $F_{n+1} \equiv 2 \pmod 8$ since $F_m \not\equiv 6 \pmod 8$ for all $m \geq 0$. It is easy to see that $F_n \equiv 1 \pmod 4$, $F_{n+2} \equiv -1 \pmod 4$ and $\mathrm{ord}_2(F_{2n+2}) = 3$. By Lemma 5.7, we know that X and Z are odd. Hence, $D = F_{n+2}^{Z} + F_{n}^{X} \equiv (-1)^{Z} + 1 \equiv 0 \pmod 4$. Since $\gcd(D, E) = 2$, we see from (5.5) that there exist relatively prime positive odd integers S and T such that

$$D = F_{n+2}^{Z} + F_{n}^{X} = 2^{3y-1}S, \qquad E = F_{n+2}^{Z} - F_{n}^{X} = 2T.$$

Since the square of an odd integer is congruent to 1 modulo 8, we see from the second equation above that

$$2T = F_{n+2}^{Z} - F_{n}^{X} \equiv F_{n+2} - F_{n} = F_{n+1} \equiv 2 \pmod 8,$$

that is, $T \equiv 1 \pmod 4$. Therefore, we find that

$$2^{3y-2}S = F_{n+2}^{Z} - T \equiv 2 \pmod 4.$$

This implies that $y = 1$, so $X = Z = 1$. □

Lemma 5.9. *If $F_{n+1} \equiv 0 \pmod 4$, then $X = y = Z = 1$.*

Proof. We assume that $F_{n+1} \equiv 0 \pmod 4$. Then $n \geq 5$ and L_{n+1} is even. It is easy to see that $F_n \equiv F_{n+2} \equiv 1 \pmod 4$ and $\beta := \mathrm{ord}_2(F_{2n+2}) \geq 3$. Since $\gcd(D, E) = 2$ and $D = F_{n+2}^{Z} + F_{n}^{X} \equiv 2 \pmod 4$, we see from (5.5) that E is divisible by $2^{\beta y - 1}$, in particular, divisible by 4 (since $\beta y - 1 \geq 3y - 1 \geq 2$).

By Lemma 5.7, we can write $X = 2X' + 1$ and $Z = 2Z' + 1$, where X' and Z' are non-negative integers. Then we find that

$$E = F_{n+2}^Z - F_n^X \equiv F_n^Z - F_n^X = \bigl(F_n^{2Z'} - F_n^{2X'}\bigr)F_n \pmod{F_{n+1}}.$$

By Cassini's identity, we easily observe that $F_n^2 \equiv \delta \pmod{F_{n+1}}$, where $\delta = (-1)^{n+1}$. Hence,

$$E \equiv \bigl(\delta^{Z'} - \delta^{X'}\bigr)F_n \pmod{F_{n+1}}.$$

Reducing this modulo 4, we have $\delta^{Z'} - \delta^{X'} \equiv 0 \pmod 4$, so $\delta^{Z'} - \delta^{X'} = 0$ since $\delta = \pm 1$. Hence, $E \equiv 0 \pmod{F_{n+1}}$. This implies that $D/2$ is prime to F_{n+1}. We can rewrite (5.5) as

$$\left(\frac{D}{2}\right)E = 2^{y-1}F_{n+1}^y\left(\frac{L_{n+1}}{2}\right)^y.$$

Since $D/2$ is prime to $2F_{n+1}$, we see that $D/2$ divides $(L_{n+1}/2)^y$; hence, E is divisible by $2^{y-1}F_{n+1}^y$. It follows that

$$\begin{aligned} 1 < \frac{D}{E} &\le \frac{2(L_{n+1}/2)^y}{2^{y-1}F_{n+1}^y} \\ &= 4\left(\frac{L_{n+1}}{4F_{n+1}}\right)^y = 4\left(\frac{1}{4} + \frac{F_n}{2F_{n+1}}\right)^y \le 4\left(\frac{9}{16}\right)^y, \end{aligned}$$

where we used the facts that $L_{n+1} = F_{n+1} + 2F_n$ and $F_n/F_{n+1} \le 5/8$. The above gives that $y \le 2$. Since $F_{n+2}^Z + F_n \le D \le 2(L_{n+1}/2)^2$, we have

$$Z \le \frac{\log\bigl(2(L_{n+1}/2)^2 - F_n\bigr)}{\log F_{n+2}} < 2,$$

so $Z = 1$. Hence, $X = y = 1$ by Lemma 4.1.

Lemmas 5.6, 5.8 and 5.9 complete the proof of Theorem F2. □

Remark 5.10. For a positive integer t, let $\{B_n(t)\}_{n\ge 0}$ be the linearly recurrence sequence defined by $B_0(t) = 0$, $B_1(t) = 1$, $B_{n+2}(t) = t\,B_{n+1}(t) - B_n(t)$. It is clear that $B_n(3) = F_{2n}$. According to **[12,** Chapter 41], the formula

$$B_n(t)^2 + B_{2n+1}(t) = B_{n+1}(t)^2$$

holds for $n \ge 0$. Also, $\{B_n(t)\}_{n\ge 0}$ has a formula similar to Cassini's identity. Similarly to the proof of Theorem F2, with the theory of linear forms in (two) logarithms (instead of the arguments in the proofs of Lemmas 5.8 and 5.9), we can prove the following result.

Theorem P 2. For each $n \geq 2$ and positive integer $t \geq 3$, the exponential Diophantine equation

$$B_n(t)^x + B_{2n+1}(t)^y = B_{n+1}(t)^z$$

has the unique solution $(x, y, z) = (2, 1, 2)$ in positive integers x, y and z.

This is a partial generalization of Theorem F2. We omit the proof.

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