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ORTHOGONALITIES, TRANSITIVITY OF NORMS AND CHARACTERIZATIONS OF HILBERT SPACES

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ABSTRACT. We introduce three concepts, called I-vector, IP-vector, and P-vector, which are related to isosceles orthogonality and Pythagorean orthogonality in normed linear spaces. Having the Banach-Mazur rotation problem in mind, we prove that an almost transitive real Banach space, whose dimension is at least three and which contains an I-vector (an IP-vector, a P-vector, or a unit vector whose pointwise James constant is $\sqrt{2}$, respectively) is a Hilbert space.

1. Introduction and basic notions. We denote by X a normed linear space of *dimension* $\dim X$, with *origin* o, *norm* $\|\cdot\|$, *unit ball* B_X and *unit sphere* S_X, and by $\mathcal{G}_X$ the group of all surjective linear isometries from X to X. Throughout this paper, the spaces under consideration are all assumed to be real. For each point $x \in X$, we set

$$\mathcal{G}_X(x) := \{T(x) : T \in \mathcal{G}_X\}.$$

If, for each pair of unit vectors x and y, there exists an isometry $T \in \mathcal{G}_X$ such that $T(x) = y$, then we say that X (or the *norm* of X) is *transitive*; see [**8,** Definition 2.1]. If there exists a unit vector x such that $\mathcal{G}_X(x)$ is dense in S_X or, equivalently, $\mathcal{G}_X(z)$ is dense in S_X for each unit vector z, then we say that X (or the *norm* of X) is *almost transitive*, cf. [**8,** Definition 2.7 and Proposition 2.8]. In this

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paper, we provide some results related to the *Banach-Mazur rotation problem* which is an unsolved problem mentioned already in Banach's book [**6**]. It asks whether each transitive and separable Banach space is a Hilbert space. The assumption of separability in the hypothesis of this problem cannot be released (cf., [**8,** Example 2.3]), and if this assumption is strengthened to that the space is finite dimensional, then this problem has an affirmative answer. For more information about various types of transitivity of Banach spaces and their relations to the Banach-Mazur rotation problem we refer to [**8**].

If the Banach-Mazur rotation problem has an affirmative answer, a natural way to confirm it could be the following: one takes additional assumptions into consideration, which guarantee that a transitive, separable Banach space is Hilbert. Making these additional assumptions step by step weaker, the direct confirmation of this famous problem might be obtained. One known additional assumption of such a type is the existence of an isometric reflection vector. A *reflection* T on X is an operator on X of the form

$$T := T_{u,u^*} : x \longmapsto x - 2u^*(x)u,$$

where $u \in X$ and $u^* \in X^*$ are two elements satisfying $u^*(u) = 1$. If a reflection T_{u,u^*}, with u as unit vector in X, is an isometry, then T is called an *isometric reflection* and u is said to be an *isometric reflection vector*. In this situation, u^* is uniquely determined by u (cf., the beginning of Section 2 in [**7**]) and called the corresponding *isometric reflection functional*. In [**20**], the following theorem was obtained.

Theorem 1.1. (cf. [**20**]). *If there exists an isometric reflection vector in an almost transitive Banach space X, then X is a Hilbert space.*

It is interesting to observe that the concept of isometric reflection is closely related to several orthogonality types in normed linear spaces.

Let x and y be two vectors in a normed linear space; x is said to be *Roberts orthogonal* to y (denoted by $x \perp_R y$) if

$$\|x + \alpha y\| = \|x - \alpha y\|$$

holds for each real number α, see [**19**]; x is said to be *Birkhoff* *orthogonal* to y (denoted by $x \perp_B y$) if

$$\|x + \alpha y\| \geq \|x\|$$

holds for each real number α, see [**9, 13**]; x is said to be *isosceles orthogonal* to y (denoted by $x \perp_I y$) if the equality

$$\|x + y\| = \|x - y\|$$

holds, see [**12**]; and x is said to be *Pythagorean orthogonal* to y if the equality

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2$$

holds, see again [**12**]. Roberts orthogonality is "strong" in the sense that one can easily verify the following two implications:

$$x \perp_R y \Longrightarrow x \perp_B y \quad \text{and} \quad x \perp_R y \Longrightarrow x \perp_I y.$$

However, unlike Birkhoff orthogonality and isosceles orthogonality, Roberts orthogonality does not have the existence property: [**12**] contains an example of a normed plane X such that, for $x, y \in X$, $x \perp_R y$ implies that $\|x\| \cdot \|y\| = 0$. See the surveys [**2, 4, 5**] for more information about these orthogonality types and the relations between them.

