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A REMARK ABOUT TWISTING SCHATTEN CLASSES

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ABSTRACT. We prove that Kalton twisting of Schatten classes is strictly singular as $B(H)$-modules. We also identify the dual construction.

1. Introduction. Nigel J. Kalton proved in [**5**] that it is possible to twist the Schatten classes, meaning that there exists a $B(H)$-module, namely Θ_p, containing a non-complemented copy of $\mathcal{S}_p$–the corresponding Schatten class–such that the quotient is again $\mathcal{S}_p$. Although not explicitly stated in this way, [**5,** Theorem 8.3] contains this fact and much more. For example, the same theorem contains the statement that every bicentralizer on the Schatten classes arises as a derivation, which is a very deep result. The proof of all these facts requires heavy machinery, of course. A natural question about Θ_p is to identify its dual. The answer is implicit in the works of Kalton by juxtaposition of [**5,** Theorem 8.3] and some results in [**7**]. So, one may conclude that $\Theta_p^* = \Theta_q$ for conjugated p, q although, as far as we know, it has never been explicitly stated. However, the necessary proofs to conclude it are not easy to follow. Furthermore, to find the precise form of the duality, one needs to go inside the proof of [**10,** Theorem 5.1] and combine it again with [**5,** Theorem 8.3]. We provide a direct computation of the dual so one can skip hard proofs and see explicitly how duality is working. A second comment related to this construction is whether Θ_p is an extremal twisting. Precisely, is the natural quotient map $\Theta_p \to \mathcal{S}_p$ strictly singular? Recall that an operator is said to be strictly singular if it is never an isomorphism when restricted to an infinite dimensional subspace. This is equivalent to saying that the corresponding bicentralizer is never trivial when restricted to an infinite dimensional subspace. We show that Θ_p is extremal in the category of $B(H)$-modules. That

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is, the quotient map $\Theta_p \to \mathcal{S}_p$ is not an isomorphism when restricted to an infinite dimensional $B(H)$-submodule, say V, if and only if

$$\max\{\mathrm{rk}\,(T) : T \in V\} = +\infty,$$

where, by $\mathrm{rk}\,(T)$, we denote the rank of T. As far as we know, this result is new. Thus, the aim of this note is to simplify and clarify a couple of points–duality and singularity–for Kalton twisting of Schatten classes. Let us sketch briefly the main definitions necessary for the paper.

Definition 1.1. Let Z and Y be quasi-normed modules over the Banach algebra A, and let $\widetilde{Y}$ be another module containing Y in the purely algebraic sense. A bicentralizer from Z to Y with ambient space $\widetilde{Y}$ is a homogeneous mapping $\Omega : Z \to \widetilde{Y}$ having the following properties.

(a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f+g) - \Omega(f) - \Omega(g) \in Y$ and $\|\Omega(f+g) - \Omega(f) - \Omega(g)\|_Y \leq Q(\|f\|_Z + \|g\|_Z)$.

(b) There is a constant C so that if $a, b \in A$ and $f \in Z$. Then $\Omega(afb) - a\Omega(f)b \in Y$ and

$$\|\Omega(afb) - a\Omega(f)b\|_Y \leq C\|a\|_A\|f\|_Z\|b\|_A.$$

We now indicate the connection between bicentralizers and extensions. Let Z and Y be quasi-Banach modules and $\Omega : Z \to \widetilde{Y}$ is a bicentralizer from Z to Y. Then

$$Y \oplus_\Omega Z = \{(g, f) \in \widetilde{Y} \times Z : g - \Omega f \in Y\}$$

is a linear subspace of $\widetilde{Y} \times Z$, and the functional $\|(g, f)\|_\Omega = \|g - \Omega f\|_Y + \|f\|_Z$ is a quasi-norm on it. Moreover, the map $i : Y \to Y \oplus_\Omega Z$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $p : Y \oplus_\Omega Z \to Z$ given as $p(g, f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces:

