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EXISTENCE OF EVENTUALLY POSITIVE SOLUTIONS OF HIGHER ORDER IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. A search of the literature reveals only a few studies on the necessary as well as sufficient conditions for the existence of eventually positive and/or monotone solutions of higher order impulsive differential equations that also allow delays. To fill this gap, we study a general class of higher order impulsive delay differential equations and establish necessary and/or sufficient conditions for the existence of eventually positive and monotone solutions. Our results are sharp in the sense that, in special cases, they are necessary and sufficient. Illustrative examples are included.

1. Introduction. Impulsive differential equations are mathematical apparatus for simulation of different dynamical processes and phenomena observed in nature (for illustration, a pendulum equation in Example 3.3 is provided in a later section, see also **[11]** and the references therein). For this reason, many impulsive differential equations are studied and their qualitative properties investigated. However, by inspecting recent studies such as (**[1]**–**[22]**) and their references, we may see that only several recent papers (see e.g., **[4, 5, 12, 19]**) are concerned with necessary and/or sufficient conditions for the existence of eventually positive and/or monotone solutions of *higher order* impulsive delay differential equations.

In particular, in **[12]**, the author is concerned with the following system:

$$x^{(n)}(t) + a(t)x^{(n-1)}(t) + \sum_{i=1}^{m} p_i(t)x(g_i(t)) = 0, \tag{1}$$

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$$t \in [0,\infty)\setminus \{t_k\}_{k\in\mathbf{N}},$$

(2) $$x^{(i)}(t_k^+) = a_k x^{(i)}(t_k), \ k \in \mathbf{N} \quad \text{and} \quad i = 0, 1, \dots, n-1,$$

under the following conditions:

(D1) a and p_i are continuous, Lebesque measurable and locally essentially bounded functions on $[0,\infty)$ for $i = 1, 2, \dots, m$;
(D2) $a_k > 0$ for $k \in \mathbf{N}$;
(D3) the function $A(0,t)$, defined by (11), is bounded for $t \geq 0$ and $\liminf_{t\to\infty} A(0,t) > 0$;
(D4) $p_i(t) \geq 0$ for $t \geq 0$ and $i = 1, 2, \dots, m$; and
(D5) for each $i = 1, 2, \dots, m$, g_i is a continuous function on $[0,\infty)$ such that $g_i(t) \leq t$ for $t \geq 0$, and $\lim_{t\to\infty} g_i(t) = +\infty$.

Theorem 2.4 in **[12]** 'provides' a sufficient condition for the existence of bounded nonoscillatory solution:

(3)
$$\int_0^\infty \int_0^{s_{n-2}} \cdots \int_0^{s_1} \frac{1}{r(s_0)} \int_{s_0}^\infty r(s) \sum_{i=1}^m \frac{p_i(s)}{A(g_i(s),s)}\, ds\, ds_0 \cdots ds_{n-2} < \infty$$

where
$$r(t) = \exp\left(\int_0^t a(s)\, ds\right).$$

However, for $n \geq 3$, we note that the function

$$\int_0^t \int_0^{s_{n-3}} \cdots \int_0^{s_1} \frac{1}{r(s_0)} \int_{s_0}^\infty r(s) \sum_{i=1}^m \frac{p_i(s)}{A(g_i(s),s)}\, ds\, ds_0 \cdots ds_{n-3}$$

is increasing and nonnegative for $t \geq 0$. So condition (3) cannot hold unless $p_i(t) = 0$ for almost every $t \geq 0$. Furthermore, we can easily construct an eventually positive solution under the condition that $p_i(t) = 0$ for almost every $t \geq 0$. So this result is only valid under the case where $n = 2$.

There are other mistakes in reference **[20]** which is concerned with second order impulsive differential equations with delays (see a later section). These mistakes in **[20]** cast doubt on the correctness of the main theorem in **[20]**.

Besides these results in **[12]**, there are similar results in reference **[7]** for equations without impulses. There the authors are concerned with the equation

$$x^{(n)}(t) + f(t, x(g(t))) = 0, \quad t \geq 0, \tag{4}$$

where f is a nondecreasing function on R and $uf(t,u) > 0$ for $u \neq 0$. Theorem 2 in **[7]** tells us that there exists an eventually positive solution x of equation (4) with $x(t)x^{(n-1)}(t) > 0$ and $x(t)x^{(n)}(t) > 0$ eventually, and $x^{(n-2)}(t)$ is bounded above by a constant, if and only if

$$\int_a^\infty tf(t, cg^{n-2}(t))\, dt < \infty$$

for some $a \geq 0$ and $c > 0$. Other similar theorems can also be found in **[7]**.

Next, in reference **[5]**, the authors are concerned with the problem of existence of nonoscillatory solutions of system

$$x'''(t) + f(t, x(t)) = 0, \quad t \in [0, \infty) \setminus \{t_k\}_{k \in \mathbf{N}}, \tag{5}$$

$$x(t_k^+) = x(t_k), \quad k \in \mathbf{N}, \tag{6}$$

$$x'(t_k^+) = x'(t_k), \quad k \in \mathbf{N}, \tag{7}$$

$$x''(t_k^+) = x''(t_k) - I_{2k}(x(t_k)), \quad k \in \mathbf{N} \tag{8}$$

where f is a continuous function on $[0, \infty) \times R$ with $uf(t,u) > 0$ for $u \neq 0$ and $t \geq 0$, and I_{2k} are continuous functions on R for $k \in N$ with $uI_k(u) > 0$ for $u \neq 0$. Theorem 2 in **[5]** provides a sufficient condition that $|f(t,u)| \leq |f(t,v)|$ and $|I_{2k}(u)| \leq |I_{2k}(v)|$ for $|u| \leq |v|$, and

$$\int_0^\infty t^2 \left| f(t,c) \right| \, dt + \sum_{k=1}^\infty t_k^2 \left| I_{2k}(c) \right| < \infty$$

for some $c \neq 0$ such that the system (5)–(8) has a bounded nonoscillatory solution.

The above mistakes and list of results motivate us to study general classes of higher order impulsive delay differential equations and establish necessary and/or sufficient conditions for the existence of eventually positive and monotone solutions, which may be used to complete, to generalize or to patch up several recent results in the literature.

To this end, we first recall some usual notation. $\mathbf{R}$ and $\mathbf{N}$ will be used to denote the set of real numbers and positive integers, respectively, while $\mathbf{R}^+$ denotes the interval $(0, +\infty)$. We set $\mathbf{N}_n = \{1, 2, \ldots, n\}$ and

$\overline{\mathbf{N}}_n = \{0, 1, \ldots, n\}$. We let

$$\Upsilon = \{t_1, t_2, \ldots\}$$

be a set of real numbers such that $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = +\infty$. Assuming that I_1 and I_2 are intervals in $\mathbf{R}$, we define

$$\begin{aligned} PC(I_1, I_2) = \{\varphi : I_1 \longrightarrow I_2 \mid \varphi &\text{ is continuous in each interval} \\ &I_1 \cap (t_k, t_{k+1}] \text{ for } k \in \mathbf{N} \cup \{0\} \text{ and with} \\ &\text{discontinuity of the first kind only}\}, \end{aligned}$$

$$\begin{aligned} PC^{(1)}(I_1, I_2) = \{\varphi \in PC(I_1, I_2) \mid &\varphi(t) \\ &\text{is continuously differentiable almost everywhere on } I_1\}, \end{aligned}$$

and

$$PC^{(i)}(I_1, I_2) = \left\{\varphi \in PC(I_1, I_2) \mid \varphi^{(j)} \in PC^{(1)}(I_1, \mathbf{R}) \quad \text{for } j \in \mathbf{N}_{i-1}\right\}$$

for $i \geq 2$. Also, $y'(t)$ will be used to denote the left derivative of the function $y(t)$ at t. We need an order relation in the space $PC([T, \infty), [0, \infty))$: If y_1 and y_2 belong to $PC([T, \infty), [0, \infty))$, we say that $y_1 \leq y_2$ if and only if $y_1(t) \leq y_2(t)$ almost everywhere on $[T, \infty)$. A partially ordered subset of $PC([T, \infty), [0, \infty))$ is called a complete lattice if all its subsets have both a supremum and an infimum.

Let $n \geq 2$ be given. We investigate the following nonlinear delay differential systems with impulses

$$\left(r(t)x^{(n-1)}(t)\right)' + F\left(t, x^{(n-1)}(g_{n-1}(t)), x^{(n-2)}(g_{n-2}(t)), \ldots, x(g_0(t))\right) = 0, \tag{9}$$

$$x^{(i)}(t_k^+) = I_{(i)k}\left(x^{(i)}(t_k)\right), \tag{10}$$

where $t \in [0, \infty)\backslash\Upsilon$, $k \in \mathbf{N}$ and $i \in \overline{\mathbf{N}}_{n-1}$, under some or all of the following conditions:

- (A1) for any $t \geq 0$, $F(t, \mu)$ is a continuous function on $\mathbf{R}^n$, and for any $\mu \in \mathbf{R}^n$, $F(t, \mu)$ belongs to $PC([0, \infty), \mathbf{R})$;
- (A2) for each $i \in \overline{\mathbf{N}}_{n-1}$, g_i is a continuous function on $[0, \infty)$ with $g_i(t) \leq t$ for $t \geq 0$, and $\lim_{t\to\infty} g_i(t) = +\infty$;

(A3) $0 < t_1 < t_2 < \cdots$ are fixed numbers with $\lim_{k\to\infty} t_k = +\infty$;
(A4) r is a positive differentiable function on $[0, \infty)$;
(A5) for each $i \in \overline{\mathbf{N}}_{n-1}$ and $k \in \mathbf{N}$, $I_{(i)k}$ is a continuous function on $\mathbf{R}$ such that $\mu I_{(i)k}(\mu) > 0$ for $\mu \neq 0$;
(A6) there exists $M_0 \geq 1 \geq m_0 > 0$ such that

$$m_0 \leq \prod_{s\leq t_k<t} \tfrac{I_{(0)k}(\delta_k)}{\delta_k} \leq M_0$$

where $k \in \mathbf{N}$ and $t \geq s \geq 0$ with $[s,t) \cap \Upsilon \neq \emptyset$ for any sequence $\{\delta_k \neq 0\}_{k\in\mathbf{N}}$;
(A7) for each $i \in \mathbf{N}_{n-2}$, there exists $M_i \geq 1$ such that

$$\prod_{s\leq t_k<t} \tfrac{I_{(i)k}(\delta_k)}{\delta_k} \leq M_i,$$

where $k \in \mathbf{N}$ and $t \geq s \geq 0$ with $[s,t) \cap \Upsilon \neq \emptyset$ for any sequence $\{\delta_k \neq 0\}_{k\in\mathbf{N}}$.

We remark that condition (A5) means that, for each $i \in \overline{\mathbf{N}}_{n-1}$ and $k \in N$, $I_{(i)k}(\mu) > 0$ for $\mu > 0$ and $I_{(i)k}(\mu) < 0$ for $\mu < 0$. These are needed in order to 'propagate' the positivity of solutions 'into the future' and hence are important for the existence of positive solutions. We further remark that

$$\prod_{s\leq t_k<t} \tfrac{I_{(0)k}(\delta_k)}{\delta_k} := \prod_{\{k:s\leq t_k<t\}} \tfrac{I_{(0)k}(\delta_k)}{\delta_k},$$

etc., and hence conditions (A6) and (A7) require the product of the rates of change of values of solutions at t_k has positive upper bound and/or positive lower bound. We will give a specific example to illustrate conditions (A6) and (A7) in Example 3.1.