Concerning the relation of isometric reflection vectors and orthogonality types, a result in [**11**] shows that a unit vector u is an isometric reflection vector if and only if there exists a *homogeneous hyperplane* (i.e., a hyperplane containing the origin o) H such that u is Roberts orthogonal to any vector from H. This fact leads us to the following natural question: can we replace the existence of an isometric reflection vector in Theorem 1.1 by other conditions which are related to orthogonality types? We present several such replacements.

2. I-vectors. A unit vector $x \in S_X$ is said to be an *I-vector* if there exists a homogeneous hyperplane H_x such that $x \perp_I S_X \cap H_x$ (i.e., x is isosceles orthogonal to every vector from $S_X \cap H_x$). It is clear that an isometric reflection vector in S_X is always an I-vector, while the converse is obviously not true (consider an arbitrary normed plane, where each unit vector is an I-vector).

For the discussion in this section we also need the concept of Singer orthogonality. In a normed linear space, a vector x is said to be

Singer orthogonal to another vector y (denoted by $x \perp_S y$) if either $\|x\| \cdot \|y\| = 0$ or $x/\|x\| \perp_I y/\|y\|$. One of our main tools is the following lemma.

Lemma 2.1. (cf. **[16]**). *If the Singer orthogonality in a normed linear space X with $\dim X \geq 3$ is additive, i.e., $x \perp_S y + z$ holds whenever $x \perp_S y$ and $x \perp_S z$, then X is an inner product space.*

We also need the *existence and uniqueness property of isosceles orthogonality*, which is summarized in the following lemma.

Lemma 2.2. (cf. **[1, 14]**). *Let X be a normed plane, x a point in $X \backslash \{o\}$, and M_x the length of the maximal line segment contained in S_X and parallel to the line passing through $-x$ and x (when there is no such segment, M_x is set 0). Then, for each number $\gamma \in [0, 2\|x\|/M_x]$ ($\gamma \in [0, +\infty)$, when $M_x = 0$) there exists a unique point $y \in \gamma S_X$ (except for the sign) such that $x \perp_I y$. Particularly, for each number $\gamma \in [0, \|x\|]$, there exists a unique point $y \in \gamma S_X$ (except for the sign) such that $x \perp_I y$.*

Lemma 2.3. *Let X be a normed linear space, x a unit vector in X, and C the set of unit vectors which are isosceles orthogonal to x. Then* $\operatorname{span}(\{x\} \cup C) = X$.

Proof. Otherwise there exists a unit vector $y \in X \backslash (\operatorname{span}(\{x\} \cup C))$. Then, by Lemma 2.2, there exists a unit vector $z \in \operatorname{span}\{x, y\} \cap C$. This implies that $y \in \operatorname{span}(\{x\} \cup C)$, a contradiction. $\square$

Lemma 2.4. *Let X be a normed linear space. If $x_0 \in S_X$ is an I-vector and H_{x_0} is a homogeneous hyperplane such that $x_0 \perp_I S_X \cap H_{x_0}$, then each unit vector z satisfying $x \perp_I z$ belongs to H_{x_0}.*

Proof. Suppose the contrary, namely, that there exists a unit vector $z \notin H_{x_0}$ satisfying $x_0 \perp_I z$. Then there exist a vector $y \in H_{x_0} \cap S_X$ and two numbers α and β such that $z = \alpha x_0 + \beta y$. By Lemma 2.2, in the two-dimensional subspace spanned by x_0 and y there exists a unique (except for the sign) unit vector isosceles orthogonal to x_0, which implies that either $z = y$ or $z = -y$. This is a contradiction. $\square$

Lemma 2.4 shows that, if x_0 is an I-vector, then there exists a unique homogeneous hyperplane H_{x_0} such that $x_0 \perp_I S_X \cap H_{x_0}$.