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus_\Omega Z \xrightarrow{\ p\ } Z \longrightarrow 0$$

with relatively open maps. This already implies that $Y \oplus_\Omega Z$ is complete, i.e., a quasi-Banach space. Actually, only quasi-linearity (a) is necessary here. The estimate in (b) implies that the multiplication

$a(g,f)b = (agb, afb)$ makes $Y \oplus_\Omega Z$ into a quasi-Banach bimodule over A in such a way that the arrows in the exact sequence become homomorphisms. We say that Ω induces a trivial extension if and only if $\|\Omega(f) - h(f)\|_Y \leq K\|f\|_Z$ for some morphism $h : Z \to \widetilde{Y}$. In this case, we say that Ω is a trivial bicentralizer. In particular, if h is a morphism of A-modules, we say that Ω is a A-trivial bicentralizer. In our setting, both notions agree. The following lemma is known to specialists in twisted sums:

Lemma 1.2. *If a bicentralizer Ω from $\mathcal{S}_p$ to $\mathcal{S}_p$ with $1 < p < \infty$ (and ambient space $B(H)$) is trivial, then it is $B(H)$-trivial.*

Details of the proof can be found in [**2**].

2. The singularity. Kalton twisting of Schatten classes is done by constructing a nontrivial bicentralizer. The precise one, called *Kalton bicentralizer* and denoted by $\Omega_p : \mathcal{S}_p \to B(H)$, is defined as follows: given an operator $T \in \mathcal{S}_p$ with spectral form $T = \sum a_i(T) f_i \otimes e_i$,

$$\Omega_p(T) := \sum a_i(T) \log\left(\frac{\|T\|_{\mathcal{S}_p}}{a_i(T)}\right) f_i \otimes e_i.$$

We are ready to prove the singularity of the Kalton bicentralizer:

Proposition 2.1. *Let V be a $B(H)$-submodule of $\mathcal{S}_p$ for $1 < p < \infty$. The following conditions are equivalent*:

(1) *The restriction of Ω_p to V is not trivial.*
(2) $\max\{rk(T) : T \in V\} = +\infty$.

Proof. We prove (2) implies (1): Let $T \in V$ be a norm one operator with spectral representation $T = \sum_{i=1}^N a_i(T) f_i \otimes e_i$. It is possible to find for every i an operator $P_i \in B(H)$ that $P_i(T) = f_i \otimes e_i$. Assume that (1) does not hold and pick, by Lemma 1.2, a morphism of $B(H)$-modules $\Lambda : V \to \mathcal{S}_p$ such that $\|\Omega_p - \Lambda\| \leq K$. Since $\Omega_p(f_i \otimes e_i) = 0$, then $\|\Lambda(f_i \otimes e_i)\| \leq K$ and, since Λ is a morphism of $B(H)$-modules, $\Lambda(f_i \otimes e_i) = \varphi_i \otimes e_i$.

Claim A.

$$\mathbb{E}\left\|\Lambda\Big(\sum_{i=1}^{N} r_i f_i \otimes e_i\Big)\right\|_{\mathcal{S}_p} = \mathbb{E}\left\|\sum_{i=1}^{N} r_i \varphi_i \otimes e_i\right\|_{\mathcal{S}_p} \leq CKN^{1/p}$$

holds for $1 \leq p < \infty$ where C is a universal constant depending at most on p.

Once the claim is proved we will find that, on the other hand,

$$\begin{aligned}\mathbb{E}\left\|\Omega_p\Big(\sum_{i=1}^{N} r_i f_i \otimes e_i\Big)\right\|_{\mathcal{S}_p} &\overset{(a)}{=} \mathbb{E}\left\|\log N^{1/p}\sum_{i=1}^{N} r_i f_i \otimes e_i\right\|_{\mathcal{S}_p}\\ &\overset{(b)}{=} N^{1/p}\log N^{1/p}\end{aligned}$$

