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ASYMPTOTIC BEHAVIOR IN NEUTRAL DIFFERENCE EQUATIONS WITH SEVERAL RETARDED ARGUMENTS

G.E. CHATZARAKIS AND G.N. MILIARAS

ABSTRACT. In this paper, we study the asymptotic behavior of the solutions of a neutral type difference equation of the form

$$\Delta\left[x(n)+\sum_{j=1}^{w} c_j x(\tau_j(n))\right]+p(n)x(\sigma(n))=0, \quad n\geq 0$$

where $\tau_j(n)$, $j=1,\ldots,w$, are general retarded arguments, $\sigma(n)$ is a general deviated argument (retarded or advanced), $c_j\in\mathbb{R}$, $j=1,\ldots,w$, $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, and Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$.

We also examine the convergence of the solutions when these are continuous and differentiable with respect to c_j, $j=1,\ldots,w$.

1. Introduction. A neutral difference equation (NDE) is a difference equation in which the higher order difference of the unknown sequence appears in the equation both with and without delays or advances. See, for example, [**1, 4, 5, 13**] and the references cited therein. We should note that the theory of neutral difference equations presents complexities, and results which are true for non-neutral difference equations may not be true for neutral equations [**23**].

1.1. Applications of NDEs. Apart from mathematical interest, the study of those equations is motivated by their applications. Neutral difference equations arise in several areas of applied mathematics, including circuit theory [**3**], bifurcation analysis [**2**], population dynamics

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[**11**], dynamical behavior of delayed network systems [**32**], signal processing [**2**] and so on. Neutral difference equations also come up in the study of vibrating masses attached to an elastic bar as, for example, the Euler equation is used in some variational problems and in the theory of automatic control. See, also, Driver [**9**], Hale [**14**], Brayton and Willoughby [**5**], and the references cited therein. For the general theory of difference equations the reader is referred to the monographs [**1, 4, 13, 17**].

Below, we present two applications indicating the relevance of the equation we study in this paper to real world problems. The two examples are taken from the areas of signal processing and population dynamics.

1.1.1. An application in signal processing. A filter is a system that functions to extract the data from noise, in a signal. The simplest filters are the FIR (finite impulse response) filters that discriminate data from noise by computing the "running average" of the input signal. The term *running average* simply means regularly sampling the input signal at a fixed time interval Δt and computing the average of the most recent values of the input signal.

Another class of filters is the IIR (infinite impulse response) filters. In contrast to FIR filters that merely involve previous values of the input signal, IIR filters involve previous values of the input signal, as well as previous computed values of the output signal in the computation of the present output $y(n)$. Since the output is fed back to be combined with the input, these systems are examples of the general class of feedback systems.

The defining equation for an IIR filter is the difference equation (see [**20**])

$$y(n) = \sum_{i=1}^{n_1} a_i y(n-i) + \sum_{i=0}^{n_2} b_i x(n-i),$$

where $x(n)$ is the input signal, $y(n)$ is the output signal, a_i, $1 \le i \le n_1$ and b_i, $0 \le i \le n_2$ are real constants. Thus,

$$y(n+1) = \sum_{i=1}^{n_1} a_i y(n+1-i) + \sum_{i=0}^{n_2} b_i x(n+1-i).$$

From the above equations, it follows that

$$\Delta y(n) = \sum_{i=1}^{n_1} a_i \Delta y(n-i) + \sum_{i=0}^{n_2} b_i \Delta x(n-i)$$

and

$$\begin{aligned}\sum_{\ell=1}^{L} c_\ell \Delta y(n-\rho_\ell) &= \sum_{\ell=1}^{L} c_\ell \sum_{i=1}^{n_1} a_i \Delta y(n-\rho_\ell-i)\\ &\quad + \sum_{\ell=1}^{L} c_\ell \sum_{i=0}^{n_2} b_i \Delta x(n-\rho_\ell-i),\end{aligned}$$

where c_ℓ, $1 \le \ell \le L$, are real constants and ρ_ℓ, $1 \le \ell \le L$ are positive integers.

Combining the last two equations we obtain

$$\begin{aligned}\Delta\left[y(n) + \sum_{\ell=1}^{L} c_\ell y(n-\rho_\ell)\right] &= \sum_{i=1}^{n_1} a_i \left[\Delta y(n-i) + \sum_{\ell=1}^{L} c_\ell \Delta y(n-\rho_\ell-i)\right]\\ &\quad + \sum_{i=0}^{n_2} b_i \left[\Delta x(n-i) + \sum_{\ell=1}^{L} c_\ell \Delta x(n-\rho_\ell-i)\right]\end{aligned}$$

or

$$\begin{aligned}\Delta\left[y(n) + \sum_{\ell=1}^{L} c_\ell y(n-\rho_\ell)\right] + \sum_{j=0}^{n_1+\max_{1\le\ell\le L}\rho_\ell} p(j) y(n-j)\\ = \sum_{j=0}^{n_2+\max_{1\le\ell\le L}\rho_\ell} q(j) x(n-j),\end{aligned}$$

where $p(j)$, $0 \le j \le n_1 + \max_{1\le\ell\le L}\rho_\ell$ and $q(j)$, $0 \le j \le n_2 + \max_{1\le\ell\le L}\rho_\ell$ are real constants.

That means that the functional characteristics/properties of the class of IIR filters are described by a neutral difference equation having the above form, which is the non-homogeneous form of the equation we study in this paper.

1.1.2. An application in population dynamics. Neutral difference equations have been and are used to describe/model the dynamics of a population or species that increases from the reproduction of its members and decrease from competition, both within the population and competition from another population or species. The neutral difference equation we study in the paper is a more general form of the neutral difference equation that has been (commonly) used to model the change–growth or depletion–of species or populations.