Lemma 2.5. *Let X be a normed linear space, and $x \in S_X$ an I-vector. Then, for each $T \in \mathcal{G}_X$, $T(x)$ is also an I-vector.*

Proof. Let H_x be the homogeneous hyperplane associated to the I-vector x. For each point $z \in S_X \cap T(H_x)$, it is clear that $y := T^{-1}(z) \in S_X \cap H_x$. It follows that $\|x+y\| = \|x-y\|$. Since T is a linear isometry, $\|T(x)+T(y)\| = \|T(x)-T(y)\|$ holds, that is, $T(x) \perp_I z$. This implies that $T(x)$ is an I-vector. $\square$

Lemma 2.6. *Let X be a Banach space. Then the set of I-vectors is closed.*

Proof. The cases when the set of I-vectors is empty or finite are trivial. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of I-vectors converging to a vector x_0. Clearly, $x_0 \in S_X$. We denote by C the set of unit vectors that are orthogonal to x_0.

By Lemma 2.3, $\operatorname{span}(\{x_0\} \cup C) = X$. Thus, $\operatorname{span} C$ contains a homogeneous hyperplane. Next we show that $\operatorname{span} C \neq X$. It suffices to show that $\operatorname{span} C \cap S_X \subseteq C$. Let H_n be the homogeneous hyperplane satisfying $x_n \perp_I S_X \cap H_n$. Then, for any two vectors u and v in C and each number $n \in \mathbb{N}$, there exist two unit vectors u_n, $v_n \in H_n$ and four numbers α_n, $\alpha_n' \in \mathbb{R}$ and β_n, $\beta_n' \geq 0$ such that

$$u = \alpha_n x_n + \beta_n u_n \quad \text{and} \quad v = \alpha_n' x_n + \beta_n' v_n.$$

By choosing subsequences, if necessary, we may assume that there exist four numbers α_0, α_0', β_0, and β_0' such that

$$\lim_{n\to\infty} \alpha_n = \alpha_0, \qquad \lim_{n\to\infty} \alpha_n' = \alpha_0',$$
$$\lim_{n\to\infty} \beta_n = \beta_0, \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n' = \beta_0'.$$

We claim that $\beta_0, \beta_0' > 0$. Take, for example, the case of β_0. If $\beta_0 = 0$, then from the inequalities

$$|\alpha_n| - \beta_n \leq 1 = \|u\| = \|\alpha_n x_n + \beta_n u_n\| \leq |\alpha_n| + \beta_n$$

it follows that $|\alpha_0| = 1$. Thus,

$$\|u - x_0\| = \|u + x_0\| = \lim_{n\to\infty} \|u - \alpha_n x_n\| = \lim_{n\to\infty} \|\beta_n u_n\| = 0,$$

which is in contradiction to the inequality

$$\|u + x_0\| = \frac{1}{2}(\|u + x_0\| + \|u - x_0\|) \geq 1.$$

Now set

$$u_0 := \lim_{n\to\infty} \frac{1}{\beta_n}(u - \alpha_n x_n) = \lim_{n\to\infty} u_n$$

and

$$v_0 := \lim_{n\to\infty} \frac{1}{\beta'_n}(v - \alpha'_n x_n) = \lim_{n\to\infty} v_n.$$

Thus, $u = \alpha_0 x_0 + \beta_0 u_0$ and $v = \alpha'_0 x_0 + \beta'_0 v_0$. Clearly, we have the equalities

$$\|x_0 + u_0\| = \lim_{n\to\infty} \|x_n + u_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \|x_0 - u_0\|$$

and

$$\|x_0 + v_0\| = \lim_{n\to\infty} \|x_n + v_n\| = \lim_{n\to\infty} \|x_n - v_n\| = \|x_0 - v_0\|.$$

Equivalently, we have $x_0 \perp_I u_0$ and $x_0 \perp_I v_0$. Then, by Lemma 2.2 and the fact that β_0, $\beta'_0 > 0$, we have $u = u_0$ and $v = v_0$. Thus, for each pair of real numbers γ and η such that $\|\gamma u + \eta v\| \neq 0$, we have

$$\begin{aligned}\left\|x_0 + \frac{\gamma u + \eta v}{\|\gamma u + \eta v\|}\right\| &= \lim_{n\to\infty} \left\|x_n + \frac{\gamma u_n + \eta v_n}{\|\gamma u_n + \eta v_n\|}\right\| \\ &= \lim_{n\to\infty} \left\|x_n - \frac{\gamma u_n + \eta v_n}{\|\gamma u_n + \eta v_n\|}\right\| \\ &= \left\|x_0 - \frac{\gamma u + \eta v}{\|\gamma u + \eta v\|}\right\|.\end{aligned}$$