making $\|\Omega_p - \Lambda\| < +\infty$ impossible for all $T \in V$ if (2) holds. To check the last equalities (a) and (b) displayed above, notice that for any $t \in [0,1]$ the operator $T_t = \sum_{i=1}^{N} r_i(t) f_i \otimes e_i$ is a diagonal operator. Since $r_i(t) = \pm 1$, it follows that $\{\pm 1 f_i\}$ is still an orthonormal basis. Thus, $a_i(T_t) = 1$, and consequently, $\|T_t\|_p = N^{1/p}$. And now, one just needs to apply the definition of Ω_p to every T_t to obtain

$$\begin{aligned}\Omega_p(T_t) &= \sum a_i(T_t)\log\left(\frac{\|T_t\|}{a_i(T_t)}\right) r_i(t) f_i \otimes e_i\\ &= \sum \log N^{1/p} r_i(t) f_i \otimes e_i.\end{aligned}$$

Finally, one just needs to integrate to obtain equality (a). Equality (b) is immediate by the definition of norm in $\mathcal{S}_p$. We are ready to prove Claim A.

Proof of Claim A. Consider $\varphi_i = \sum_j a_{ij} e_j$, and thus $\sum_{i=1}^{N} \varphi_i \otimes e_i = \sum_{ij} a_{ij} e_j \otimes e_i$. We know that

$$\Big(\sum_j a_{ij}^2\Big)^{1/2} = \Big\|\sum_j a_{ij} e_j \otimes e_i\Big\| = \Big\|\Lambda(f_i \otimes e_i)\Big\| \leq K.$$

Then, to prove Claim A, it is enough to show that

$$\mathbb{E}\left\|\sum_{ij} a_{ij} r_i e_j \otimes e_i\right\|_{\mathcal{S}_p} \leq C\bigg(\sum_{i=1}^{N}\Big\|\sum_j a_{ij} e_j \otimes e_i\Big\|_{\mathcal{S}_p}^{p}\bigg)^{1/p}$$

for $1 \leq p < \infty$ and some universal constant C depending at most on p.

For $1 \le p \le 2$, the result follows by noting that the corresponding $\mathcal{S}_p$ has type p, so we just have to deal with the case $2 \le p < \infty$. We need a tool slightly better than the type, namely, the noncommutative version of the Khintchine inequality for Schatten classes. More precisely, in **[11]**, it was proved that, for $2 \le p < \infty$, the following holds:

$$(**) \qquad \mathbb{E}\left\| \sum_{i=1}^{N} r_i A_i \right\|_{\mathcal{S}_p} \approx \left\| \left(\sum_{i=1}^{N} A_i^* A_i \right)^{1/2} \right\|_{\mathcal{S}_p} + \left\| \left(\sum_{i=1}^{N} A_i A_i^* \right)^{1/2} \right\|_{\mathcal{S}_p}.$$

In our case, set $A_i := \sum_j a_{ij} e_j \otimes e_i$. It is clear, since A_i is a row matrix, that $A_i^* A_i = A_i A_i^* = \sum_j a_{ij}^2 e_i \otimes e_i$ and

$$\left(\sum_j a_{ij}^2 \right)^{1/2} = \|A_i\|_{\mathcal{S}_p} \le K.$$

Then the right side of $(**)$ turns into

$$\begin{aligned} 2\left\| \left(\sum_{i=1}^{N} A_i^* A_i \right)^{1/2} \right\|_{\mathcal{S}_p} &= 2\left\| \left(\sum_{i=1}^{N} \left(\sum_j a_{ij}^2 \right) e_i \otimes e_i \right)^{1/2} \right\|_{\mathcal{S}_p} \\ &= 2\left\| \sum_{i=1}^{N} \left(\sum_j a_{ij}^2 \right)^{1/2} e_i \otimes e_i \right\|_{\mathcal{S}_p} \\ &= 2\left\| \sum_{i=1}^{N} \|A_i\| \, e_i \otimes e_i \right\|_{\mathcal{S}_p} \\ &= 2\left(\sum_{i=1}^{N} \|A_i\|^p \right)^{1/p} \le 2KN^{1/p}, \end{aligned}$$