If we denote $x(n)$ to be the size of a population at the time unit n, then according to the *Malthus model* of population dynamics, the size/growth of the population at $n+1$ is described by an equation of the form

$$x(n+1) - x(n) = q(n)x(n-m), \quad n \geq 0,$$

where $q(n)$ is a time dependent growth rate. The delay m indicates that it takes m periods of time for a newborn in the population to mature and start reproducing.

However, a population's growth is checked by various environmental agents and factors, primarily the limited resources that lead to intra and extra population competition. Except from competition, various other causes, like disease, act to check the growth of a population. We can describe the affect on a population, namely, the decrease in a population, due to disease and intra population competition by the difference equation

$$c\,[x(n-k+1) - x(n-k)] = p(n)x(n-r), \quad n \geq 0,$$

where c is a real constant, $p(n)$ is a time dependent depletion rate and k, r are nonnegative integers such that $r \geq k$. The value of delay r is chosen so that $r \geq m+1$, thus indicating that, on the average, the relatively more aged members of the population are more susceptible to competition and disease than the younger ones.

Combining the above equations, the net change in the size of the population can be described by the neutral difference equation

$$\Delta\,[x(n) - cx(n-k)] - q(n)x(n-m) + p(n)x(n-r) = 0, \quad n \geq 0.$$

If the rate $q(n)$ of the population's reproduction is slow relative to the rate $p(n)$ at which the population is depleted due to, say, an epidemic disease and $-1 < c < 1$, then in the short term, the change in the population can be described by the neutral difference equation

$$\Delta\,[x(n) - cx(n-k)] + p(n)x(n-r) = 0, \quad n \geq 0,$$

which is a special case of the neutral difference equation we study in this paper (see Section 3, Theorem 3.1, Part III).

1.2. NDE with several retarded arguments. Consider the neutral difference equation in which the difference of the unknown sequence appears in the equation both with and without more than one delay

$$\Delta\left[x(n)+\sum_{j=1}^{w} c_j x(\tau_j(n))\right]+p(n)x(\sigma(n))=0, \quad n\geq 0, \tag{E}$$

where $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $c_j\in\mathbb{R}$, $j=1,\dots,w$, $(\tau_j(n))_{n\geq 0}$, $j=1,\dots,w$ is an increasing sequence of integers that satisfies

$$\begin{gathered}\tau_j(n)\leq n-1,\ j=1,\dots,w \quad \forall\, n\geq 0,\ \lim_{n\to\infty}\tau_j(n)=+\infty \\ \text{and} \\ \tau_\ell(n)<\tau_m(n+1), \quad \forall\, \ell,m\in[1,w]\cap\mathbb{N}\end{gathered} \tag{1.1}$$

and $(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that

$$\begin{gathered}\sigma(n)\leq n-1 \quad \forall\, n\geq 0,\ \lim_{n\to\infty}\sigma(n)=+\infty, \\ \text{or} \\ \sigma(n)\geq n+1 \quad \forall\, n\geq 0.\end{gathered} \tag{1.2}$$

The keen interest in equation (E) is motivated by the fact that it represents a general form of first order neutral difference equations (see [**6**])

$$\Delta\left[x(n)+cx(\tau(n))\right]+p(n)x(\sigma(n))=0, \quad n\geq 0, \tag{E$_1$}$$

where $(p(n))_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $c\in\mathbb{R}$, $(\tau(n))_{n\geq 0}$ is an increasing sequence of integers which satisfies

$$\tau(n)\leq n-1 \quad \text{for all } n\geq 0 \text{ and } \lim_{n\to\infty}\tau(n)=+\infty \tag{1.1$'$}$$

and $(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that (1.2) holds.

Define

$$k=-\min_{\substack{n\geq 0\\ 1\leq j\leq w}}\{\tau_j(n),\sigma(n)\} \quad \text{if } \sigma(n) \text{ is a retarded argument.}$$

(Clearly, k is a positive integer.)

By a *solution* of the neutral difference equation (E), we mean a sequence of real numbers $(x(n))_{n\geq -k}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers c_{-k}, $c_{-k+1}, \ldots,$ c_{-1}, c_0, there exists a unique solution $(x(n))_{n\geq -k}$ of (E) which satisfies the initial conditions $x(-k) = c_{-k}$, $x(-k+1) = c_{-k+1}, \ldots, x(-1) = c_{-1}$, $x(0) = c_0$.

A solution $(x(n))_{n\geq -k}$ of the neutral difference equation (E) is called *oscillatory* if, for every positive integer n_0 there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $(x(n))_{n\geq -k}$ is *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory.*

If $\sigma(n)$ is an advanced argument, then:

By a *solution* of the neutral difference equation (E), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E) for all $n \geq 0$.

A solution $(x(n))_{n\geq 0}$ of the neutral difference equation (E) is called *oscillatory* if, for every positive integer n_0, there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $(x(n))_{n\geq 0}$ is oscillatory if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory.*

In the special case where $\tau_j(n) = n - a_j$, $\sigma(n) = n \pm b$, $a_j, b \in \mathbb{N}$ and $\tau(n) = n - a$, $a \in \mathbb{N}$, equations (E) and (E_1) take the forms:

$$(\mathrm{E}') \qquad \Delta\left[x(n) + \sum_{j=1}^{w} c_j x(n-a_j)\right] + p(n)x(n \pm b) = 0, \quad n \geq 0$$

$$(\mathrm{E}_1') \qquad \Delta\left[x(n) + cx(n-a)\right] + p(n)x(n \pm b) = 0, \quad n \geq 0,$$

respectively.