Thus, $\operatorname{span} C \cap S_X \subseteq C$, and therefore $\operatorname{span} C \neq X$. This implies that $\operatorname{span} C$ is contained in a homogeneous hyperplane. It follows that $H_{x_0} := \operatorname{span} C$ is a hyperplane, and $C = H_{x_0} \cap S_X$ is an easy consequence of the existence and uniqueness property of isosceles orthogonality on the unit sphere. Hence, x_0 is an I-vector. $\square$

Since an isometric reflection vector is an I-vector and the converse is not true, the following theorem improves Theorem 1.1.

Theorem 2.7. *Let X be a Banach space with* $\dim X \geq 3$. *If X is almost transitive and contains an I-vector x_0, then X is a Hilbert space.*

Proof. By Lemma 2.1, we need only to show that Singer orthogonality is additive on X. Since the norm on X is almost transitive, the closure of $\mathcal{G}(x_0)$ is S_X. Moreover, since the set of I-vectors is closed, $\mathcal{G}(x_0) = S_X$, which means that each unit vector of X is an I-vector. Assume that x, y and z are three points in X satisfying $x \perp_S y$ and $x \perp_S z$. If $\|x\| \cdot \|y\| \cdot \|z\| = 0$, then it is clear that $x \perp_S y + z$. In the following, we assume that $\|x\| \cdot \|y\| \cdot \|z\| \neq 0$ and $\|y + z\| \neq 0$. By the definition of Singer orthogonality we have

$$\frac{x}{\|x\|} \perp_I \frac{y}{\|y\|} \quad \text{and} \quad \frac{x}{\|x\|} \perp_I \frac{z}{\|z\|}.$$

Since $x/\|x\|$ is an I-vector, there exists a homogeneous hyperplane H_x such that $x/\|x\| \perp_I H_x \cap S_X$. From Lemma 2.4, it follows that $y/\|y\|$, $z/\|z\| \in H_x$. Thus,

$$\frac{x}{\|x\|} \perp_I \frac{y+z}{\|y+z\|}.$$

This implies that $x \perp_S y + z$. Thus, Singer orthogonality is additive on X. □

3. IP- and P-vectors in normed linear spaces. A unit vector x is said to be an *IP-vector* if, for each unit vector y isosceles orthogonal to x, the equality $\|x + y\| = \|x - y\| = \sqrt{2}$ holds (or, equivalently, y and $-y$ are both Pythagorean orthogonal to x).

Lemma 3.1. *Let X be a normed linear space and x an IP-vector in X. Then, for each linear isometry $T \in \mathcal{G}_X$, $T(x)$ is also an IP-vector.*

Proof. Denote by C the set of unit vectors that are isosceles orthogonal to x. Let z be an arbitrary unit vector which is isosceles orthogonal to $T(x)$. Then

$$\left\|x + T^{-1}(z)\right\| = \|T(x) + z\| = \|T(x) - z\| = \left\|x - T^{-1}(z)\right\|.$$

This implies that $T^{-1}(z) \in C$. Since x is an IP-vector, $\|T(x)+z\| = \sqrt{2}$. Hence $T(x)$ is also an IP-vector. □

Lemma 3.2. *The set of IP-vectors in a normed linear space is closed.*

Proof. We denote by $\mathcal{IP}$ the set of IP-vectors in X. The cases when $\mathcal{IP}$ is empty or finite are clear. Suppose again that $\{x_n\}_{n=1}^{\infty}$ is an arbitrary converging sequence contained in $\mathcal{IP}$ and $x_0 = \lim\limits_{n\to\infty} x_n$.