and Claim A is proved. To prove (1) implies (2); assume (2) does not hold and pick $T \in V$ of finite rank. We claim now that the least constant $c(T)$ making

$$\|\Omega_p(T)\|_p \le c(T)\|T\|_p$$

is exactly $c(T) = \log(\mathrm{rk}\,(T)^{1/p})$, and thus, since (2) does not hold,

$$\sup_{T \in V} c(T) < \infty.$$

This last result means that $\Omega_{p|V}$ is trivial, and the proof is done. So we just need to prove our claim.

This claim can be found in **[3]** under a more general form. Let us reproduce the argument for the sake of completeness. First, observe that Ω_p is a homogeneous map in the sense

$$\Omega_p(\lambda T) = \lambda \Omega_p(T)$$

with $\lambda \in \mathbb{K}$ and $T \in \mathcal{S}_p$. To prove it, notice that, by writing $T \in \mathcal{S}_p$ in spectral form and using a similar argument as in the previous proof for the orthonormal basis, one has: $a_i(\lambda T) = |\lambda| a_i(T)$. Now by putting the definition of $\Omega_p(\lambda T)$ and comparing it with $\lambda\Omega_p(T)$, it follows that Ω_p is homogeneous. Thus, since the expression $\|\Omega_p(T)\|_p \leq c(T)\|T\|_p$ is homogeneous, one may assume that $T = \sum_{i=1}^N a_i(T) f_i \otimes e_i$ is norm one, i.e., $\sum_{i=1}^N a_i(T)^p = 1$. It only remains to prove the following.

Claim B.

$$\sup\left\{\left(\sum_{i=1}^N |a_i|^p \left(-\log|a_i|\right)^p\right)^{1/p} : \sum_{i=1}^N |a_i|^p = 1\right\} = \log N^{1/p},$$

where $1 < p < \infty$.

Proof of Claim B. To compute the supremum, we use Lagrange's multiplier theorem. Thus, we write

$$\Lambda(a_i, \lambda) = \sum_{i=1}^N |a_i|^p \left(-\log|a_i|\right)^p + \lambda\left(1 - \sum_{i=1}^N |a_i|^p\right).$$

There is no loss of generality to assume that $0 < a_i < 1$ for all i. From $d\Lambda/da_i = 0$, we get $(-\log a_i)^p - (-\log a_i)^{p-1} = \lambda$. It is routine to check that the function $\psi(a) := (-\log a)^p - (-\log a)^{p-1}$ is injective, and thus $a_i = a_{i'}$ for $i, i' \in \{1, \ldots, N\}$. Consequently, it must be $a_i = N^{-1/p}$ for $i = 1, \ldots, N$ and Claim B is proved. $\square$

This result is, in a sense, optimal. It was proven by Kalton and Peck that the map Ω_p from ℓ_p to ℓ_p (with ambient space ℓ_∞) is singular for $1 < p < \infty$, which means that its restriction to any infinite dimensional subspace is not trivial.

3. The duality theorem. We make the analogue proof of **[10,** Theorem 5.1**]**. The crucial step in the proof is the following trivial inequality.

Lemma 3.1. *The following expression holds*:

$$\left|ts\left(\log\frac{|t|}{|s|^{p-1}}\right)\right| \le \frac{p-1}{e}\left(|t|^q+|s|^p\right),$$

for $1 = 1/p + 1/q$ *and* $1 < p, q < \infty$.

This lemma corresponds to the case $n = 1$ of [**10,** Lemma 5.2].