Let $T \geq 0$ and

$$r_T = \min_{0\leq j\leq n-1} \left\{ \inf_{t\geq T} g_j(t) \right\}.$$

Definition 1.1. Let $T \geq 0$. For any $\phi \in PC^{(n-1)}([r_T, T], \mathbf{R})$, a function $x \in PC^{(n)}([T, \infty), \mathbf{R})$ is said to be a solution of system (9)–(10) on $[T, \infty)$ satisfying the initial value condition

$$x(t) = \phi(t), \quad t \in [r_T, T],$$

if $x(t)$ satisfies (9) for almost every $t \geq T$ and satisfies (10) for $t \geq T$.

Definition 1.2. Let $x = x(t)$ be a real function defined for all sufficiently large t. We say that x is eventually positive if there exists a number T such that $x(t) > 0$ for every $t \geq T$. We say that x is nonoscillatory if either $x(t)$ or $-x(t)$ is eventually positive.

Let $a_{(i)k} > 0$, $a_k > 0$, $b_k > 0$ and $b_k^* > 0$ for $k \in \mathbf{N}$ and $i \in \overline{\mathbf{N}}_{n-1}$. We define functions

$$A_i(s,t) = \begin{cases} \prod_{s \leq t_k < t} a_{(i)k} & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases} \tag{11}$$

and

$$A(s,t) = \begin{cases} \prod_{s \leq t_k < t} a_k & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset, \end{cases}$$

$$B(s,t) = \begin{cases} \prod_{s \leq t_k < t} b_k & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases}$$

and

$$B^*(s,t) = \begin{cases} \prod_{s \leq t_k < t} b_k^* & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases}$$

for $t \geq s \geq 0$ and $i \in \overline{\mathbf{N}}_{n-2}$. Assume that there exist $m_i > 0$ for $i \in \mathbf{N}_{n-2}$ such that

$$A_i(s,t) \geq m_i \quad \text{for } t \geq s \geq 0 \text{ and } i \in \mathbf{N}_{n-2}. \tag{A}$$

We remark that oscillatory, nonoscillatory, monotone and periodic solutions, etc., are major concerns in the theory of impulsive differential equations with delays. Yet simple questions such as the uniqueness of eventually positive solutions are difficult due to the nonlinear nature caused by the impulses. Therefore, much has to be done before the solution structures of these equations can be clarified.

2. Main results. We first establish a necessary condition for the existence of eventually positive and monotone solutions of systems with nonlinear impulses.

Theorem 2.1. *Let $\beta > \alpha > 0$ be given. Assume that* (A1)–(A7) *hold and $I_{(n-1)k}(\mu)/\mu \leq b_k^*$ for $\mu \neq 0$ and $k \in \mathbf{N}$, and that there is a* *function* $\Phi \in PC([0,\infty),\mathbf{R})$ *such that*

$$F(t,\mu_1,\mu_2,\ldots,\mu_n) \geq \Phi(t) \geq 0 \tag{12}$$

where $\beta \geq \mu_n \geq \alpha$, $(-1)^{i+1}\mu_i \geq 0$ *and* $t \geq 0$ *for* $i \in \mathbf{N}_{n-1}$. *Let* $x(t)$ *be a solution of the system*

(13)

$$\left(r(t)x^{(n-1)}(t)\right)' + F\left(t, x^{(n-1)}(g_{n-1}(t)), x^{(n-2)}(g_{n-2}(t)), \ldots, x(g_0(t))\right) \leq 0,$$
$$t \in \mathbf{R}_0 \backslash \Upsilon,$$

and (10) *such that* $\alpha \leq x(t) \leq \beta$ *eventually. If one of the following conditions holds*

(i) n *is even and* $(-1)^{i+1}x^{(i)}(t) > 0$ *eventually for* $i \in \mathbf{N}_{n-1}$; *or*
(ii) n *is odd and* $(-1)^{i}x^{(i)}(t) > 0$ *eventually for* $i \in \mathbf{N}_{n-1}$;

then

$$\int_{\tau}^{\infty}\int_{s_0}^{\infty}\int_{s_1}^{\infty} \cdots \int_{s_{n-3}}^{\infty}\int_{s_{n-2}}^{\infty} \frac{\Phi(s_{n-1})}{r(s_{n-2})B^*(s_{n-2},s_{n-1})}\, ds_{n-1}ds_{n-2}\cdots ds_0 < \infty \tag{14}$$

for some $\tau \geq 0$.

Proof. Assume that condition (i) holds. Without loss of generality, we can assume that $\alpha \leq x(t) \leq \beta$ and $(-1)^{i+1}x^{(i)}(t) > 0$ for $t \geq r_0$. Then we have

$$F\left(t, x^{(n-1)}(g_{n-1}(t)), x^{(n-2)}(g_{n-2}(t)), \ldots, x(g_0(t))\right) \geq \Phi(t) \tag{15}$$

for $t \geq 0$. We define functions

$$A_{(i)}(x)(s,t) = \begin{cases} \prod_{s \leq t_k < t} \frac{I_{(i)k}(x^{(i)}(t_k))}{x^{(i)}(t_k)} & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases} \tag{16}$$

for $t \geq s \geq 0$ and $i \in \overline{\mathbf{N}}_{n-1}$. By (A6), (A7) and $I_{(n-1)k}(\mu)/\mu \leq b_k^*$ for $\mu \neq 0$ and $k \in N$, we see that

$$\begin{aligned} &A_{(0)}(x)(s,t) \geq m_0, \\ &0 < A_{(i)}(x)(s,t) \leq M_i \end{aligned} \tag{17}$$

and

$$0 < A_{(n-1)}(x)(s,t) \le B^*(s,t)$$

for $t \ge s \ge 0$ and $i \in \mathbf{N}_{n-2}$. Let

$$z(t) = \frac{r(t)x^{(n-1)}(t)}{x(t)}$$

for $t \ge 0$. By assumption, we see that $z(t) > 0$ for $t \ge 0$, and

$$z(t_k^+) = \frac{r(t_k)I_{(n-1)k}(x^{(n-1)}(t_k))}{I_{(0)k}(x(t_k))} = c_k z(t_k) \tag{18}$$

for $k \in \mathbf{N}$, where

$$c_k = \frac{x(t_k)}{I_{(0)k}(x(t_k))} \frac{I_{(n-1)k}(x^{(n-1)}(t_k))}{x^{(n-1)}(t_k)}.$$

By (15), we see that

$$z'(t) \le \frac{-F\big(t, x^{(n-1)}(g_{n-1}(t)), x^{(n-2)}(g_{n-2}(t)), \ldots, x(g_0(t))\big)}{x(t)} \le -\frac{\Phi(t)}{\beta} \tag{19}$$

for $t \ge 0$. Let

$$C(s,t) = \begin{cases} \prod_{s \le t_k < t} c_k & \text{if } [s,t) \cap \Upsilon \ne \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases}$$

for $t \ge s \ge 0$. We divide (19) by $C(0,t)$. Then

$$\left(\frac{z(t)}{C(0,t)}\right)' \le -\frac{1}{\beta}\frac{\Phi(t)}{C(0,t)} \tag{20}$$

for $t \ge 0$. Let $d > 0$ be given. In view of (18), we note that the function $z(t)/C(0,t)$ is continuous for $t \ge 0$. We integrate (20) from t to d. Then

$$-\frac{z(t)}{C(0,t)} \le \frac{z(d)}{C(0,d)} - \frac{z(t)}{C(0,t)} \le -\frac{1}{\beta}\int_t^d \frac{\Phi(s)}{C(0,s)}\,ds$$

for $t \ge 0$. Since d is arbitrary, we have

$$\frac{r(t)x^{(n-1)}(t)}{\alpha} \ge z(t) \ge \frac{1}{\beta}\int_t^\infty \frac{\Phi(s)}{C(t,s)}\,ds$$

for $t \geq 0$, from which it follows that

$$x^{(n-1)}(t) \geq \frac{\alpha}{\beta r(t)} \int_t^\infty \frac{\Phi(s)}{C(t,s)}\,ds \tag{21}$$

for $t \geq 0$. We divide (21) by $A_{n-2}(x)(0,t)$ and then integrate it from t to d. We may see that

$$\begin{aligned} -x^{(n-2)}(t) &\geq x^{(n-2)}(d) - x^{(n-2)}(t) \\ &\geq \frac{\alpha}{\beta}\frac{1}{M_{n-2}} \int_t^d \frac{1}{r(s_{n-2})} \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{C(s_{n-2},s_{n-1})}\,ds_{n-1}\,ds_{n-2} \end{aligned}$$

for $0 \leq t \leq d$. Since d is arbitrary, we further see by (17) that

$$-x^{(n-2)}(t) \geq \frac{\alpha}{\beta}\frac{1}{M_{n-2}} \int_t^\infty \frac{1}{r(s_{n-2})} \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{C(s_{n-2},s_{n-1})}\,ds_{n-1}\,ds_{n-2}$$

for $t \geq 0$. We use similar arguments $n-2$ times. Then

$$\begin{aligned} x(t) \geq{}& \frac{\alpha}{\beta}\Big(\prod_{i=1}^{n-2}\tfrac{1}{M_i}\Big) m_0 \textstyle\int_0^t \int_{s_0}^\infty \int_{s_1}^\infty \\ &\cdots \int_{s_{n-3}}^\infty \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{r(s_{n-2})C(s_{n-2},s_{n-1})}\,ds_{n-1}\,ds_{n-2}\cdots ds_0 \\ \geq{}& \frac{\alpha}{\beta}\Big(\prod_{i=1}^{n-2}\tfrac{1}{M_i}\Big) m_0^2 \textstyle\int_0^t \int_{s_0}^\infty \int_{s_1}^\infty \\ &\cdots \int_{s_{n-3}}^\infty \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{r(s_{n-2})B^*(s_{n-2},s_{n-1})}\,ds_{n-1}\,ds_{n-2}\cdots ds_0 \end{aligned}$$

for $t \geq 0$. Since $x(t)$ is bounded, we see that (14) holds.

Assume that condition (ii) holds. Without loss of generality, we may assume that $\alpha \leq x(t) \leq \beta$ and $(-1)^i x^{(i)}(t) > 0$ for $t \geq r_0$. Similarly, we have (15), (16) and (17). In view of (13) and (15),

$$\Big(r(t)x^{(n-1)}(t)\Big)' \leq -\Phi(t) \quad \text{for } t \geq 0. \tag{22}$$

Let $d > 0$ be given. We divide (22) by $A_{n-1}(x)(0,t)$, and then integrate it from t to d. We have

$$\frac{r(d)x^{(n-1)}(d)}{A_{(n-1)}(x)(0,d)} - \frac{r(t)x^{(n-1)}(t)}{A_{(n-1)}(x)(0,t)} \leq -\int_t^d \frac{\Phi(s_{n-1})}{A_{(n-1)}(x)(0,s_{n-1})}\,ds_{n-1}$$

for $d \geq t \geq 0$. Since $x^{(n-1)}(d) > 0$ and d is arbitrary, we can see that

$$\begin{aligned} x^{(n-1)}(t) &\geq \frac{1}{r(t)} \int_t^\infty \frac{\Phi(s_{n-1})}{A_{(n-1)}(x)(t, s_{n-1})}\, ds_{n-1} \\ &\geq \frac{1}{r(t)} \int_t^\infty \frac{\Phi(s_{n-1})}{B^*(t, s_{n-1})}\, ds_{n-1} \end{aligned} \tag{23}$$

for $t \geq 0$.