In the last few decades the asymptotic behavior of neutral difference equations has been extensively researched and developed. Hence, a large number of related papers have been published. See [**2, 3, 5**–**8, 10, 12, 15, 16, 18, 19, 22, 24**–**31**], and the references cited therein. Most of these papers concern the special case of the neutral delay difference equation (E_1') where the algebraic characteristic equation provides useful information about oscillation and stability, while only a small number of papers have dealt with the general case of the neutral difference equation (E_1).

Regarding equation (E_1), recently, Chatzarakis and Miliaras [**8**] established the following theorem:

Theorem 1.1 [**8**]. *For equation* (E_1)*, the following statements hold true*:

(I) *Every nonoscillatory solution is unbounded if* $c < -1$.
(II) *Every solution oscillates if* $c = -1$.
(III) *Every nonoscillatory solution tends to zero if* $-1 < c < 1$.
(IV) *Every nonoscillatory solution is bounded if* $c \geq 1$.
Furthermore, if any solution of (E_1) *is continuous with respect to c, then the following statements hold*:
(V) *Every solution is eventually zero, if* $c \leq -1$.
(VI) *Every solution tends to zero, if* $-1 < c \leq 1$.
(VII) *If additionally, any solution of* (E_1) *has continuous derivatives of any order and convergent Taylor series for every* $c \in \mathbb{R}$*, then the solution is zero.*

In the same article [**8**], Chatzarakis and Miliaras established the following corollary for equation (E_1'):

Corollary 1.1 [**8**]. *For equation* (E_1')*, the following statements hold true*:

(i) *Every nonoscillatory solution tends to* $\pm\infty$ *if* $c < -1$.
(ii) *Every solution oscillates if* $c = -1$.
(iii) *Every nonoscillatory solution tends to zero if* $c > -1$.
Furthermore, if any solution of (E_1') *is continuous with respect to c, then the following statements hold*:
(iv) *Every solution is eventually zero, if* $c \leq -1$.
(v) *If additionally, any solution of* (E_1') *has continuous derivatives of any order and convergent Taylor series for every* $c \in \mathbb{R}$*, then the solution is zero.*

The objective in this paper is to investigate the convergence and divergence of solutions of equation (E) in the case of general delay arguments $\tau_j(n)$, $j = 1, 2, \ldots, w$, and a general deviated (retarded or advanced) argument $\sigma(n)$, depending on real constants c_j, $j = 1, \ldots, w$.

2. Some preliminaries. Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we can restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, for all $n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\sigma(n)) > 0, \qquad x(\tau_j(n)) > 0, \quad j = 1, 2, \dots, w \quad \text{for all } n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), equation (E) becomes

$$\Delta z(n) + p(n)x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = -p(n)x(\sigma(n)) \leq -px(\sigma(n)) < 0, \quad \text{for all } n \geq n_0,$$

which means that the sequence $(z(n))$ is eventually strictly decreasing, regardless of the values of the real constants c_j.

Let the domain of τ_j be the set $D(\tau_j) = \mathbb{N}_{n_j^*} = \{n_j^*, n_j^*+1, n_j^*+2, \dots\}$, where n_j^* is the smallest natural number that τ_j is defined. Set

$$n_* = \max_{1\leq j\leq w} n_j^*.$$

Then τ_j, $j = 1, 2, \dots, w$, is defined in the set $\mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, \dots\}$.

Let the subsequences

$$\begin{gathered} x(\tau_{\rho(n)}(n)) = \max\{x(\tau_1(n)), x(\tau_2(n)), \dots, x(\tau_w(n))\} \\ \text{and} \\ x(\tau_{\varphi(n)}(n)) = \min\{x(\tau_1(n)), x(\tau_2(n)), \dots, x(\tau_w(n))\}, \end{gathered} \tag{2.3}$$

where $\rho(n), \varphi(n)$ are sequences that take values in the set $\{1, 2, \dots, w\}$. Clearly, condition (1.1) guarantees that $(x(\tau_{\rho(n)}(n)))$ and $(x(\tau_{\varphi(n)}(n)))$ are subsequences of $(x(n))$.

Notice that

$$\tau_{j_1}(\tau_{j_2}(\cdots \tau_{j_\ell}(n))) = \tau_{j_1}(n_s) \quad \text{where } n_s = \tau_{j_2}(\cdots \tau_{j_\ell}(n)). \tag{2.4}$$

The following lemma provides some tools which are useful for the main results:

Lemma 2.1. *Assume that* $(x(n))_{n\geq -k}$ *is a positive solution of* (E). *Then the following statements hold*:

(i) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty,$$

then

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))), \quad A \in \mathbb{R}. \tag{2.5}$$

(ii) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty,$$

then

$$z(n) < 0 \quad \textit{eventually.} \tag{2.6}$$

(iii) *If the constants* c_j *are all nonpositive or nonnegative and* $c \geq -1$, *then*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty. \tag{2.7}$$

Proof. Summing up (2.2) from n_0 to n, $n \geq n_0$,we obtain

$$z(n+1) - z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)) = 0,$$

or

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{2.8}$$

For the above relation, there are only two possible cases:

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \tag{2.8.a}$$

or

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.8.b}$$

Assume that (2.8.a) holds. Since $p(n) \geq p > 0$, we have

$$+\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \geq p \sum_{i=n_0}^{\infty} x(\sigma(i)).$$

The last inequality guarantees that

$$\sum_{i=n_0}^{\infty} x(\sigma(i)) < +\infty$$

and, consequently,

$$\lim_{n\to\infty} x(\sigma(n)) = 0. \tag{2.9}$$

Also, (2.8.a) guarantees that $\lim_{n\to\infty} z(n)$ exists as a real number. Set

$$\lim_{n\to\infty} z(n) = A \in \mathbb{R}.$$

Since $(z(\sigma(n)))$ is a subsequence of $(z(n))$, we have $\lim_{n\to\infty} z(\sigma(n)) = A$, or

$$\lim_{n\to\infty} \left[x(\sigma(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) \right] = A.$$

Using (2.9), we obtain

$$\lim_{n\to\infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) = A.$$

Thus,

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))).$$

The proof of part (i) of the lemma is complete.