Theorem 3.2. *There exists an isomorphism* φ *making the following diagram commute*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}_q & \xrightarrow{j} & \Theta_q & \xrightarrow{q} & \mathcal{S}_q & \longrightarrow & 0 \equiv \frac{1}{1-p}\Omega_q, \\
 & & \big\downarrow{\scriptstyle \mathrm{tr}} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \mathrm{tr}} & & \\
0 & \longrightarrow & (\mathcal{S}_p)^* & \xrightarrow{j^*} & (\Theta_p)^* & \xrightarrow{q^*} & (\mathcal{S}_p)^* & \longrightarrow & 0 \equiv (\Omega_p)^*,
\end{array}$$

where tr *denotes trace duality and* $1 = 1/p + 1/q$ *for* $1 < p, q < \infty$.

Proof. Let us denote by $\mathfrak{F}$ the space of finite rank operators acting between a Hilbert space and by $\mathfrak{F}_r$ when endowed with the corresponding $\mathcal{S}_r$ norm. Given $T \in \mathfrak{F}$ with spectral decomposition $T = \sum a_i(T)e_i \otimes f_i$, let us write by simplicity:

$$\Omega_p(T) = \sum a_i(T)\log\left(\frac{\|T\|_{\mathcal{S}_p}}{a_i(T)}\right)e_i \otimes f_i,$$
$$\Omega_q(T) = \frac{1}{1-p}\sum a_i(T)\log\left(\frac{\|T\|_{\mathcal{S}_q}}{a_i(T)}\right)e_i \otimes f_i.$$

We define the map $\varphi : \mathfrak{F}_q \oplus_{\Omega_q(T)} \mathfrak{F}_q \to \Theta_p^*$ by the formula

$$(\varphi(S,T))\,(V,W) = \mathrm{tr}\,(SW) + \mathrm{tr}\,(TV).$$

Let us rewrite the expression above as:

$$(*) \qquad \mathrm{tr}\,(SW) + \mathrm{tr}\,(TV) = \mathrm{tr}\,(T(V - \Omega_p(W))) + \mathrm{tr}\,(T\Omega_p(W)) \\ + \mathrm{tr}\,(\Omega_q(T)W) + \mathrm{tr}\,((S - \Omega_q(T))W).$$

Setting $T = \sum a_i(T)e_i \otimes f_i$ and $W = \sum a_j(W)u_j \otimes v_j$, we easily get

$$T\Omega_p(W) = \sum a_j(W)a_i(T)\log\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)(e_i|v_j)u_j \otimes f_i,$$

$$\Omega_q(T)W = \frac{1}{1-p}\sum a_j(W)a_i(T)\log\left(\frac{\|T\|_{\mathcal{S}_q}}{a_i(T)}\right)(e_i|v_j)u_j \otimes f_i,$$
$$\mathrm{tr}\,(T\Omega_p(W)) = \sum a_j(W)a_i(T)\log\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)(e_i|v_j)(f_i|u_j),$$
$$\mathrm{tr}\,(\Omega_q(T)W) = \frac{1}{1-p}\sum a_j(W)a_i(T)\log\left(\frac{\|T\|_{\mathcal{S}_q}}{a_i(T)}\right)(e_i|v_j)(f_i|u_j).$$

We need to prove that expression $(*)$ is bounded. To this end, we write

$$\begin{aligned}|\mathrm{tr}\,(SW) + \mathrm{tr}\,(TV)| \leq{}& |\mathrm{tr}\,(T(V - \Omega_p(W)))| + |\mathrm{tr}\,(T\Omega_p(W)) \\ &+ \mathrm{tr}\,(\Omega_q(T)W)| + |\mathrm{tr}\,((S - \Omega_q(T))W)|.\end{aligned}$$

The quantities $|\mathrm{tr}\,(T(V - \Omega_p(W)))|$ and $|\mathrm{tr}\,((S - \Omega_q(T))W)|$ can be easily bounded. Assume for a moment that

$$|\mathrm{tr}\,(T\Omega_p(W)) + \mathrm{tr}\,(\Omega_q(T)W)| \leq \|W\|\|T\|. \tag{1}$$