We divide (23) by $A_{(n-2)}(0, t)$, and then integrate it from t to d. Since $x^{(n-2)}(d) < 0$ and d is arbitrary, we can see that

$$\begin{aligned} -x^{(n-2)}(t) \geq \int_t^\infty & \frac{1}{A_{(n-2)}(t, s_{n-2}) r(s_{n-2})} \\ & \times \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{B^*(s_{n-2}, s_{n-1})}\, ds_{n-1}\, ds_{n-2} \end{aligned}$$

for $t \geq 0$, from which and (17), it follows that

$$-x^{(n-2)}(t) \geq \frac{1}{M_{n-2}} \int_t^\infty \frac{1}{r(s_{n-2})} \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{B^*(s_{n-2}, s_{n-1})}\, ds_{n-1}\, ds_{n-2}$$

for $t \geq 0$. We use similar arguments $n-2$ times. Then

$$\begin{aligned} x(t) \geq & \left(\prod_{i=1}^{n-2} \frac{1}{M_i} \right) m_0 \int_0^t \int_{s_0}^\infty \\ & \cdots \int_{s_{n-3}}^\infty \int_{s_{n-2}}^\infty \frac{\Phi(s_{n-1})}{r(s_{n-2}) B^*(s_{n-2}, s_{n-1})}\, ds_{n-1}\, ds_{n-2} \cdots ds_0 \end{aligned}$$

for $t \geq 0$. Since $x(t)$ is bounded, we see that (14) holds. The proof is complete. □

Remark 2.2. We will establish sufficient conditions such that conditions (i) and (ii) in Theorem 2.1 hold. This is important to show that Theorem 2.1 is non-vacuous. In addition, by inspecting the proof of Theorem 2.1, we may see that condition (A6) in the hypotheses of Theorem 2.1 may be replaced by

$$\prod_{s \leq t_k < t} \frac{I_{(0)k}(\delta_k)}{\delta_k} \geq m_0$$

where $k \in \mathbf{N}$ and $t \geq s \geq 0$ with $[s,t) \cap \Upsilon \neq \emptyset$ for any sequence $\{\delta_k \neq 0\}_{k\in\mathbf{N}}$.

Next, we establish a sufficient condition for the existence of eventually positive and monotone solutions of systems with linear impulses.

Theorem 2.3. *Assume that* (A1)–(A6) *and* (A) *hold,* $I_{(i)k}(\mu) = a_{(i)k}\mu$ *for* $\mu \in \mathbf{R}$, $i \in \overline{\mathbf{N}}_{n-1}$ *and* $k \in \mathbf{N}$, *and that there exist positive numbers* α, β *and* γ_i *for* $i \in \mathbf{N}_{n-1}$ *with* $\beta > M_0/(m_0\alpha)$. *Assume that*

$$\tag{24} \int_\tau^\infty \int_{s_0}^\infty \int_{s_1}^\infty \cdots \int_{s_{n-3}}^\infty \int_{s_{n-2}}^\infty \frac{Q(s_{n-1})}{r(s_{n-2})A_{n-1}(s_{n-2},s_{n-1})}\, ds_{n-1}\, ds_{n-2} \cdots ds_0 < \infty$$

for some $\tau \geq 0$ *where*

$$Q(t) = \max\big\{F(t,\gamma_1,-\gamma_2,\ldots,(-1)^{n-2}\gamma_{n-1},\beta), \\ F(t,\gamma_1,-\gamma_2,\ldots,(-1)^{n-2}\gamma_{n-1},\alpha)\big\}.$$

(1) *If* n *is even and one of the following conditions holds*:
(1-1) *for any* $t \geq 0$, $F(t,\mu_1,\mu_2,\ldots,\mu_n) \geq F(t,\nu_1,\nu_2,\ldots,\nu_n) \geq 0$ *where* $\beta \geq \mu_n \geq \nu_n \geq \alpha$ *and* $\gamma_i \geq (-1)^{n-i+1}\mu_i \geq (-1)^{n-i+1}\nu_i \geq 0$ *for* $i \in \mathbf{N}_{n-1}$; *or*
(1-2) *for any* $t \geq 0$, $F(t,\mu_1,\mu_2,\ldots,\mu_n) \geq F(t,\nu_1,\nu_2,\ldots,\nu_n) \geq 0$ *where* $\beta \geq \nu_n \geq \mu_n \geq \alpha$ *and* $\gamma_i \geq (-1)^{n-i+1}\mu_i \geq (-1)^{n-i+1}\nu_i \geq 0$ *for* $i \in \mathbf{N}_{n-1}$;
then the system (9)–(10) *has an eventually positive solution* $x(t)$ *such that* $\alpha \leq x(t) \leq \beta$ *and* $0 \leq (-1)^{i+1}x^{(i)}(t) \leq \gamma_i$ *eventually for* $i \in \mathbf{N}_{n-1}$.
(2) *If* n *is odd and one of the following conditions holds*:
(2-1) *for any* $t \geq 0$, $F(t,\mu_1,\mu_2,\ldots,\mu_n) \geq F(t,\nu_1,\nu_2,\ldots,\nu_n) \geq 0$ *where* $\beta \geq \mu_n \geq \nu_n \geq \alpha$, *and* $\gamma_i \geq (-1)^{n-i}\mu_i \geq (-1)^{n-i}\nu_i \geq 0$ *for* $i \in \mathbf{N}_{n-1}$; *or*
(2-2) *for any* $t \geq 0$, $F(t,\mu_1,\mu_2,\ldots,\mu_n) \geq F(t,\nu_1,\nu_2,\ldots,\nu_n) \geq 0$ *where* $\beta \geq \nu_n \geq \mu_n \geq \alpha$ *and* $\gamma_i \geq (-1)^{n-i}\mu_i \geq (-1)^{n-i}\nu_i \geq 0$ *for* $i \in \mathbf{N}_{n-1}$;

then system (9)–(10) *has an eventually positive solution* $x(t)$ *such that* $\alpha \le x(t) \le \beta$ *and* $0 \le (-1)^i x^{(i)}(t) \le \gamma_i$ *eventually for* $i \in \mathbf{N}_{n-1}$

Furthermore, if

$$F(t, \mu_1, \mu_2, \dots, \mu_n) > 0, \quad \textit{almost everywhere } t \ge 0, \tag{25}$$

where $\gamma_i \ge (-1)^{i+1}\mu_i \ge 0$ *for* $i \in \mathbf{N}_{n-1}$ *and* $\beta \ge \mu_n > 0$, *then* $x^{(i)}(t) \ne 0$ *eventually for* $i \in \mathbf{N}_{n-1}$.

Proof. For the sake of convenience, let

$$\Phi_1(t) = \int_t^\infty \frac{Q(s)}{r(t)A_{n-1}(t,s)}\, ds$$

for $t \ge 0$. Let $\eta = \beta/M_0 - \alpha/m_0$,

$$\varepsilon_0 = \min\left\{\Big(\prod_{1\le i\le n-2} m_i\Big)\eta,\ m_0\Big(\prod_{0\le i\le n-2} m_i\Big)\eta,\right.$$
$$\left.\Big(M_0 \prod_{0\le i\le n-2} m_i\Big)\eta,\ \Big(\frac{1}{M_0}\prod_{0\le i\le n-2} m_i\Big)\eta\right\}$$

$$\varepsilon_k = \prod_{k\le i\le n-2} m_i \gamma_{n-k}, \quad k \in \mathbf{N}_{n-2},$$

and

$$\varepsilon_{n-1} = \gamma_1.$$

In view of (24), there exists $T \in \Upsilon$ such that $T > \tau$,

$$\int_T^\infty \int_{s_0}^\infty \int_{s_1}^\infty \cdots \int_{s_{n-3}}^\infty \Phi_1(s_{n-2})\, ds_{n-2} \cdots ds_0 < \varepsilon_0, \tag{26}$$

$$\int_T^\infty \int_{s_k}^\infty \cdots \int_{s_{n-3}}^\infty \Phi_1(s_{n-2})\, ds_{n-2} \cdots ds_0 < \varepsilon_k, \quad k \in \mathbf{N}_{n-3}, \tag{27}$$

$$\int_T^\infty \Phi_1(s_{n-2})\, ds_{n-2} < \varepsilon_{n-2}, \tag{28}$$

and

$$\Phi_1(t) < \varepsilon_{n-1}, \quad t \ge T. \tag{29}$$

Given $y \in PC([T,\infty),[0,\infty))$. Let $P_{n-1}(y)(t) = y(t)$ for $t \geq T$. By induction, we can define

$$P_i(y)(t) = \int_t^\infty \frac{P_{i+1}(y)(s)}{A_i(t,s)}\,ds \tag{30}$$

for $t \geq T$ and $i \in \mathbf{N}_{n-2}$. Let

$$\begin{aligned} U_1(t) &= A_0(T,t)\frac{\alpha}{m_0} + \int_T^t A_0(s,t)P_1(y)(s)\,ds,\\ U_2(t) &= A_0(T,t)\frac{\beta}{M_0} - \int_t^\infty \frac{P_1(y)(s)}{A_0(t,s)}\,ds,\\ U_3(t) &= A_0(T,t)\frac{\alpha}{m_0} + \int_t^\infty \frac{P_1(y)(s)}{A_0(t,s)}\,ds, \end{aligned} \tag{31}$$

and

$$U_4(t) = A_0(T,t)\frac{\beta}{M_0} - \int_T^t A_0(s,t)P_1(y)(s)\,ds$$

for $t \geq T$ and $t_k \geq T$. For any $y \in PC([T,\infty),[0,\infty))$, we note that $P_i(y)(t) \geq 0$ and $U_j(t) \geq 0$ for $t \geq T$, $i \in \mathbf{N}_{n-1}$ and $j \in \mathbf{N}_2$. Let us define four sets

$$\begin{aligned} X_j = \{y \in PC([T,\infty),[0,\infty)) : 0 \leq P_i(y)(t) \leq \gamma_{n-i}&\\ \text{and } \alpha \leq U_j(y)(t) \leq \beta \text{ for } t \geq T \text{ and } i \in \mathbf{N}_{n-1}\}& \end{aligned}$$

for $j \in \mathbf{N}_4$. In view of (A6), we can see that $P_i(0)(t) = 0$ and $\alpha \leq U_j(0)(t) \leq \beta$ for $t \geq T$, $i \in \mathbf{N}_{n-1}$ and $j \in \mathbf{N}_4$. It follows that X_j is nonempty because of $0 \in X_j$ for $j \in \mathbf{N}_4$. Clearly, with this ordering, X_j are complete lattices for $j \in \mathbf{N}_4$. For $y \in PC([T,\infty),[0,\infty))$, we define an operator S in $PC([T,\infty),[0,\infty))$ by

$$S(y)(t) = \frac{1}{r(t)}\int_t^\infty \frac{F(s, w_{n-1}(y)(s), w_{n-2}(y)(s), \dots w_1(y)(s), w_0(y)(s))}{A_{n-1}(t,s)}\,ds$$

for $t \geq T$ where $w_i(y)(t)$ are functions on $[r_T,\infty)$ with respect to $y(t)$. We consider four cases: Case 1: n is even and condition (1-1) holds; Case 2: n is even and condition (1-2) holds; Case 3: n is odd and condition (2-1) holds; Case 4: n is odd and condition (2-2) holds.