Assume that (2.8.b) holds. Then, by taking limits on both sides of (2.8) we obtain $\lim_{n\to\infty} z(n) = -\infty$ which, in conjunction with that fact that the sequence $(z(n))$ is eventually strictly decreasing, means that

$$z(n) < 0 \quad \text{eventually.}$$

The proof of part (ii) of the lemma is complete.

Assume that $-1 \le c < 0$ and suppose, for the sake of contradiction, that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$. Then, in view of part (ii), we have $z(n) < 0$ eventually. Thus, if

$$c = \sum_{j=1}^{w} c_j \tag{2.10}$$

and, taking into account definition (2.3), we have

$$\begin{aligned}
x(n) &< -\sum_{j=1}^{w} c_j x(\tau_j(n)) \le \left(-\sum_{j=1}^{w} c_j \right) x(\tau_{\rho_1(n)}(n)) \\
&= -c x(\tau_{\rho_1(n)}(n)) \\
&< (-c) \left(-\sum_{j=1}^{w} c_j x(\tau_j(\tau_{\rho_1(n)}(n))) \right) \\
&\le (-c) \left(-\sum_{j=1}^{w} c_j \right) x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) \\
&= (-c)^2 \, x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) \\
&< \cdots < (-c)^{m(n)} \, x(\tau_{\rho_{m(n)}}(\tau_{\rho_{m(n)-1}}(\cdots \tau_{\rho_1(n)}(n)))).
\end{aligned}$$

Based on (2.4), the last inequality becomes

$$x(n) < (-c)^{m(n)} \, x(\tau_{\rho_{m(n)}}(n_*)). \tag{2.11}$$

If $-1 < c < 0$, clearly $(-c)^{m(n)} \to 0$ since $m(n) \to \infty$ as $n \to \infty$. Since $x(n) > 0$ for all large n, (2.11) guarantees that $\lim_{n\to\infty} x(n) = 0$. This implies that $\lim_{n\to\infty} z(n) = 0$, and consequently, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < \infty$, which contradicts our assumption.

If $c = -1$, from (2.11) we obtain $x(n) < x(\tau_{\rho_{m(n)}}(n_*))$, which means that $(x(n))$ is bounded, and therefore, $(z(n))$ is bounded. Thus, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < \infty$, which contradicts our assumption.

Assume that $c \geq 0$ and $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$. In view of part (ii), (2.6) holds, i.e., $z(n) < 0$ eventually. This contradicts $z(n) = x(n)+\sum_{j=1}^{w} c_j x(\tau_j(n)) > 0$. Therefore, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty$. The proof of part (iii) of the lemma is complete.

The proof of Lemma 2.1 is complete. $\square$

3. Main results. The asymptotic behavior of the solutions of the neutral type difference equation (E) is described by the theorem below. It is clear that the behavior of $x(n)$ depends on the constants $c_1, c_2, \ldots, c_w$. So, we may consider $x(n)$ to be a function of $(c_1, c_2, \ldots, c_w)$. We examine the behavior of $x(c_1, c_2, \ldots, c_w, n)$ in terms of continuity and differentiability and, as we will see, the results we will obtain are essentially different, especially in the case where the delays are constant.

Theorem 3.1. *For equation* (E), *the following statements hold*:

(I) *Every nonoscillatory solution does not converge in* $\mathbb{R}$ *if the constants* c_j *are all nonpositive and* $c < -1$.

(II) *Every solution oscillates if the constants* c_j *are all nonpositive and* $c = -1$.

(III) *Every nonoscillatory solution tends to zero if* either c_j*s are all nonpositive or the* c_j*s are all nonnegative, and* $-1 < c < 1$.

(IV) *Every nonoscillatory solution is bounded if the* c_j*s are all nonnegative and* $c \geq 1$.

If any solution of (E) *is continuous with respect to* $(c_1, c_2, \ldots, c_w)$ *where the* c_j*s are either nonpositive or nonnegative, then the following statements hold*:

(V) *Every solution is eventually zero, if* $c = -1$.

(VI) *Every solution tends to zero, if* $-1 < c \leq 1$.

Proof. Assume that $(x(n))_{n \geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n \geq -k}$ is also a solution of (E), we can restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, for all $n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\sigma(n)) > 0, \qquad x(\tau_j(n)) > 0, \quad j = 1, 2, \ldots, w, \ \forall\, n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), equation (E) becomes

$$\Delta z(n) + p(n)x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account the fact that $p(n) \geq p > 0$, we have

$$\Delta z(n) = -p(n)x(\sigma(n)) \leq -px(\sigma(n)) < 0 \quad \text{for all } n \geq n_0,$$

which means that the sequence $(z(n))$ is strictly decreasing eventually, regardless of the values of the real constants c_j's.

Assume that the c_j's are all nonpositive and $c < -1$. If (2.8.a) holds, then, in view of part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))), \quad A \in \mathbb{R},$$

which guarantees that $A \leq 0$.