Let y_0 be an arbitrary unit vector isosceles orthogonal to x_0. For each natural number $n \geq 1$, denote by C_n the set of unit vectors isosceles orthogonal to x_n. Clearly, C_n is symmetric with respect to the origin o. By Lemma 2.3, $\operatorname{span}(\{x_n\} \cup C_n) = X$ holds for each $n \geq 1$. Then, for each $n \geq 1$, there exist a vector $y_n \in C_n$ and two numbers $\alpha_n \in \mathbb{R}$ and $\beta_n \geq 0$ such that $y_0 = \alpha_n x_n + \beta_n y_n$. By choosing subsequences, if necessary, we may assume that $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α_0 and $\{\beta_n\}_{n=1}^{\infty}$ converges to a number $\beta_0 \geq 0$. We show that $\beta_0 > 0$. Otherwise, from

$$|\alpha_n| - \beta_n \leq 1 = \|y_0\| \leq |\alpha_n| + \beta_n$$

it would follow that $|\alpha_0| = 1$. Thus,

$$\|y_0 + x_0\| = \|y_0 - x_0\| = \lim_{n\to\infty} \|y_0 - \alpha_n x_n\| = 0,$$

which is impossible. Therefore,

$$y_0' := \lim_{n\to\infty} \frac{1}{\beta_n}(y_0 - \alpha_n x_n) = \frac{1}{\beta_0}(y_0 - \alpha_0 x_0) = \lim_{n\to\infty} y_n$$

exists. It is clear that $y_0 = \alpha_0 x_0 + \beta_0 y_0'$ and $x_0 \perp_I y_0'$. By Lemma 2.2 and the fact that $\beta_0 > 0$, we obtain $y_0' = y_0$. Thus,

$$\|x_0 + y_0\| = \lim_{n\to\infty} \|x_n + y_n\| = \sqrt{2}.$$

This completes the proof. □

The following lemma proved in **[10]** is helpful for our investigations.

Lemma 3.3. (cf. **[10]**). *Let X be a normed linear space with* $\dim X \geq 3$ *and $\varepsilon \in (0,2)$ a fixed number. Then X is an inner product space* *if and only if*

$$\begin{aligned}\delta_X(\varepsilon) &:= \inf\left\{1-\frac{1}{2}\|x+y\| : \ x,y\in S_X, \|x-y\|=\varepsilon\right\}\\ &\geq 1-\frac{1}{2}\sqrt{4-\varepsilon^2}.\end{aligned}$$

Since it was proved in **[18]** that

$$\delta_X(\varepsilon)\leq 1-\frac{1}{2}\sqrt{4-\varepsilon^2}$$

holds for each $\varepsilon\in(0,2)$, the inequality sign in Lemma 3.3 can be replaced by the equality sign. In **[3]**, it was proved that, when $\lambda=\varepsilon/\sqrt{4-\varepsilon^2}$, the equality

$$\delta_X(\varepsilon)= 1-\frac{1}{2}\sqrt{4-\varepsilon^2}$$

is equivalent to the implication

$$x,y\in S_X,\quad x\perp_I y\Rightarrow \|x+\lambda y\|^2=1+\lambda^2.$$

By putting these facts together, we obtain the following lemma.

Lemma 3.4. (cf. [**5,** page 170]). *A normed linear space X with* dim $X\geq 3$ *is an inner product space if and only if*

$$x,y\in S_X,\qquad x\perp_I y\Longrightarrow \|x+y\|=\sqrt{2}.$$

Theorem 3.5. *Let X be a Banach space with* $\dim X\geq 3$. *If X is almost transitive and contains an IP-vector, then X is a Hilbert space.*

Proof. By Lemma 3.4, we only need to show that, for each pair of unit vectors x and y which are isosceles orthogonal, $\|x+y\|=\sqrt{2}$ holds. It suffices to show that each unit vector is an IP-vector. By Lemma 3.1, $\mathcal{G}(x_0)$ is a set of IP-vectors. Since the norm is almost transitive and the set of IP-vectors is closed, $S_X=\mathcal{G}(x_0)$. This completes the proof. □

Let x be a unit vector in a normed linear space X. If there exists a homogeneous hyperplane H_x such that $x\perp_P S_X\cap H_x$, then we say that x is a *P-vector*.

Lemma 3.6. *Each P-vector in a normed linear space X is an IP-vector as well as an I-vector.*

Proof. Let x be a P-vector and H_x a homogeneous hyperplane such that $x \perp_P S_X \cap H_x$. For each vector $y \in S_X \cap H_x$, it is clear that $-y \in S_X \cap H_x$. Thus, $x \perp_P y$ and $x \perp_P -y$ hold simultaneously. It follows that

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 = 2 = \|x+y\|^2 .$$

This implies that $x \perp_I S_X \cap H_x$. Therefore, x is an I-vector.