Case 1. We assume that

$$w_0(y)(t) = \begin{cases} U_1(y)(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \le T, \end{cases}$$

and

$$w_i(y)(t) = \begin{cases} (-1)^{i+1} P_i(y)(g_i(t)) & \text{if } g_i(t) > T, \\ 0 & \text{if } g_i(t) \le T, \end{cases} \tag{32}$$

for $t \ge T$, $i \in \mathbf{N}_{n-1}$ and $y \in X_1$. For any $y \in X_1$, it is obvious that

$$0 \le (-1)^{i+1} w_i(y)(t) \le \gamma_{n-i} \quad \text{and} \quad \alpha \le w_0(y)(t) \le \beta \tag{33}$$

where $t \ge T$ for $i \in \mathbf{N}_{n-1}$. Then

$$\begin{aligned} F(t, w_{n-1}(y)(t), w_{n-2}(y)(t), \dots, w_0(y)(t)) \\ \le F(t, \gamma_1, -\gamma_2, \cdots, \gamma_{n-1}, \beta) \le Q(t) \end{aligned}$$

where $t \ge T$. By (A), (27), (28) and (29), we see that

$$0 \le P_{n-1}(S(y))(t) = S(y)(t) \le \Phi_1(t) \le \gamma_1,$$

$$\begin{aligned} 0 &\le P_{n-2}(S(y))(t) \\ &= \int_t^\infty \frac{P_{n-1}\left(S(y)\right)(s)}{A_{n-2}(t,s)}\, ds \\ &\le \frac{1}{m_{n-2}} \int_t^\infty \Phi_1(s)\, ds \le \gamma_2 \end{aligned}$$

and

$$0 \le P_i(S(y))(t) \le \gamma_{n-i}, \quad i \in \mathbf{N}_{n-3},$$

where $t \ge T$. By (A6), (A) and (26), we further see that

$$\begin{aligned} \alpha \le U_1(S(y))(t) &\le M_0 \left(\frac{\alpha}{m_0} + \int_T^t P_1(S_1(y))(s)\, ds \right) \\ &\le \frac{M_0 \alpha}{m_0} + \left(M_0 \prod_{1 \le i \le n-2} \frac{1}{m_i} \right) \int_0^\infty \int_{s_0}^\infty \\ &\quad \cdots \int_{s_{n-3}}^\infty \Phi_1(s_{n-2}) ds_{n-2} \cdots ds_0 \\ &\le \frac{M_0 \alpha}{m_0} + \left(M_0 \prod_{1 \le i \le n-2} \frac{1}{m_i} \right) \varepsilon_0 \le \beta \end{aligned}$$

for $t \geq T$. So $S(X_1) \subseteq X_1$. Given $y_1, y_2 \in X_1$ with $y_1 \leq y_2$. By induction, it is obvious that $U_1(y_1) \leq U_1(y_2)$ and $P_i(y_1) \leq P_i(y_2)$ for any $i \in \overline{\mathbf{N}}_{n-1}$. So

$$0 \leq (-1)^{i+1} w_i(y_1)(t) \leq (-1)^{i+1} w_i(y_2)(t) \leq \gamma_i$$

and

$$\alpha \leq w_0(y_1)(t) \leq w_0(y_2)(t) \leq \beta$$

where $t \geq T$ for $i \in \mathbf{N}_{n-1}$, from which it follows that

$$\begin{aligned} &F(t, w_{n-1}(y_1)(t), w_{n-2}(y_1)(t), \ldots, w_0(y_1)(t)) \\ &\qquad\qquad \leq F(t, w_{n-1}(y_2)(t), w_{n-2}(y_2)(t), \ldots, w_0(y_2)(t)) \end{aligned}$$

for $t \geq T$. Then $S(y_1) \leq S(y_2)$, which implies that S is increasing in X_1. By the Knaster-Tarski fixed point theorem, there exists $z_1 \in X_1$ such that $S(z_1) = z_1$. Let

$$x_1(t) = \begin{cases} U_1(z_1)(t) & \text{if } t > T, \\ \alpha & \text{if } r_T \leq t \leq T. \end{cases}$$

We assert that x_1 is an eventually positive solution of system (9)–(10) such that $\alpha \leq x_1(t) \leq \beta$ and $0 \leq (-1)^{i+1} x_1^{(i)}(t) \leq \gamma_i$ eventually for $i \in \mathbf{N}_{n-1}$. Indeed, let $T_1 > T$ be such that $r_{T_1} > T$. Then $g_j(t) > T$ for $t \geq T_1$ and $j \in \overline{\mathbf{N}}_{n-1}$. Clearly, $x_1 \in PC^{(n)}([T_1, \infty), [\alpha, \beta])$ and

$$0 \leq (-1)^{i+1} x_1^{(i)}(t) = P_i(z_1)(t) \leq \gamma_{n-i}$$

for $t \geq T_1$ and $i \in \mathbf{N}_{n-1}$. Furthermore,

$$x_1^{(i)}(g_i(t)) = (-1)^{i+1} P_i(z_1)(g_i(t)) = w_i(z_1)(t)$$

where $t \geq T_1$ for $i \in \mathbf{N}_{n-1}$. Then

(34)
$$x_1^{(n-1)}(t) = z_1(t) = \frac{1}{r(t)} \int_t^\infty \frac{F\Big(s, x_1^{(n-1)}(g_{n-1}(s)), x_1^{(n-2)}(g_{n-2}(s)), \ldots, x_1(g_0(s))\Big)}{A_{n-1}(t,s)}\, ds$$

for $t \geq T_1$. It follows that

$$\Big(r(t) x_1^{(n-1)}(t)\Big)' = -F\Big(t, x_1^{(n-1)}(g_{n-1}(t)), x_1^{(n-2)}(g_{n-2}(t)), \ldots, x_1(g_0(t))\Big)$$

for almost every $t \geq T_1$. In view of (30) and (31), it is easy to see that

$$x_1^{(i)}(t_k^+) = (-1)^{i+1} P_i(z_1)(t_k^+) = a_{(i)k}(-1)^{i+1} P_i(z_1)(t_k) = a_{(i)k} x_1^{(i)}(t_k)$$

and

$$x_1(t_k^+) = U_1(z_1)(t_k^+) = a_{(0)k} U_1(z_1)(t_k) = a_{(0)k} x_1(t_k^+)$$

for $t_k \geq T_1$ and $i \in \mathbf{N}_{n-1}$. Therefore, $x_1(t)$ is an eventually positive solution of system (9)–(10) such that $\alpha \leq x(t) \leq \beta$ and $0 \leq (-1)^{i+1} x^{(i)}(t) \leq \gamma_i$ eventually for $i \in \mathbf{N}_{n-1}$. Assume that (25) holds. Since $x_1(g_0(t)) > 0$ for $t \geq T_1$, by (34), we note that $z_1(t) = S(z_1)(t) > 0$ for $t \geq T_1$. It follows that $(-1)^{i+1} x_1^{(i)}(t) = P_i(z_1)(t) > 0$ for $t \geq T_1$ and $i \in \mathbf{N}_{n-1}$.

Case 2. We assume that

$$w_0(y)(t) = \begin{cases} U_2(y)(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \leq T, \end{cases}$$

and $w_i(y)(t)$ are defined by (32) for $t \geq T$, $i \in \mathbf{N}_{n-1}$ and $y \in X_2$. We note that (33) holds and

$$\begin{aligned} F(t, w_{n-1}(y)(t), w_{n-2}(y)(t), \ldots w_1(y)(t), w_0(y)(t)) \\ \leq F(t, \gamma_1, -\gamma_2, \ldots \gamma_{n-1}, \alpha) \leq Q(t) \end{aligned}$$

for $t \geq T$. Similarly, we have

$$0 \leq P_i(S(y))(t) \leq \gamma_{n-i}$$

and

$$\begin{aligned} \beta \geq U_2(S(y))(t) &\geq \frac{m_0\beta}{M_0} - \frac{1}{m_0} \int_t^\infty P_1(y)(s)\, ds, \\ &\geq \frac{m_0\beta}{M_0} - \Bigg(\prod_{0 \leq i \leq n-2} \frac{1}{m_i} \Bigg) \int_T^\infty \int_{s_0}^\infty \int_{s_1}^\infty \\ &\quad \cdots \int_{s_{n-3}}^\infty \Phi_1(s_{n-2}) ds_{n-2} \cdots ds_0 \geq \alpha \end{aligned}$$

for $t \geq T$ and for $i \in \mathbf{N}_{n-1}$. So $S(X_2) \subseteq X_2$. Given $y_1, y_2 \in X_2$ with $y_1 \leq y_2$. Similarly, we have $P_i(y_1) \leq P_i(y_2)$ and $U_2(y_1) \geq U_2(y_2)$ for any $i \in \mathbf{N}_{n-1}$. It follows that

$$0 \leq (-1)^{i+1} w_i(y_1)(t) \leq (-1)^{i+1} w_i(y_2)(t) \leq \gamma_i$$

and

$$\alpha \le w_0(y_2)(t) \le w_0(y_1)(t) \le \beta$$

where $t \ge T$ for $i \in \mathbf{N}_{n-1}$, from which it follows that $S(y_1)(t) \le S(y_2)(t)$ for $t \ge T$, which implies that S is increasing in X_2. By the Knaster-Tarski fixed point theorem, there exists $z_2 \in X_2$ such that $S(z_2) = z_2$. Let

$$x_2(t) = \begin{cases} U_2(z_2)(t) & \text{if } t > T \\ \alpha & \text{if } r_T \le t \le T. \end{cases}$$

Let $T_1 > T$ be such that $r_{T_1} > T$. Similarly, we can check that $x_2(t)$ is an eventually positive solution of system (9)–(10) such that $\alpha \le x_2(t) \le \beta$ and $0 \le (-1)^{i+1} x_2^{(i)}(t) \le \gamma_i$ eventually for $i \in \mathbf{N}_{n-1}$. Assume that (25) holds. Similar to the previous Case 1, we can see that $(-1)^{i+1} x_2^{(i)}(t) = P_i(z_2)(t) > 0$ for $t \ge T_1$ and $i \in \mathbf{N}_{n-1}$.

Case 3. We assume that

$$w_0(y)(t) = \begin{cases} U_3(y)(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \le T, \end{cases}$$

and

$$w_i(y)(t) = \begin{cases} (-1)^i P_i(y)(g_i(t)) & \text{if } g_i(t) > T, \\ 0 & \text{if } g_i(t) \le T, \end{cases} \tag{35}$$

for $t \ge T$, $i \in \mathbf{N}_{n-1}$ and $y \in X_3$. We note that $0 \le (-1)^i w_i(y)(t) \le \gamma_{n-i}$ and $\alpha \le w_0(y)(t) \le \beta$ for $t \ge T$. Similarly, we have

$$0 \le P_i(S(y))(t) \le \gamma_{n-i} \quad \text{and} \quad \alpha \le U_3(S(y))(t) \le \beta$$

for $t \ge T$ and for $i \in \mathbf{N}_{n-1}$. So $S(X_3) \subseteq X_3$. Given $y_1, y_2 \in X_3$ with $y_1 \le y_2$, similarly, we have $S(y_1)(t) \le S(y_2)(t)$ for $t \ge T$, which implies that S is increasing in X_3. By the Knaster-Tarski fixed point theorem, there exists $z_3 \in X_3$ such that $S(z_3) = z_3$. Let

$$x_3(t) = \begin{cases} U_3(z_3)(t) & \text{if } t > T \\ \alpha & \text{if } r_T \le t \le T. \end{cases}$$

Let $T_1 > T$ be such that $r_{T_1} > T$. Then we can check that $x_3(t)$ is an eventually positive solution of system (9)–(10) such that $\alpha \le x_3(t) \le \beta$ and $0 \le (-1)^i x_3^{(i)}(t) \le \gamma_i$ eventually for $i \in \mathbf{N}_{n-1}$. Assume that (25) holds. Similar to the previous cases, we can see that $(-1)^i x_3^{(i)}(t) > 0$ for $t \geq T_1$ and $i \in \mathbf{N}_{n-1}$.