If $A = 0$, then

$$\lim_{n\to\infty} z(n) = 0.$$

Taking into account that the sequence $(z(n))$ is eventually strictly decreasing, it follows that, eventually, $z(n) > 0$. Thus, from (2.3), (2.4) and (2.10), we have

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} c_j x(\tau_j(n)) \geq \left(-\sum_{j=1}^{w} c_j \right) x(\tau_{\varphi_1(n)}(n)) \\ &= -cx(\tau_{\varphi_1(n)}(n)) > \cdots > (-c)^{m(n)}\, x(\tau_{\varphi_{m(n)}}(n_*)). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} x(n) \geq \lim_{n\to\infty} \left[(-c)^{m(n)}\, x(\tau_{\varphi_{m(n)}}(n_*)) \right] = +\infty,$$

which contradicts (2.9). Therefore, $A < 0$. Now, since $\lim_{n\to\infty} x(\sigma(n)) = 0$, $(x(n))$ has more than one accumulation point. Thus, $(x(n))$ does not converge in $\mathbb{R}$.

Assume that (2.8.b) holds. Then, by taking limits on both sides of (2.8) we obtain

$$\lim_{n\to\infty} z(n) = -\infty,$$

or

$$\lim_{n\to\infty}\left[x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n))\right] = -\infty,$$

or

$$\lim_{n\to\infty}\left[x(n) + \left(\sum_{j=1}^{w} c_j\right) x(\tau_{\rho(n)}(n))\right] = -\infty,$$

or

$$\lim_{n\to\infty}\left[x(n) + cx(\tau_{\rho(n)}(n))\right] = -\infty.$$

Since $c < -1$, the last relation guarantees that

$$\lim_{n\to\infty} x(\tau_{\rho(n)}(n)) = +\infty,$$

which means that $(x(n))$ is unbounded. Therefore, $(x(n))$ does not converge in $\mathbb{R}$. The proof of part (I) of the theorem is complete.

Assume that the constants c_j are all nonpositive and $c = -1$. Then, from parts (iii) and (i) of Lemma 2.1, we have

$$\begin{aligned}\lim_{n\to\infty} z(n) &= \lim_{n\to\infty}\left[x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n))\right] = A\\ &= \lim_{n\to\infty} \sum_{j=1}^{w} c_j x(\tau_j(\sigma(n)))\end{aligned}$$

which guarantees that $A \leq 0$.

Assume that $A < 0$. Then there exists a natural number n_λ such that $z(n) < 0$, for all $n \geq n_\lambda$. Thus, from (2.3), (2.4) and (2.10), we have

$$\begin{aligned}x(n) &< -\sum_{j=1}^{w} c_j x(\tau_j(n)) \leq \left(-\sum_{j=1}^{w} c_j\right) x(\tau_{\rho_1(n)}(n))\\ &= x(\tau_{\rho_1(n)}(n)) < -\sum_{j=1}^{w} c_j x(\tau_j(\tau_{\rho_1(n)}(n)))\end{aligned}$$

$$\leq \Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n)))$$
$$= x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) < \cdots < x(\tau_{\rho_{m(n_\ell)}}(n_s))$$

which means that $(x(n))$ is bounded.

Let

$$M = \limsup_{n\to\infty} x(n). \tag{3.1}$$

Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that

$$\lim_{n\to\infty} x(\theta(n)) = M.$$

Thus,

$$\lim_{n\to\infty} \Big[x(\theta(n)) + \sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \Big] = A,$$

or

$$-\lim_{n\to\infty} \Big[\sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) \Big] = M - A.$$

Therefore, for every ε with $0 < \varepsilon < -A$, there exists $n_3 \in \mathbb{N}$ such that

$$-\sum_{j=1}^{w} c_j x(\tau_j(\theta(n))) + \varepsilon \geq M - A, \quad \text{for all } n \geq n_3$$

or

$$\Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon \geq M - A,$$

or

$$x(\tau_{\rho(\theta(n))}(\theta(n))) + \varepsilon \geq M - A,$$

or

$$M + \varepsilon \geq \limsup_{n\to\infty} \big[x(\tau_{\rho(\theta(n))}(\theta(n))) \big] + \varepsilon > M - A$$

or

$$\varepsilon \geq -A.$$

This result contradicts that $\varepsilon < -A$, and therefore $A = 0$, i.e., $\lim_{n\to\infty} z(n) = 0$. Furthermore, taking into account the fact that the sequence $(z(n))$ is eventually strictly decreasing, we conclude that $z(n) > 0$, or equivalently,

$$x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)) > 0,$$

or

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} c_j x(\tau_j(n)) \geq \Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\varphi_1(n)}(n)) \\ &= x(\tau_{\varphi_1(n)}(n)) > \cdots > x(\tau_{\varphi_{m(n)}}(n_*)). \end{aligned}$$

Since $(x(n))$ has a lower bound greater than zero, it cannot have any subsequence that tends to zero. Thus, $\lim_{n\to\infty} x(\sigma(n)) = 0$ is not valid, and therefore $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \infty$, which comes to contradiction with our previous assumptions. Thus, if $c = -1$, $(x(n))$ oscillates. The proof of part (II) of the theorem is complete.

Assume that the constants c_j are all nonpositive and $-1 < c < 0$. In view of parts (i) and (iii) of Lemma 2.1, by a similar procedure as in part (II), we conclude that $A \leq 0$.

Assume that $A < 0$. Then there exists a natural number n_λ such that $z(n) < 0$, for all $n \geq n_\lambda$, and therefore

$$\begin{aligned} x(n) &< -\sum_{j=1}^{w} c_j x(\tau_j(n)) \leq \Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\rho_1(n)}(n)) \\ &= (-c) x(\tau_{\rho_1(n)}(n)) < \cdots < (-c)^{m(n_\ell)} x(\tau_{\rho_{m(n_\ell)}}(n_s)), \end{aligned}$$

which means that $(x(n))$ tends to zero as $n \to \infty$. Thus, $(z(n))$ tends to zero as $n \to \infty$, i.e., $A = 0$, which contradicts the assumption that $A < 0$. Hence, $A = 0$.