Suppose that z is an arbitrary unit vector isosceles orthogonal to x. From Lemma 2.4, it follows that $z \in H_x$. Thus,

$$\|x+z\| = \|x-z\| = \sqrt{2},$$

which implies that x is an IP-vector. □

The following result is an immediate consequence of Theorem 3.5 and Lemma 3.6.

Corollary 3.7. *If a Banach space X with $\dim X \geq 3$ is almost transitive and contains a P-vector, then X is a Hilbert space.*

4. The pointwise James constant. In this section, we improve Theorem 3.5 by weakening the hypothesis with the help of the pointwise James constant. The *James* (*or non-square*) *constant* $J(X)$ of a normed linear space X is defined by

$$J(X) := \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

In **[15]**, the equality

$$J(X) = \sup\{\|x+y\| : x, y \in S_X, x \perp_I y\}$$

was proved. For each unit vector x in a normed linear space X we call the constant

$$J_X(x) := \sup\{\|x+y\| : \ y \in S_X, x \perp_I y\}$$

the *pointwise James constant* of x, cf., **[21]**.

Lemma 4.1. *Let X be a normed linear space, and let $T \in \mathcal{G}_X$. If x is a unit vector such that $J_X(x) = \sqrt{2}$, then $J_X(T(x)) = \sqrt{2}$.*

Proof. Denote by C the set of unit vectors which are isosceles orthogonal to x and by C' the set of unit vectors which are isosceles orthogonal to $T(x)$. For each vector $z \in C'$, we have

$$\left\|x - T^{-1}(z)\right\| = \|T(x) - z\| = \|T(x) + z\| = \left\|x + T^{-1}(z)\right\|,$$

which implies that $T^{-1}(z) \in C$. Thus,

$$\|T(x) + z\| = \left\|x + T^{-1}(z)\right\| \leq J_X(x) = \sqrt{2}.$$

It follows that $J_X(T(x)) \leq \sqrt{2}$.

Since $J_X(x) = \sqrt{2}$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset C$ such that

$$\lim_{n\to\infty} \|x + z_n\| = \lim_{n\to\infty} \|x - z_n\| = \sqrt{2}.$$

Clearly, $\{T(z_n)\}_{n=1}^{\infty} \subset C'$ and

$$\lim_{n\to\infty} \|T(x) + T(z_n)\| = \lim_{n\to\infty} \|x + z_n\| = \sqrt{2}.$$

Thus, $J_X(T(x)) = \sqrt{2}$. □

Lemma 4.2. *Let X be a Banach space. Then the set*

$$A := \{x : \; x \in S_X, J_X(x) = \sqrt{2}\}$$

is closed.

Proof. The cases when A is empty or finite are clear. Again, let $\{x_n\}_{n=1}^{\infty} \subset A$ be a sequence converging to a unit vector x. We denote by C the set of unit vectors that are isosceles orthogonal to x and, for each integer $n \geq 1$, we denote by C_n the set of unit vectors that are isosceles orthogonal to x_n.

For each integer $n \geq 1$, there exists a vector $y_n \in C_n$ such that

$$0 \leq \sqrt{2} - \|x_n \pm y_n\| \leq \frac{1}{n}.$$

From the inequalities

$$\|x_n \pm y_n\| - \|x - x_n\| \leq \|x \pm y_n\| \leq \|x_n \pm y_n\| + \|x - x_n\|$$

it follows that

$$\lim_{n\to\infty} \|x \pm y_n\| = \sqrt{2}.$$

Replacing y_n by $-y_n$, if necessary, we may assume that there exist two numbers $\alpha_n, \beta_n \geq 0$ and a vector $z_n \in C$ such that $y_n = \alpha_n x + \beta_n z_n$. By the Monotonicity lemma (cf., [**17,** Proposition 31]), we obtain

$$\|x - y_n\| \leq \|x - z_n\| = \|x + z_n\| \leq \|x + y_n\|,$$

from which it follows that

$$\lim_{n\to\infty} \|x + z_n\| = \sqrt{2}.$$

Thus, $J_X(x) \geq \sqrt{2}$.