Case 4. We assume that

$$w_0(y)(t) = \begin{cases} U_4(y)(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \leq T \end{cases}$$

and $w_i(y)(t)$ are defined by (35) for $t \geq T$, $i \in \mathbf{N}_{n-1}$ and $y \in X_4$. Similarly, we may verify that $S(X_4) \subseteq X_4$ and S is increasing in X_4. By the Knaster-Tarski fixed point theorem, there exists $z_4 \in X_4$ such that $S(z_4) = z_4$. Let

$$x_4(t) = \begin{cases} U_4(z_4)(t) & \text{if } t > T \\ \alpha & \text{if } r_T \leq t \leq T. \end{cases}$$

Let $T_1 > T$ be such that $r_{T_1} > T$. Then we may check that $x_4(t)$ is an eventually positive solution of system (9)–(10) such that $\alpha \leq x_4(t) \leq \beta$ and $0 \leq (-1)^i x_4^{(i)}(t) \leq \gamma_i$ eventually for $i \in \mathbf{N}_{n-1}$. Assume that (25) holds. Similar to the previous cases, we may see that $(-1)^i x_4^{(i)}(t) > 0$ for $t \geq T_1$ and $i \in \mathbf{N}_{n-1}$. The proof is complete. □

In general, it is difficult to establish the converse result for a higher order system with nonlinear impulses. However, when $n = 2$, we can utilize a similar technique in the proof of Theorem 2.3 to establish a sufficient condition.

Theorem 2.4. *Assume that* $n = 2$, (A1)–(A6) *hold,* $b_k \leq I_{(1)k}(\mu)/\mu \leq b_k^*$ *for* $\mu \neq 0$ *and* $k \in \mathbf{N}$, *and*

$$\frac{I_{(0)k}(\mu)}{\mu} \leq \frac{I_{(0)k}(\nu)}{\nu} \quad \text{and} \quad \frac{I_{(1)k}(\mu)}{\mu} \geq \frac{I_{(1)k}(\nu)}{\nu} \text{ if } 0 < \mu \leq \nu \tag{36}$$

for any $k \in \mathbf{N}$. *Assume that there are positive numbers* α, β *and* γ *such that* $\beta > M_0/(m_0\alpha)$ *and one of the following conditions holds*:

(i) *for any* $t \geq 0$ *and* $F(t, \mu_1, \mu_2) \geq F(t, \nu_1, \nu_2) \geq 0$ *where* $\beta \geq \mu_2 \geq \nu_2 \geq \alpha$, *and* $\gamma \geq \mu_1 \geq \nu_1 \geq 0$ *for* $i \in \mathbf{N}_{n-1}$; *or*

(ii) $I_{(0)k}(\mu) = a_{(0)k}\mu$ *for* $\mu \in \mathbf{R}$ *and* $k \in \mathbf{N}$, *and for any* $t \geq 0$, $F(t, \mu_1, \mu_2) \geq F(t, \nu_1, \nu_2) \geq 0$ *where* $\beta \geq \nu_2 \geq \mu_1 \geq \alpha$ *and* $\gamma \geq \mu_1 \geq \nu_1 \geq 0$.

If

$$\int_{\tau}^{\infty}\int_{t}^{\infty}\frac{Q(s)}{r(t)B(t,s)}\,ds\,dt<\infty \tag{37}$$

for some $\tau \geq 0$, where

$$Q(t)=\max\left\{F(t,\gamma,\beta),F(t,\gamma,\alpha)\right\},$$

then the system (9)–(10) has an eventually positive solution x such that $\alpha \leq x(t) \leq \beta$ and $0 \leq x'(t) \leq \gamma$ eventually. Furthermore, if

$$F(t,\mu_1,\mu_2)>0, \tag{38}$$

where $\gamma \geq \mu_1 \geq 0$, $\beta \geq \mu_2 > 0$ and almost every $t \geq 0$, then $x'(t) > 0$ eventually.

Proof. Let $\varepsilon_1 = \gamma$ and

$$\varepsilon_0=\min\left\{\frac{\beta}{M_0}-\frac{1}{m_0}\alpha,\frac{m_0}{M_0}\left(m_0\beta-M_0\alpha\right)\right\}.$$

In view of (37), there exists $T \in \Upsilon$ such that $T > \tau$,

$$\int_{t}^{\infty}\frac{Q(s)}{r(t)B(t,s)}\,ds<\varepsilon_1$$

and

$$\int_{t}^{\infty}\int_{s}^{\infty}\frac{Q(v)}{r(s)B(s,v)}dvds<\varepsilon_0 \text{ for } t\geq T. \tag{39}$$

We define a function

$$\Gamma_k(\mu)=\begin{cases}\frac{I_{(1)k}(\mu)}{\mu} & \text{if } \mu\neq 0,\\ b_k^* & \text{if } \mu=0\end{cases} \quad \text{for } \mu\in\mathbf{R}. \tag{40}$$

Clearly, $\Gamma_k(\mu) > 0$ for $\mu \in \mathbf{R}$ and $k \in \mathbf{N}$. Given $y \in PC([T,\infty),[0,\infty))$, let

$$D_1(y)(s,t)=\begin{cases}\prod_{s\leq t_k<t}\Gamma_k\left(y(t_k)\right) & \text{if } [s,t)\cap\Upsilon\neq\emptyset\\ 1 & \text{if } [s,t)\cap\Upsilon=\emptyset\end{cases} \tag{41}$$

for $t \geq s \geq T$. Then $D_1(y)(s,t) > 0$ for $t \geq s \geq T$. For $y \in PC([T,\infty),[0,\infty))$, we define an operator $\widetilde{S}$ in $PC([T,\infty),[0,\infty))$

by

$$\widetilde{S}(y)(t) = \frac{1}{r(t)} \int_t^\infty \frac{F(s, w_1(y)(s), w_0(y)(s))}{D_1(y)(t,s)}\, ds$$

for $t \geq T$ where $w_0(y)(t)$ and $w_1(y)(t)$ are functions on $[r_T, \infty)$. Assume that condition (i) holds. By induction, we can define

$$v_1(y)(k) = D_0(y)(T, t_k)\frac{\alpha}{m_0} + \int_T^{t_k} D_0(y)(s, t_k) y(s)\, ds$$

and

$$D_0(y)(s,t) = \begin{cases} \prod_{s \leq t_k < t} \frac{I_{(0)k}(v_1(y)(k))}{v_1(y)(k)} & \text{if } [s,t) \cap \Upsilon \neq \emptyset, \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases}$$

where $t \geq s \geq T$ and $t_k \geq T$ for $y \in PC([T,\infty),[0,\infty))$. Let

$$V_1(y)(t) = D_0(y)(T,t)\frac{\alpha}{m_0} + \int_T^t D_0(y)(s,t) y(s)\, ds,$$

where $t \geq T$ for $y \in PC([T,\infty),[0,\infty))$. For any $y \in PC([T,\infty),[0,\infty))$, we note that $v_1(y)(k) > 0$ for $t_k \geq T$. So $D_0(y)(s,t)$ is well-defined. We further note that $V_1(y)(t_k) = v_1(y)(k)$. Let us define

$$Y_1 = \{y \in PC([T,\infty),[0,\infty)) : 0 \leq y(t) \leq \gamma \\ \text{and } \alpha \leq V_1(y)(t) \leq \beta \text{ for } t \geq T\}.$$

In view of (A6), we see that $\alpha \leq V_1(0)(t) \leq \beta$ for $t \geq 0$, which implies $0 \in Y_1$. So Y_1 is nonempty. Clearly, with this ordering, Y_1 is a complete lattice. Let

$$w_0(y)(t) = \begin{cases} y(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \leq T, \end{cases}$$

and

$$w_1(y)(t) = \begin{cases} V_1(y)(g_1(t)) & \text{if } g_1(t) > T, \\ 0 & \text{if } g_1(t) \leq T, \end{cases}$$

for $t \geq r_T$. In order to use the Knaster-Tarski fixed point theorem, we need to show that $\widetilde{S}(Y_1) \subseteq Y_1$ and $\widetilde{S}$ is increasing in Y_1. For any $y \in PC([T,\infty),[0,\infty))$, by (A6) and assumption, we note that $D_0(y)(s,t) \leq M_0$ and $I_{(1)k}(y(t_k)) \geq b_k y(t_k)$ where $t \geq s \geq T$ and

$t_k \geq T$. It follows that $D_1(y)(s,t) \geq B(s,t)$ where $t \geq s \geq T$ for $y \in PC([T,\infty),[0,\infty))$. Given $y \in Y_1$, it is obvious that

$$0 \leq w_1(y)(t) \leq \gamma \quad \text{and} \quad \alpha \leq w_0(y)(t) \leq \beta$$

where $t \geq T$. We have

$$F(t, w_1(y)(t), w_0(y)(t)) \leq F(t,\gamma,\beta) \leq Q(t)$$

where $t \geq T$. In view of (A6), we may see that

$$0 \leq \widetilde{S}(y)(t) \leq \frac{1}{r(t)} \int_t^\infty \frac{Q(s)}{B(t,s)}\, ds \leq \gamma$$

and

$$\begin{aligned}\alpha \leq V_1(\widetilde{S}(y))(t) &\leq M_0\left(\frac{\alpha}{m_0} + \int_T^t \widetilde{S}(y)(s)\, ds\right)\\ &\leq \frac{M_0\alpha}{m_0} + M_0 \int_T^\infty \int_s^\infty \frac{Q(v)}{r(s)B(s,v)}\, dv\, ds\\ &\leq \frac{M_0\alpha}{m_0} + M_0\varepsilon_0 \leq \beta\end{aligned}$$

for $t \geq T$. So $\widetilde{S}(Y_1) \subseteq Y_1$. Given $y_1, y_2 \in Y_1$ with $y_1 \leq y_2$, by (36) and (40), we see that $D_0(y_1)(s,t) \leq D_0(y_2)(s,t)$ and $D_1(y_1)(s,t) \geq D_1(y_2)(s,t)$ for $t \geq s \geq T$. So $V_1(y_1) \leq V_1(y_2)$,

$$0 \leq w_1(y_1)(t) \leq w_1(y_2)(t) \leq \gamma$$

and

$$\alpha \leq w_0(y_1)(t) \leq w_0(y_2)(t) \leq \beta,$$

where $t \geq T$, from which it follows that

$$F(t, w_1(y_1)(t), w_0(y_1)(t)) \leq F(t, w_1(y_2)(t), w_0(y_2)(t))$$

for $t \geq T$. So we see that $\widetilde{S}(y_1) \leq \widetilde{S}(y_2)$, which implies that $\widetilde{S}$ is increasing in Y_1. By the Knaster-Tarski fixed point theorem, there exists $\widetilde{z}_1 \in Y_1$ such that $\widetilde{S}(\widetilde{z}_1) = \widetilde{z}_1$. Let

$$\widetilde{x}_1(t) = \begin{cases} V_1(\widetilde{z}_1)(t) & \text{if } t > T,\\ \alpha & \text{if } r_T \leq t \leq T.\end{cases}$$

We assert that $\widetilde{x}_1$ is an eventually positive solution of system (9)–(10) such that $\alpha \leq \widetilde{x}_1(t) \leq \beta$ and $0 \leq \widetilde{x}_1'(t) \leq \gamma$ eventually. Indeed, let