Taking into account the fact that the sequence $(z(n))$ is strictly decreasing, it is obvious that $z(n) > 0$. Hence, for every $\varepsilon > 0$, there exists a natural number n_4 such that

$$x(n) + \sum_{j=1}^{w} c_j x(\tau_j(n)) < \varepsilon, \quad \text{for all } n \geq n_4$$

or

$$
\begin{aligned}
x(n) &< -\sum_{j=1}^{w} c_j x(\tau_j(n)) + \varepsilon \leq \Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\rho_1(n)}(n)) + \varepsilon \\
&= -c x(\tau_{\rho_1(n)}(n)) + \varepsilon \\
&< -c\Big[-\sum_{j=1}^{k} c_j x(\tau_j(\tau_{\rho_1(n)}(n))) + \varepsilon \Big] + \varepsilon \\
&\leq -c\Big[\Big(-\sum_{j=1}^{w} c_j \Big) x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) + \varepsilon \Big] + \varepsilon \\
&= -c\left[-c x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) + \varepsilon \right] + \varepsilon \\
&= c^2 x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) - c\varepsilon + \varepsilon \\
&< \cdots < (-c)^{m(n_\ell)}\, x(\tau_{\rho_{m(n_\ell)}}(n_s)) + \varepsilon \\
&\quad - c\varepsilon + \cdots + (-c)^{m(n_\ell)}\, \varepsilon.
\end{aligned}
$$

As $n \to \infty$, clearly $m(n_\ell) \to \infty$, and therefore,

$$\lim_{n\to\infty} x(n) \leq \lim_{n\to\infty} \left[\varepsilon - c\varepsilon + \cdots + (-c)^{m(n_\ell)}\, \varepsilon \right] = \frac{\varepsilon}{1+c}.$$

Since ε is an arbitrary positive real number and $(x(n)) > 0$, it becomes evident that

$$\lim_{n\to\infty} x(n) = 0.$$

Let $c = 0$. In view of part (iii) of Lemma 2.1, we have

$$\sum_{i=n_0}^{\infty} p(i) x(\sigma(i)) < +\infty.$$

This guarantees that $(z(n))$ is bounded. Therefore, $(x(n))$ is bounded, since $z(n) = x(n)$. Also, since $(z(n))$ is strictly decreasing, we can infer that $\lim_{n\to\infty} z(n)$ exists, and consequently, $\lim_{n\to\infty} x(n)$ exists. Using (2.9), we conclude that

$$\lim_{n\to\infty} x(n) = 0.$$

Assume that $0 < c < 1$. Then, using parts (i) and (iii) of Lemma 2.1, we have

$$\lim_{n\to\infty}\sum_{j=1}^{k} c_j x(\tau_j(\sigma(n)) = A.$$

Therefore, for every ε with $0<\varepsilon<(1-c)A$, there exists $n_5\in\mathbb{N}$ such that, for all $n\geq n_5$,

$$\sum_{j=1}^{w} c_j x(\tau_j(\sigma(n))) + \varepsilon > A,$$

or

$$\left(\sum_{j=1}^{w} c_j\right) x(\tau_{\rho(\sigma(n))}(\sigma(n))) + \varepsilon > A,$$

or

$$cx(\tau_{\rho(\sigma(n))}(\sigma(n))) + \varepsilon > A,$$

or

$$x(\tau_{\rho(\sigma(n))}(\sigma(n))) > \frac{A-\varepsilon}{c},$$

and therefore

$$z(\tau_{\rho(\sigma(n))}(\sigma(n)) > \frac{A-\varepsilon}{c} > A,$$

which contradicts that $0<\varepsilon<(1-c)A$. Hence, $A=0$, i.e.,

$$\lim_{n\to\infty} z(n) = 0,$$

and since

$$z(n)\geq x(n) > 0,$$

we have

$$\lim_{n\to\infty} x(n) = 0.$$

The proof of part (III) of the theorem is complete.

Assume that $c\geq 1$. By part (iii) of Lemma 2.1, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) < +\infty$ which, in conjunction with (2.8), implies that $(z(n))$ is bounded. Therefore, $(x(n))$ is bounded. The proof of part (IV) of the theorem is complete.

For the rest of the proof, we assume that $x(n)$ is a continuous function with respect to $({}_1, c_2, \ldots, c_w)$, where the constants c_j are either nonpositive or nonnegative. Therefore, instead of $x(n)$, we shall write $x(c_1, c_2, \ldots, c_w, n)$.

Let

$$d_1, d_2, \ldots, d_w \le 0 \quad \text{with} \quad d_1 + d_2 + \cdots + d_w = -1.$$

Then, in view of part (II), $(x(d_1, d_2, \ldots, d_w, n))$ oscillates. On the other hand, since $x(c_1, c_2, \ldots, c_w, n)$ is continuous, we have

$$\lim_{(c_1, c_2, \ldots, c_w) \to (d_1, d_2, \cdots, d_w)} x(c_1, c_2, \ldots, c_w, n) = x(d_1, d_2, \ldots, d_w, n).$$

But $x(c_1, c_2, \ldots, c_w, n) > 0$ for all large n, and therefore its limit is always nonnegative. Thus,

$$x(d_1, d_2, \ldots, d_w, n) \ge 0 \quad \text{for all large } n,$$

which contradicts that $(x(d_1, d_2, \ldots, d_w, n))$ oscillates. Thus, $x(d_1, d_2, \ldots, d_w, n) = 0$, eventually. Thus, for every w-tuple $(d_1, d_2, \ldots, d_w)$ with $d_1 + d_2 + \cdots + d_w = -1$, we have $x(d_1, d_2, \ldots, d_w, n) = 0$, eventually. The proof of part (V) of the theorem is complete.