Next we show that $J_X(x) \leq \sqrt{2}$. Otherwise, there exists a point $z \in C$ such that

$$\|z + x\| = \|z - x\| > \sqrt{2}.$$

For each integer $n \geq 1$, there exist two numbers $\alpha_n \in \mathbb{R}$ and $\beta_n \geq 0$ and a vector $y_n \in C_n$ such that $z = \alpha_n x_n + \beta_n y_n$. From the Monotonicity lemma, it follows that

$$\begin{aligned} \|x_n + y_n\| &= \|x_n - y_n\| \\ &\in [\min\{\|x_n - z\|, \|x_n + z\|\}, \max\{\|x_n - z\|, \|x_n + z\|\}], \end{aligned}$$

and from the inequalities

$$\lim_{n\to\infty} \|x_n - z\| = \|x - z\| > \sqrt{2}$$

and

$$\lim_{n\to\infty} \|x_n + z\| = \|x + z\| > \sqrt{2}$$

it follows that

$$\lim_{n\to\infty} \|x_n + y_n\| = \lim_{n\to\infty} \|x_n - y_n\| > \sqrt{2}.$$

Thus, when n is sufficiently large, we have $J_X(x_n) > \sqrt{2}$, a contradiction. $\square$

The following theorem improves Theorem 3.5 since the pointwise James constant $J_X(x)$ of an IP-vector x is $\sqrt{2}$.

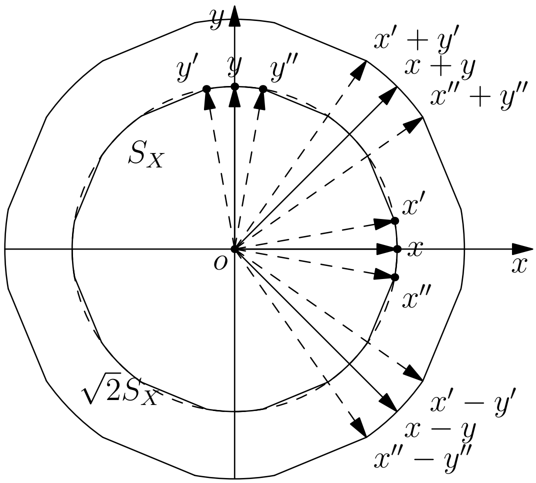

FIGURE 1. A counterexample.

Theorem 4.3. *Let X be a Banach space with* $\dim X \geq 3$*. If X is almost transitive and contains a unit vector x such that $J_X(x) = \sqrt{2}$, then X is a Hilbert space.*

Proof. By Lemma 3.4, we only need to show that, for each pair of unit vectors u and v which are isosceles orthogonal to each other, the equality $\|u + v\| = \sqrt{2}$ holds. Since X is almost transitive, the closure of $\mathcal{G}_X(x)$ is equal to S_X. By Lemma 4.2, $\mathcal{G}_X(x) = S_X$. This implies that $J_X(u) = \sqrt{2}$. Thus $\|u + v\| \leq \sqrt{2}$.

Suppose the contrary, namely, that $\|u + v\| < \sqrt{2}$. Set $u' = (u + v)/\|u + v\|$ and $v' = (u - v)/\|u - v\|$. Then $J_X(u') = J_X(v') = \sqrt{2}$. However,

$$\|u' + v'\| = \frac{2}{\|u + v\|} > \sqrt{2},$$

which is a contradiction. The proof is complete. □

Remark 4.4. A subset R of a topological space T is said to be *rare* in T if the interior of the closure of R in T is empty. Guerrero and Palacios **[7]** proved that if the set of all isometric reflection vectors in a Banach space X is not rare in S_X, then X is a Hilbert space. In this result, we cannot replace isometric reflection vectors by P-vectors. Consider the three-dimensional Banach space X whose unit sphere is obtained by rotating the closed convex curve given in Figure 1 around the y-axis. Replacing certain circular arcs by line segments, this curve is constructed from a circle centered at the origin. Then the set of P-vectors in X is not rare in S_X, and the set of unit vectors x satisfying $J_X(x) = \sqrt{2}$ is also not rare in S_X. However, X is clearly not a Hilbert space. From this point of view, our Theorem 2.7 is strictly stronger than Theorem 1.1.

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