$T_1 > T$ such that $r_{T_1} > T$. Clearly, $\widetilde{x}_1 \in PC^{(2)}([T_1, \infty), [\alpha, \beta])$ and

$$0 \le \widetilde{x}_1'(t) = \widetilde{z}_1(t) \le \gamma$$

for $t \ge T_1$. Furthermore,

$$\widetilde{x}_1'(g_1(t)) = \widetilde{z}_1(g_1(t)) = w_1(\widetilde{z}_1)(t)$$

where $t \ge T_1$. Then

$$\widetilde{x}_1'(t) = \widetilde{z}_1(t) = \frac{1}{r(t)} \int_t^\infty \frac{F\left(s, \widetilde{x}_1'(g_1(s)), \widetilde{x}_1(g_0(s))\right)}{D_1(\widetilde{z}_1)(t,s)}\, ds \tag{42}$$

for $t \ge T_1$. It follows that

$$\left(r(t)\widetilde{x}_1'(t)\right)' = -F\left(t, \widetilde{x}_1'(g_1(t)), \widetilde{x}_1(g_0(t))\right)$$

for almost every $t \ge T_1$. We assert that

$$\widetilde{x}_1^{(i)}(t_k^+) = I_{(i)k}(\widetilde{x}_1^{(i)}(t_k)) \quad \text{for } t \ge T_1 \text{ and } i \in \overline{\mathbf{N}}_1.$$

Indeed, we note that

$$\begin{aligned} \widetilde{x}_1(t_k^+) = V_1(\widetilde{z}_1)(t_k^+) &= \frac{I_{(0)k}(v_1(\widetilde{z}_1)(k))}{v_1(\widetilde{z}_1)(k)} V_1(\widetilde{z}_1)(t_k) \\ &= I_{(0)k}(V_0(\widetilde{z}_1)(t_k)) = I_{(0)k}(\widetilde{x}_1(t_k)) \end{aligned} \tag{43}$$

for $t_k \ge T_1$. Given $t_k \ge T_1$. If $\widetilde{z}_1(t_k) = 0$, we see that $\widetilde{x}_1'(t_k) = \widetilde{z}_1(t_k) =$ 0 and

$$\widetilde{x}_1'(t_k^+) = \widetilde{z}_1(t_k^+) = \Gamma_k\left(\widetilde{z}_1(t_k)\right)\widetilde{z}_1(t_k) = 0 = I_k(\widetilde{x}_1'(t_k)).$$

If $\widetilde{z}_1(t_k) = 0$, we see that

$$\widetilde{x}_1'(t_k^+) = \widetilde{z}_1(t_k^+) = \frac{I_{(1)k}(\widetilde{z}_1(t_k))}{\widetilde{z}_1(t_k)}\widetilde{z}_1(t_k) = I_{(1)k}(\widetilde{x}_1'(t_k)).$$

So we have verified our assertion. Therefore, $\widetilde{x}_1(t)$ is an eventually positive solution of system (9)–(10) such that $\alpha \le \widetilde{x}_1(t) \le \beta$ and $0 \le \widetilde{x}_1'(t) \le \gamma$ eventually. Assume that (38) holds. Since $\widetilde{x}_1(g_0(t)) > 0$ for $t \ge T_1$, by (42), we note that $\widetilde{z}_1(t) = \widetilde{S}(\widetilde{z}_1)(t) > 0$ for $t \ge T_1$. It follows that $\widetilde{x}_1'(t) = \widetilde{z}_1(t) > 0$ for $t \ge T_1$.

Assume that condition (ii) holds. By induction, we can define

$$V_2(y)(k) = A_0(T,t)\frac{\beta}{M_0} - \int_t^\infty \frac{y(s)}{A_0(y)(T,t)}\, ds$$

where $t \geq T$ for $y \in PC([T,\infty),[0,\infty))$. Let us define a set

$$Y_2 = \{y \in PC([T,\infty),[0,\infty)) : 0 \leq y(t) \leq \gamma \\ \text{and } \alpha \leq V_2(y)(t) \leq \beta \text{ for } t \geq T\}.$$

In view of (A6), we see that $\alpha \leq V_2(0)(t) \leq \beta$, which implies $0 \in Y_2$. So Y_2 is nonempty. Clearly, with this ordering, Y_2 is a complete lattice. Let

$$w_0(y)(t) = \begin{cases} y(g_0(t)) & \text{if } g_0(t) > T, \\ \alpha & \text{if } g_0(t) \leq T, \end{cases}$$

and

$$w_1(y)(t) = \begin{cases} V_2(y)(g_i(t)) & \text{if } g_1(t) > T, \\ 0 & \text{if } g_1(t) \leq T, \end{cases}$$

for $t \geq r_T$. In order to use the Knaster-Tarski fixed point theorem, we need to show that $\widetilde{S}(Y_2) \subseteq Y_2$ and $\widetilde{S}$ is increasing in Y_2. We have

$$F(t, w_1(y)(t), w_0(y)(t)) \leq F(t,\gamma,\alpha) \leq Q(t),$$

$$0 \leq \widetilde{S}(y)(t) \leq \gamma,$$

and

$$\beta \geq V_2(\widetilde{S}(y))(t) \geq \frac{m_0\beta}{M_0} - \frac{1}{m_0}\int_T^\infty \int_s^\infty \frac{Q(v)}{r(s)B(s,v)}\,dv\,ds \geq \alpha,$$

where $t \geq T$. So $\widetilde{S}(Y_2) \subseteq Y_2$. Given $y_1, y_2 \in Y_2$ with $y_1 \leq y_2$, by (36) and (40), we see that $D_1(y_1)(s,t) \geq D_1(y_2)(s,t) > 0$ for $t \geq s \geq T$. So $V_2(y_1) \geq V_2(y_2)$,

$$0 \leq w_1(y_1)(t) \leq w_1(y_2)(t) \leq \gamma$$

and

$$\alpha \leq w_0(y_2)(t) \leq w_0(y_1)(t) \leq \beta$$

where $t \geq T$, from which it follows that $\widetilde{S}(y_1) \leq \widetilde{S}(y_2)$, which implies that $\widetilde{S}$ is increasing in Y_2. By the Knaster-Tarski fixed point theorem, there exists $\widetilde{z}_2 \in Y_2$ such that $\widetilde{S}(\widetilde{z}_2) = \widetilde{z}_2$. Let

$$\widetilde{x}_2(t) = \begin{cases} V_2(\widetilde{z}_2)(t) & \text{if } t > T, \\ \alpha & \text{if } r_T \leq t \leq T. \end{cases}$$

Similarly, we can verify that $\widetilde{x}_2(t)$ is an eventually positive of system (9)–(10) such that $\alpha \leq \widetilde{x}_2(t) \leq \beta$ and $0 \leq \widetilde{x}_2'(t) \leq \gamma$ eventually. Hence, $x_2'(t) > 0$ eventually if (38) holds. □

By Theorems 2.1 and 2.3, it is easy to see that the condition (14) or (24) is a necessary as well as sufficient condition for special impulsive delay systems. Therefore, our previous results are sharp.

Corollary 2.5. *Assume that* (A1)–(A7) *and* (A) *hold and* $I_{(i)k}(\mu) = a_{(i)k}\mu$ *for* $\mu \in \mathbf{R}$, $i \in \overline{\mathbf{N}}_{n-1}$ *and* $k \in \mathbf{N}$. *Let* $\alpha > 0$, $\beta > 0$ *and* $\gamma_i > 0$ *for* $i \in \mathbf{N}_{n-1}$ *such that* $\beta > M_0/(m_0\alpha)$. *Assume that there exist* $\rho \in PC([0,\infty),[0,\infty))$ *with* $\rho(t) \neq 0$ *for almost every* $t \geq 0$, *and continuous function* f *defined on* $\mathbf{R}^n$ *such that* $f(\mu_1, \mu_2, \ldots, \mu_n) > 0$ *where* $(-1)^{i+1}\mu_i \geq 0$ *and* $\mu_n > 0$ *for* $i \in \mathbf{N}_n$ *such that*

$$F(t, \mu_1, \mu_2, \ldots, \mu_n) = \rho(t) f(\mu_1, \mu_2, \ldots, \mu_n) \qquad \text{where } t \geq 0 \text{ and } \mu_i \in \mathbf{R} \text{ for } i \in \mathbf{N}_n.$$

(i) *If* n *is even and one of the conditions* (1-1) *or* (1-2) *in Theorem* 2.3 *holds, then*

$$\int_\tau^\infty \int_{s_0}^\infty \int_{s_1}^\infty \cdots \int_{s_{n-3}}^\infty \int_{s_{n-2}}^\infty \frac{\rho(s_{n-1})}{r(s_{n-2})A_{n-1}(s_{n-2}, s_{n-1})}\, ds_{n-1}\, ds_{n-2} \cdots ds_0 < \infty \tag{44}$$

for some $\tau \geq 0$ *if, and only if, the system* (9)–(10) *has an eventually positive solution* $x(t)$ *such that* $\alpha \leq x(t) \leq \beta$ *and* $0 < (-1)^{i+1}x^{(i)}(t) \leq \gamma_i$ *eventually for* $i \in \mathbf{N}_{n-1}$.

(ii) *If* n *is odd and one of the conditions* (2-1) *or* (2-2) *in Theorem* 2.3 *holds, then* (44) *holds if, and only if, the system* (9)–(10) *has an eventually positive solution* $x(t)$ *such that* $\alpha \leq x(t) \leq \beta$ *and* $0 < (-1)^i x^{(i)}(t) \leq \gamma_i$ *eventually for* $i \in \mathbf{N}_{n-1}$.

Similarly, by Theorems 2.1 and 2.4, we have the following conclusion.

Corollary 2.6. *Assume that* $n = 2$ *and that* (A1)–(A6) *and* (36) *hold,* $b_k \leq I_{(1)k}(\mu)/\mu \leq b_k^*$ *for* $\mu \neq 0$ *and* $k \in \mathbf{N}$, *and* $\delta B(t,s) \geq B^*(t,s)$ *where* $t \geq s \geq 0$ *for some* $\delta > 0$. *Let* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$ *be* *such that* $\beta > M_0/(m_0\alpha)$. *Assume that* $F(t,\mu_1,\mu_2) = \rho(t)f(\mu_1,\mu_2)$ *for* $t \geq 0$, *and* $\mu_1,\mu_2 \in \mathbf{R}$ *where* ρ *and* f *satisfy the following assumptions*:

(i) $\rho \in PC([0,\infty),[0,\infty))$ *such that* $\rho(t) \neq 0$ *for almost every* $t \geq 0$;
(ii) f *is a continuous function on* $\mathbf{R}^2$ *such that* $f(\mu_1,\mu_2) > 0$ *for* $\mu_1 \geq 0$ *and* $\mu_2 > 0$;
(iii) $f(\mu_1,\mu_2) \geq f(\nu_1,\nu_2)$ *for* $\gamma \geq \mu_1 \geq \nu_1 \geq 0$ *and* $\beta \geq \mu_2 \geq \nu_2 \geq \alpha$.

Then

$$\int_\tau^\infty \int_t^\infty \frac{\rho(s)}{r(t)B(t,s)}\,ds\,dt < \infty \tag{45}$$

for some $\tau \geq 0$ *if, and only if, the system* (9)–(10) *has an eventually positive solution* $x(t)$ *such that* $\alpha \leq x(t) \leq \beta$ *and* $0 < x'(t) \leq \gamma$ *eventually.*

As an application, we show that Theorem 2.3 or Theorem 2.4 will yield Theorem 2.4 in **[12]** for $n = 2$. For ease of discussion, we rewrite Theorem 2.4 in **[12]** in the case where $n = 2$ as follows.