Let a point

$$(c_1, c_2, \ldots, c_w) \quad \text{with } -1 < c_1 + c_2 + \cdots + c_w = c \le 1,$$

and a point

$$(d_1, d_2, \ldots, d_w) \quad \text{with } d_1 + d_2 + \cdots + d_w = 1.$$

Taking into account part (III), it suffices to show part (VI), only for $c = 1$. From parts (i) and (iii) of Lemma 2.1, we have

$$\lim_{n \to \infty} \sum_{j=1}^{w} c_j x(c_1, c_2, \ldots, c_w, \tau_j(\sigma(n))) = A.$$

Thus, for every $\varepsilon > 0$, there exists $n_6 \in \mathbb{N}$ such that

$$\sum_{j=1}^{w} c_j x(c_1, c_2, \cdots, c_w, \tau_j(\sigma(n))) \ge A + \varepsilon, \quad \text{for all } n \ge n_6$$

or

$$\left(\sum_{j=1}^{w} c_j \right) x(c_1, c_2, \ldots, c_w, \tau_{\rho(n)}(n)) \ge A + \varepsilon,$$

or

$$x(c_1, c_2, \dots, c_w, \tau_{\rho(n)}(n)) \geq \frac{A+\varepsilon}{c}.$$

Assume, for the sake of contradiction, that $A > 0$. Taking into account the fact that

$$\lim_{n\to\infty} x(c_1, c_2, \dots, c_w, n) = 0$$

when

$$0 < c_1 + c_2 + \cdots + c_w < 1,$$

there exists $n_7 \in \mathbb{N}$ such that

$$x(c_1, c_2, \dots, c_w, x(\tau_{\rho(n)}(n))) < \frac{A}{2}, \quad \text{for all } n \geq n_7,$$

since $\lim_{n\to\infty} x(c_1, c_2, \dots, c_w, x(\tau_{\rho(n)}(n))) = 0$. But $(A+\varepsilon)/c > A/c > A/2$, since $c < 1$. Therefore, $A = 0$, i.e.,

$$\lim_{n\to\infty} z(d_1, d_2, \dots, d_w, n) = 0,$$

which means that

$$\lim_{n\to\infty} x(d_1, d_2, \dots, d_w, n) = 0.$$

The proof of part (VI) of the theorem is complete.

The proof of Theorem 3.1 is complete. □

As a consequence of Theorem 3.1, we postulate the following corollary:

Corollary 3.2. *For equation* (E′), *the following hold*:

(i) *Every nonoscillatory solution is unbounded if* $c < -1$.
(ii) *Every solution oscillates if* $c = -1$.
(iii) *Every nonoscillatory solution tends to zero if* $c > -1$.
If any solution of (E_1) *is continuous with respect to* c, *then the following statements are true*:
(iv) *Every solution is eventually zero, if* $c \leq -1$.
(v) *If additionally, any solution of* (E_1) *has continuous derivatives of any order and convergent Taylor series for every* $c_j \in \mathbb{R}$, *then the solution is zero.*

Proof. Assume that the constants c_j are all nonpositive and $c < -1$. If $\sum_{i=n_0}^{\infty} p(i)x(n \pm b) = S_0 < +\infty$, then, in view of part (i), (2.4) is satisfied, i.e.,

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x(n - a_j \pm b),$$

which guarantees that $A \leq 0$.

Let $A < 0$. Since $\lim_{n\to\infty} x(n \pm b) = 0$, it is obvious that $\lim_{n\to\infty} x(n) = 0$, and consequently $\lim_{n\to\infty} x(n - a_j) = 0$. Thus, $\lim_{n\to\infty} z(n) = 0$, which contradicts $A < 0$. Hence, $A = 0$.

Taking into account that the sequence $(z(n))$ is eventually strictly decreasing, it is obvious that $z(n) > 0$ eventually, or

$$\begin{aligned} x(n) &> -\sum_{j=1}^{w} c_j x(n - a_j) \geq \Big(-\sum_{j=1}^{w} c_j \Big) x(n - a_{\varphi_1(n)}) \\ &= -cx(n - a_{\varphi_1(n)}) > \cdots > (-c)^{m(n)}\, x(n_* - a_{\varphi_{m(n)}}). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} x(n) \geq \lim_{n\to\infty} \Big[(-c)^{m(n)}\, x(n_* - a_{\varphi_{m(n)}}) \Big] = +\infty,$$

which contradicts (2.9). Therefore, $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$. Now, summing up (2.2) from n_0 to n, we obtain

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(n \pm b). \tag{2.8}$$

Taking limits on both sides of (2.8) we obtain

$$\lim_{n\to\infty} z(n) = -\infty,$$

or

$$\lim_{n\to\infty} \Big[x(n) + \sum_{j=1}^{w} c_j x(n - a_j) \Big] = -\infty,$$

or

$$\lim_{n\to\infty} \Big[x(n) + \Big(\sum_{j=1}^{w} c_j \Big) x(n - a_{\rho(n)}) \Big] = -\infty,$$

or

$$\lim_{n\to\infty}\left[x(n)+cx(n-a_{\rho(n)})\right]=-\infty.$$

Since $c<-1$, the last relation guarantees that

$$\lim_{n\to\infty}x(n-a_{\rho(n)})=+\infty,$$

which means that $(x(n))$ is unbounded. The proof of part (i) of the corollary is complete.

Part (ii) follows directly from part (I) of Theorem 3.1.