Theorem 2.7. (**[12,** Theorem 2.4]). *Assume that* $n = 2$ *and* (D1)–(D5) *hold. If*

$$\int_0^\infty \frac{1}{r(t)} \int_t^\infty \frac{\sum_{i=1}^m r(s)p_i(s)}{A(g_i(s),s)}\,ds\,dt < \infty \tag{46}$$

where

$$r(t) = \exp\bigg(\int_0^t a(s)\,ds\bigg), \tag{47}$$

then system

$$x''(t) + a(t)x'(t) + \sum_{i=1}^m p_i(t)x(g_i(t)) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{48}$$

$$x^{(i)}(t_k^+) = a_k x^{(i)}(t_k), \quad k \in \mathbf{N} \text{ and } i = 0,1, \tag{49}$$

has a bounded nonoscillatory solution $x(t)$ *such that* $|x(t)|$ *has positive lower bound.*

If we multiply (48), where $n = 2$, by $r(t)$, the equation (1) can be transformed into

$$(r(t)x'(t))' + \sum_{j=1}^{m} r(t)p_j(t)x(g_j(t)) = 0, \quad t \in [0, \infty)\backslash\Upsilon. \tag{50}$$

We see easily that the system (48)–(49) is a special case of system (9)–(10). Since condition (D3) holds, there exist $m > 0$ and $M > 0$ such that $m \leq A(0,t) \leq M$ for $t \geq 0$. For any $i \in \mathbf{N}_m$, then

$$0 < \frac{m}{M} \leq A(s,t) = \frac{A(0,t)}{A(0,s)} \leq \frac{M}{m}, \tag{51}$$

for any $t \geq s \geq 0$. Let $\alpha = 1$ and $\beta = M^2/m^2$. We may note that condition (37) (or (24)) is equivalent to the condition

$$\int_\tau^\infty \frac{1}{r(t)} \int_t^\infty \sum_{i=1}^{m} r(s)p_i(s)\, ds\, dt < \infty. \tag{52}$$

In view of (51), condition (46) is equivalent to condition (52). Therefore, we can utilize Theorem 2.3 or Theorem 2.4 to yield Theorem 2.4 in **[12]** in the case $n = 2$.

A partial converse of Theorem 2.4 in **[12]** is obtained by means of Theorem 2.1.

Theorem 2.8. *Assume that* (D1)–(D5) *hold and* $r(t)$ *is defined by* (47). *Let* $\alpha > 0$ *and* $\beta > 0$ *be given, and let* $x(t)$ *be a nonoscillatory solution of system* (48)–(49) *such that* $\alpha \leq |x(t)| \leq \beta$ *and* $x(t)x'(t) > 0$ *eventually. Then condition* (46) *holds.*

Proof. Since system (48)–(49) is linear, we can assume without loss of generality that $x(t)$ is an eventually positive solution. So $\alpha \leq x(t) \leq \beta$ and $x'(t) > 0$ eventually. Assume that $\sum_{i=1}^{m} p_i(t) = 0$ for almost every $t \geq 0$. Then (46) holds. Assume that $\sum_{i=1}^{m} p_i(t) > 0$ for almost every $t \geq 0$. Then

$$r(t)\sum_{i=1}^{m} p_i(t)\mu \geq r(t)\sum_{i=1}^{m} p_i(t)\alpha > 0$$

for almost every $t \geq 0$ and $\mu \in [\alpha, \beta]$. So the condition (12) is satisfied. By Theorem 2.1, (46) holds. $\square$

Remark 2.9. By studying the proof of Theorem 2.4 in [**12**], we may find that an eventually positive and bounded solution $x(t)$ also satisfies $x(t)x'(t) > 0$ eventually. Therefore, our Theorem 2.8 is indeed the true converse of Theorem 2.4 in [**12**] stated for the existence of such a solution.

3. Examples. We illustrate our results by means of several examples.

Example 3.1. Let $\{a_k\}_{k\in\mathbf{N}}$ and $\{b_k\}_{k\in\mathbf{N}}$ be positive sequences such that

$$a_k = b_k = \begin{cases} 0.5 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even} \end{cases} \tag{53}$$

for $k \in \mathbf{N}$. Assume that the functions g_0 and g_1 satisfy condition (A2). Let $d_1 > 0$ and $d_2 > 0$ be given. We consider the second order system

$$\big(e^t x'(t)\big)' + \frac{1}{e^t - 1}\Big(\big(x'(g_1(t))\big)^{d_1} + \big(x(g_0(t))\big)^{d_2}\Big) = 0, \tag{54}$$

$$x(t_k^+) = a_k x(t_k) \quad \text{and} \quad x'(t_k^+) = b_k x'(t_k), \tag{55}$$

where $t \in [0,\infty)\backslash\Upsilon$ and $k \in \mathbf{N}$. Clearly,

$$0.5 \le A(s,t) \le 2 \quad \text{and} \quad \frac{1}{B(s,t)} \le 2$$

for any $t \ge s \ge 0$. We integrate $1/(e^t - 1)$ from t to ∞, and then we get

$$\int_t^\infty \frac{1}{e^s - 1} ds = \int_{e^t}^\infty \frac{1}{u^2 - u} du = \left[\ln\left|\frac{u-1}{u}\right|\right]_{e^t}^\infty = \ln\left|\frac{e^t}{e^t - 1}\right|$$

for $t \ge 0$. Clearly, $\ln|e^t/(e^t - 1)|$ is decreasing for $t > 0$. For any $\tau > 0$, we have

$$\begin{aligned}
&\int_\tau^\infty \frac{1}{e^{s_0}} \int_{s_0}^\infty \frac{1}{B(s_0,s_1)} \frac{1}{e^{s_1} - 1}\, ds_1\, ds_0 \\
&\quad\le 2\int_\tau^\infty \frac{1}{e^{s_0}} \int_{s_0}^\infty \frac{1}{e^{s_1} - 1}\, ds_1\, ds_0 \\
&\quad\le 2\int_\tau^\infty \frac{1}{e^{s_0}} \ln\left|\frac{e^{s_0}}{e^{s_0} - 1}\right| ds_0
\end{aligned}$$

$$\leq 2\ln\left|\frac{e^{\tau}}{e^{\tau}-1}\right|\int_{\tau}^{\infty}\frac{1}{e^{s}}\,ds$$
$$= 2\ln\left|\frac{e^{\tau}}{e^{\tau}-1}\right|e^{-\tau} < \infty.$$

Thus (37) holds. We may take arbitrary $\alpha > 0$, $\beta > 4\alpha$ and $\gamma_1 > 0$. So condition (i) in Theorem 2.4 holds. By Theorem 2.4, we see that the system (54)–(55) has an eventually positive solution $x(t)$ such that $\alpha \leq x(t) \leq \beta$ and $0 < x'(t) \leq \gamma_1$ eventually. If we change the equation (54) into

$$(56)\qquad \left(e^{t}x'(t)\right)' + \frac{1}{e^{t}-1}\left(\left(x'(g_1(t))\right)^{d} + \frac{x(g_0(t))}{x^2(g_0(t))+1}\right) = 0$$

where $d > 0$. We may take $\alpha = 1$, $\beta = 4$ and $\gamma_1 = 1$. So condition (ii) in Theorem 2.4 holds. By Theorem 2.4, we see that the system (56)–(55) has an eventually positive solution $x(t)$ such that $1 \leq x(t) \leq 4$ and $0 < x'(t) \leq 1$ eventually.

Example 3.2. Let $a_{(i)k} > 0$ for $k \in \mathbf{N}$ and $i \in \overline{\mathbf{N}}_2$. Assume that $g_i(t)$ satisfies condition (A2) for $i \in \overline{\mathbf{N}}_2$ and that there exist $M_0 > 0$ and $m_i > 0$ for $i \in \overline{\mathbf{N}}_2$ such that $M_0 \geq A_0(s,t) \geq m_0$ and $A_i(s,t) \geq m_i$ for $i \in \mathbf{N}_2$. We consider the third order system

$$(57)\quad \left(e^{2t}x''(t)\right)' + \frac{\exp\left(x''(g_2(t))\right)}{(1+t)^2} - te^{-t}x'(g_1(t)) + \frac{e^{-t}x(g_0(t))}{x^2(g_0(t))+1} = 0,$$

(58)

$$x(t_k^+) = a_{(0)k}x(t_k),\quad x'(t_k^+) = a_{(1)k}x'(t_k)\quad \text{and}\quad x''(t_k^+) = a_{(2)k}x''(t_k)$$

where $t \in [0,\infty)\backslash\Upsilon$ and $k \in \mathbf{N}$. We let $r(t) = e^{2t}$ and

$$F(t,\mu_1,\mu_2,\mu_3) = \frac{e^{\mu_1}}{(1+t)^2} - te^{-t}\mu_2 + \frac{e^{-t}\mu_3}{\mu_3^2+1}$$

for $t \geq 0$, and $\mu_1,\mu_2,\mu_3 \in \mathbf{R}$. Then

$$\int_0^{\infty}\int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty}\frac{1}{A_2(s,\eta)}\left(\frac{1}{(1+\eta)^2} - \eta e^{-\eta} + e^{-\eta}\right)d\eta\,ds\,dt$$
$$\leq \int_0^{\infty}\int_t^{\infty}e^{-2s}\int_s^{\infty}\left(\frac{1}{(1+\eta)^2} - \eta e^{-\eta} + e^{-\eta}\right)d\eta\,ds\,dt$$

$$= \int_0^\infty \int_t^\infty e^{-2s}\left(\frac{1}{(1+s)} - se^{-s}\right) ds\, dt$$
$$\le \int_0^\infty \int_t^\infty \left(\frac{e^{-2s}}{(1+t)} - se^{-3s}\right) ds\, dt$$
$$= \int_0^\infty \left(\frac{e^{-2t}}{2(1+t)} - \left(\frac{t}{3} - \frac{1}{9}\right)e^{-3t}\right) dt < \infty.$$

We note that the function

$$\frac{\mu}{\mu^2+1}$$

is increasing on $[0,1]$ and is decreasing on $[1,\infty)$. We can take $\beta > 1$ such that $\beta > M_0/m_0$. By Theorem 2.3, the system (57)–(58) has eventually positive solutions $x(t)$ such that $1 \le x(t) \le \beta$, $x'(t) < 0$ and $x''(t) > 0$ eventually. If we can choose $\alpha, \beta \in (0,1]$ such that $\beta > M_0/m_0\alpha$, by Theorem 2.3, we can also find an eventually positive solution $\widetilde{x}(t)$ of system (57)–(58) such that $\alpha \le x(t) \le \beta$ eventually.