As we have proved in parts (III) and (IV) of the Theorem 3.1, if $c>-1$, then

$$\lim_{n\to\infty}x(\sigma(n))=0,$$

and consequently,

$$\lim_{n\to\infty}x(n\pm b)=0,$$

which means that

$$\lim_{n\to\infty}x(n)=0.$$

The proof of part (iii) of the corollary is complete.

In view of part (V) of Theorem 3.1, it suffices to consider the case $c<-1$.

Let $M>0$. Then there exists an index n_5 such that $x(c_1,c_2,\dots,c_w, n-a_{\rho(n)})>M$, for all $n\ge n_5$. Since the function $x(c_1,c_2,\dots,c_w,n)$ is continuous, so is $x(c_1,c_2,\dots,c_w,n-a_{\rho(n)})$. Therefore,

$$\begin{aligned}\lim_{(c_1,c_2,\dots,c_w)\to(d_1,d_2,\dots,d_w)} & x(c_1,c_2,\dots,c_w,n-a_{\rho(n)})\\ &= x(d_1,d_2,\dots,d_w,n-a_{\rho(n)})=0\end{aligned}$$

for all $n\ge n_5$. Hence, there exists $h>0$ so that, if $c_1+c_2+\cdots+c_w> -1-h$, then $x(c_1,c_2,\dots,c_w,n-a_{\rho(n)})<M$, for all $n\ge n_5$. But that is a contradiction to $x(c_1,c_2,\dots,c_w,n-a_{\rho(n)})>M$.

This implies that there exists an open ball $B((a_1,a_2,\dots,a_w),r)$ such that

$$x(c_1,c_2,\dots,c_w,n)=0$$

for all $(c_1,c_2,\dots,c_w)\in B((d_1,d_2,\dots,d_w),r)$ eventually. Let

$$R = \sup\{r \mid x(c_1, c_2, \ldots, c_w, n) = 0 \quad \text{for all } (c_1, c_2, \ldots, c_w) \in B((d_1, d_2, \ldots, d_w), r)\}\,.$$

Let

$$(b_1, b_2, \ldots, b_w) \quad \text{with } b_1 + b_2 + \cdots + b_w < -1$$

be the intersection of the line connecting the origin of $\mathbb{R}^w$ and $(d_1, d_2, \ldots, d_w)$ with the surface of $B((d_1, d_2, \ldots, d_w), R)$. Then, for all $(c_1, c_2, \ldots, c_w) \in B((d_1, d_2, \ldots, d_w), R)$, we have

$$\lim_{(c_1,c_2,\ldots,c_w)\to(b_1,b_2,\ldots,b_w)} x(c_1, c_2, \ldots, c_w, n) = x(b_1, b_2, \ldots, b_w, n)$$

by continuity. But, $x(c_1, c_2, \ldots, c_w, n) = 0$. Thus,

$$x(b_1, b_2, \ldots, b_w, n) = 0, \quad \text{eventually.}$$

Thus, by a similar procedure, we obtain an open ball $B((b_1, b_2, \ldots, b_w), R^{'})$ such that

$$x(c_1, c_2, \ldots, c_w, n) = 0$$
$$\text{for every } (c_1, c_2, \ldots, c_w) \in B\left((b_1, b_2, \ldots, b_w), R'\right) \text{ eventually.}$$

Repeating the above procedure infinitely many times, we conclude that, for every point $(c_1, c_2, \ldots, c_w)$ with $c_1+c_2+\cdots+c_w < -1$ which belongs to the line connecting the origin of $\mathbb{R}^w$ and $(d_1, d_2, \ldots, d_w)$, we have

$$x(c_1, c_2, \ldots, c_w, n) = 0, \quad \text{eventually.}$$

Therefore, every point $(c_1, c_2, \ldots, c_w)$ with $c_1 + c_2 + \cdots + c_w \leq -1$, on any line passing through the origin, satisfies $x(c_1, c_2, \ldots, c_w, n) = 0$, eventually. Thus,

$$x(c_1, c_2, \ldots, c_w, n) = 0 \quad \text{eventually}$$

for every $(c_1, c_2, \ldots, c_w)$ with $c_1 + c_2 + \cdots + c_w \leq -1$. The proof of part (iv) of the corollary is complete.

Finally, since $x(c_1, c_2, \ldots, c_w, n)$ has a convergent Taylor series, we have

$$x(0, \ldots, a_j, \ldots, 0, n) = \sum_{m=0}^{\infty} \frac{\frac{\partial^m}{\partial c_j^m} x(0, \ldots, -1, \ldots, 0, n)}{m!} (a_j + 1)^m = 0, \text{for every } d_j \in \mathbb{R}.$$

Assume that $d_j \leq 0$ and pick $d_\ell \leq 0$ too. Then we have

$$x(0,\ldots,d_j,0,\ldots,d_\ell,\ldots,0,n) = \sum_{m=0}^{\infty} \frac{\frac{\partial^m}{\partial c_\ell^m} x(0,\ldots,a_j,\ldots,0,n)}{m!} d_\ell^m = 0.$$

After $w-2$ steps, we obtain that $x(d_1,d_2,\ldots,d_w,n) = 0$ when $d_1,d_2,\ldots,d_w \leq 0$. By a similar procedure, we have $x(d_1,d_2,\ldots,d_w,n) = 0$ when $d_1,d_2,\ldots,d_w \geq 0$. Therefore, $x(c_1,c_2,\ldots,c_w,n) = 0$. The proof of part (v) of the corollary is complete.

The proof of Corollary 3.2 is complete.

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Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), 14121, N. Heraklio, Athens, Greece
Email address: **geaxatz@otenet.gr, gea.xatz@aspete.gr**

American University of Athens, Andrianou 9, 11525, P. Psychico, Athens, Greece
Email address: **gmiliara@yahoo.gr, gmiliaras@aua.edu**