Example 3.3. Assume that $F(\mu) = -\mathrm{sgn}\,(\mu)e^{\mu}$ for $\mu \in R$. Let $r(t) = \exp(k/(mL))$ for $t \ge 0$. We consider the pendulum equation with impulses

$$(59) \qquad (r(t)\theta'(t))' + r(t)\frac{g}{L}\sin(\theta(t)) + \frac{1}{mL}r(t)F\left(\theta(t-\tau)\right) = 0, \quad t \in \mathbf{R}_0\backslash\mathbf{N},$$

$$(60) \qquad \theta(t_k^+) = \theta(t_k), \quad k \in \mathbf{N},$$

$$(61) \qquad \theta'(t_k^+) = J_k(\theta'(t_k)), \quad k \in \mathbf{N},$$

where

$$J_k(\mu) = \begin{cases} \mathrm{sgn}\,(\mu)\,\{\mu(2-\mu)\} & \text{if } |\mu| \le 1 \\ \mu & \text{if } |\mu| > 1, \end{cases} \qquad k \in \mathbf{N}.$$

We are interested in whether oscillatory motion may disappear (such a disappearance corresponds to an ill-functioned pendulum). We let $b_k = 1$, $b_k^* = 2$, $\Phi_1(t) = 0$, $\Phi_2(t) = r(t)/(mL)$,

$$F_1(t,\mu_1,\mu_2) = \frac{g}{L}r(t)\sin(\mu_2) \text{ and } F_2(t,\mu_1,\mu_2) = \mathrm{sgn}\,(\mu_2)\frac{r(t)}{mL}e^{\mu_2}$$

for $t \ge 0$, $\mu_1, \mu_2 \in R$ and $k \in N$. Then

$$F_1(t,\mu_1,\mu_2) \ge \Phi_1(t) \quad \text{and} \quad F_2(t,\mu_1,\mu_2) \ge \Phi_2(t)$$

for $t \ge 0$, $\mu_1 \ge 0$ and $\pi \ge \mu_2 > 0$. By elementary analysis, we can see that $b_k \le J_k(\mu)/\mu \le b_k^*$ for $\mu \ne 0$ and $k \in N$. So $B^*(t,s) \le 2^{s+1}/2^t$ for $s \ge t \ge 0$. Then

$$\begin{aligned}\int_\varepsilon^\infty \int_t^\infty \frac{1}{B^*(t,s)}\, ds\, dt &\ge \int_\varepsilon^\infty 2^{t-1} \int_t^\infty 2^{-s} ds\, dt \\ &= \frac{1}{2\ln 2}\int_\varepsilon^\infty 1\, dt = \infty,\end{aligned}$$

for any $\varepsilon \ge 0$. It follows that

$$\int_\varepsilon^\infty \int_t^\infty \frac{\Phi_1(s)+\Phi_2(s)}{r(t)B^*(t,s)}\, ds\, dt = \frac{1}{mL}\int_\varepsilon^\infty \int_t^\infty \frac{1}{B^*(t,s)}\, ds\, dt = \infty$$

for any $\varepsilon \ge 0$. By Theorem 2.1, the system (59)–(61) cannot have a solution $\theta(t)$ such that $0 < \theta(t) \le \pi$ and $\theta'(t) < 0$, eventually.

4. Discussion. In this section, we intend to point out the mistakes of Lemmas 2.1 and 2.2 in **[20]**. Because the proof of the main theorem in **[20]** needs these two results, it is reasonable to doubt the correctness of the main theorem in **[20]**. More specifically, in **[20]**, the authors investigated the system

$$x''(t) + r(t)x'(t) + (p(t)-q(t))\,x(t-\tau) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{62}$$

$$x(t_k^+) = I_k(x(t_k)), \quad k \in \mathbf{N}, \tag{63}$$

$$x'(t_k^+) = J_k(x'(t_k)), \quad k \in \mathbf{N}, \tag{64}$$

under the following conditions:

(i) $r(t)$, $p(t)$ and $q(t)$ are continuous function on $[0,\infty)$ such that $r(t) \ge 1$ and $p(t)-q(t) > 0$ for $t \ge 0$. Furthermore, $\inf_{t\ge 0}\{Q(t)\} > 0$ where $Q(t) = \int_{t_j}^t q(s)\, ds$;

(ii) For all $k \in \mathbf{N}$, I_k and J_k are continuous functions on $\mathbf{R}$, and there exist positive numbers a_k, a_k^*, b_k and b_k^* such that $a_k \le I_k(\mu)/\mu \le a_k^*$ and $b_k \le J_k(\mu)/\mu \le b_k^*$ for $\mu \ne 0$; and

(iii)

$$\lim_{n\to\infty} \sum_{m=1}^{n-1} \prod_{k=m}^{n-1} \prod_{l=0}^{m-1} a_k^* b_l \int_{t_{m-1}}^{t_m} \exp\left(\int_{t_0}^{v} r(s)\, ds\right) dv = \infty.$$

We note that there is an obvious mistake in assumption (i). By the definition of the function Q, it is impossible that $\inf_{t\geq 0}\{Q(t)\} > 0$. There are two more mistakes in Lemma 2.1 and Lemma 2.2 in **[20]**. For ease of discussion, we list Lemma 2.1 and Lemma 2.2 in **[20]** as follows.

Lemma 2.1 [20]. *Assume that $x(t)$ is a solution of system* (62)–(64). *Suppose that there exists $T \geq 0$ such that $x(t) > 0$ for $t \geq T$. If* (i), (ii) *and* (iii) *are satisfied, then $x'(t_k) > 0$ and $x'(t) > 0$ where $t \in [t_k, t_{k+1})$ and $k \in \mathbf{N}$.*

Lemma 2.2 [20]. *Let $x(t)$ be a solution of system* (62)–(64), *$b_0 = 1$ and $b_k \leq 1$ for $k \in \mathbf{N}$. Assume that* (i), (ii) *and* (iii) *hold, and that for all $k \in \mathbf{N}$, $x(t) > 0$, and*

$$r(t) + \int_{t_k}^{t} q(s)\,ds + \int_{t_k}^{t} r'(s)\,ds < 1$$

for $t \in (t_k, t_{k+1}]$. Then $Z(t) > 0$ and $Z(t_k^+) \leq b_k Z(t_k)$ for $k \in \mathbf{N}$ where

$$Z(t) = x'(t) - \int_0^t q(s)x(s-\tau)\,ds + \int_0^t r(s)x'(s)\,ds.$$

We give a counterexample to illustrate that Lemma 2.1 and Lemma 2.2 in **[20]** are incorrect. Let $t_k = k/10$ for $k \in \mathbf{N}$. Let $r(t) = 0.8$, $\tau = 1$, $I_k(\mu) = J_k(\mu) = \mu$, $p(t) = \delta + 1$ and $q(t) = 1$ for $t \geq t_0$, where $\delta = 0.15e^{-0.5}$. So system (62)–(64) is:

$$x''(t) + 0.8x'(t) + \delta x(t-1) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{65}$$

$$x(t_k^+) = x(t_k), \quad k \in \mathbf{N}, \tag{66}$$

$$x'(t_k^+) = x'(t_k), \quad k \in \mathbf{N}. \tag{67}$$

Then

$$p(t) - q(t) = \delta \equiv 0.15e^{-0.5} > 0.$$

Let

$$a_k = a_k^* = b_k = b_k^* = 1 \quad \text{for all } k \in \mathbf{N},$$
$$p(t) = \delta + 1 \quad \text{and} \quad q(t) = 1 \text{ for } t \geq t_0.$$

Clearly, conditions (i), (ii) and (iii) hold. We note that $x(t) = e^{-0.5t}$ is a positive solution of (65) and $x'(t) < 0$, eventually. So Lemma 2.1 in [**20**] is not true. For any $k \in \mathbf{N}$,

$$r(t) + \int_{t_k}^{t} q(s)\,ds + \int_{t_k}^{t} r'(s)ds = 0.8 + t - t_k \le 0.9 < 1, \quad t_k < t \le t_{k+1}.$$

So all hypotheses of Lemma 2.2 in [**20**] hold. By definition of $Z(t)$, we see that

$$\begin{aligned} Z(t) &= x'(t) - \int_0^t q(s)x(s-1)\,ds + \int_0^t r(s)x'(s)\,ds \\ &= -0.5e^{-0.5t} - \int_0^t e^{-0.5(s-1)}ds - 0.4\int_0^t e^{-0.5s}ds \\ &= -0.5e^{-0.5t} + 2e^{0.5}\left(e^{-0.5t} - 1\right) + 0.8\left(e^{-0.5t} - 1\right) \\ &= \left(2e^{0.5} + 0.3\right)e^{-0.5t} - 0.8 - 2e^{0.5}. \end{aligned}$$

Then $Z(t) < 0$ for all sufficiently large t. So Lemma 2.2 in [**20**] is not true either.

However, we may utilize our Theorems 2.1 and 2.4 to obtain oscillatory results for the same system (62)–(64). We assume that $R(t) = \exp(\int_0^t r(s)\,ds)$ for $t \ge 0$. We multiply (62) by $R(t)$, and then the system (62)–(64) becomes

$$(R(t)x'(t))' + R(t)\left(p(t) - q(t)\right)x(t-\tau) = 0,\ t \in [0,\infty)\backslash\Upsilon, \tag{68}$$

$$x(t_k^+) = I_k(x(t_k)),\ k \in \mathbf{N}, \tag{69}$$

$$x'(t_k^+) = J_k(x'(t_k)), \quad k \in \mathbf{N}. \tag{70}$$

Assume that there exists $m_0 > 0$ and $M_0 > 0$ such that $m_0 \le A(s,t) \le M_0$ for $t \ge s \ge 0$.

(i) Assume that

$$\frac{I_k(\mu)}{\mu} \le \frac{I_k(\nu)}{\nu} \quad \text{and} \quad \frac{J_k(\mu)}{\mu} \ge \frac{J_k(\nu)}{\nu} \text{ if } \nu \ge \mu > 0 \tag{71}$$

for $k \in \mathbf{N}$. By Theorem 2.4, a sufficient condition such that system (68)–(70) has an eventually positive solution is:

$$\int_{\tau}^{\infty} \frac{1}{R(t)} \int_{t}^{\infty} \frac{R(s)\,(p(s)-q(s))}{B(t,s)}\, ds\, dt < \infty,$$

for some $\tau \geq 0$.

(ii) Assume that system (68)–(70) has a bounded and nonoscillatory solution $x(t)$ with $x(t)x'(t) > 0$ eventually. Since system (68)–(70) is linear, we can assume without loss of generality that $x(t) > 0$ and $x'(t) > 0$ for $t \geq T$ and some $T > 0$. So we have

$$x(t) \geq A(T,t)x(T) \geq m_0 x(T) > 0 \quad \text{where } t \geq T.$$

By Theorem 2.1,

$$\int_{\tau}^{\infty} \int_{t}^{\infty} \frac{R(s)\,(p(s)-q(s))}{R(t)B^*(t,s)}\, ds\, dt < \infty. \tag{72}$$

Furthermore, we note that condition (72) implies the condition

$$\int_{\tau}^{\infty} \int_{t}^{\infty} \frac{p(s)-q(s)}{B^*(t,s)}\, ds\, dt < \infty$$

because $R'(t) = r(t)R(t) > 0$ for $t \geq 0$.

(iii) By the previous discussions, we may see that if there exists $\delta > 0$ such that $\delta B(t,s) \geq B^*(t,s)$ for $t \geq s \geq 0$, then system (68)–(70) has a bounded and eventually positive solution $x(t)$ with $x(t)x'(t) > 0$ eventually if, and only if, (72) holds. Furthermore, if we replace condition (71) by

$$\frac{I_k(\mu)}{\mu} \leq \frac{I_k(\nu)}{\nu}$$

and

$$\frac{J_k(\mu)}{\mu} \geq \frac{J_k(\nu)}{\nu} \quad \text{if } |\nu| \geq |\mu| > 0 \text{ and } \mu\nu > 0$$

for $k \in \mathbf{N}$, then the system (68)–(70) has a bounded and nonoscillatory solution $x(t)$ with $x(t)x'(t) > 0$ eventually if, and only if, (72) holds.

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