Then observe

$$\begin{aligned}|\mathrm{tr}\,(SW) + \mathrm{tr}\,(TV)| &\leq |\mathrm{tr}\,(T(V - \Omega_p(W)))| + |\mathrm{tr}\,(T\Omega_p(W)) \\ &\quad + \mathrm{tr}\,(\Omega_q(T)W)| + |\mathrm{tr}\,((S - \Omega_q(T))W)| \\ &\leq \|(V - \Omega_p(W)\|_{\mathcal{S}_p}\|T\|_{\mathcal{S}_q} + \|W\|_{\mathcal{S}_p}\|T\|_{\mathcal{S}_q} \\ &\quad + \|W\|_{\mathcal{S}_p}\|(S - \Omega_q(T)\|_{\mathcal{S}_q} \\ &\leq \|(S,T)\|_{\Theta_q}\|(V,W)\|_{\Theta_p}.\end{aligned}$$

Therefore, the expression $(*)$ defines a bounded operator. So all the rest is to convince us that the bound (1) holds for arbitrary $T, W \in \mathfrak{F}$. To this end, we may bound the left side of expression (1) by:

$$\begin{aligned}(**)\qquad &= \left|\sum a_j(W)a_i(T)(e_i|v_j)(f_i|u_j)\log\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)\right. \\ &\quad \left. + \frac{1}{1-p}\sum a_j(W)a_i(T)(e_i|v_j)(f_i|u_j)\log\left(\frac{\|T\|_{\mathcal{S}_q}}{a_i(T)}\right)\right| \\ &= \left|\sum a_j(W)a_i(T)(e_i|v_j)(f_i|u_j)\log\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\left(\frac{a_i(T)}{\|T\|_{\mathcal{S}_q}}\right)^{1/p-1}\right)\right| \\ &= \frac{1}{p-1}\left|\sum a_j(W)a_i(T)(e_i|v_j)(f_i|u_j)\right.\end{aligned}$$

$$\times \log\left(\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)^{p-1}\frac{a_i(T)}{\|T\|_{\mathcal{S}_q}}\right)\Bigg|$$
$$\le \frac{1}{p-1}\sum_{i,j}|a_j(W)a_i(T)(e_i|v_j)(f_i|u_j)$$
$$\times \log\left(\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)^{p-1}\frac{a_i(T)}{\|T\|_{\mathcal{S}_q}}\right)\Bigg|$$
$$\le \frac{1}{p-1}\left(\sum_{i,j}|c_{i,j}||(e_i|v_j)|^2\right)^{1/2}\left(\sum_{i,j}|c_{i,j}||(f_i|u_j)|^2\right)^{1/2},$$

where

$$c_{i,j} := a_j(W)a_i(T)\log\left(\left(\frac{\|W\|_{\mathcal{S}_p}}{a_j(W)}\right)^{p-1}\frac{a_i(T)}{\|T\|_{\mathcal{S}_q}}\right).$$

Now recall that, by Lemma 3.1,

$$c_{i,j} \le \frac{p-1}{e}\|W\|\|T\|\left(\frac{a_i(T)^q}{\|T\|^q}+\frac{a_j(W)^p}{\|W\|^p}\right).$$

Thus, we find that the last expression $(**)$ is bounded by

$$\frac{\|W\|\|T\|\frac{p-1}{e}}{p-1}\left(\sum_{i,j}\left(\frac{a_i(T)^q}{\|T\|^q}+\frac{a_j(W)^p}{\|W\|^p}\right)|(e_i|v_j)|^2\right)^{1/2}$$
$$\left(\sum_{i,j}\left(\frac{a_i(T)^q}{\|T\|^q}+\frac{a_j(W)^p}{\|W\|^p}\right)|(f_i|u_j)|^2\right)^{1/2}.$$

To finish, let us observe the following upper bounds for the last expression:

$$\frac{\|W\|\|T\|}{e}\left(\sum_{i,j}\frac{a_i(T)^q}{\|T\|^q}|(e_i|v_j)|^2+\sum_{i,j}\frac{a_j(W)^p}{\|W\|^p}|(e_i|v_j)|^2\right)^{1/2}$$
$$\left(\sum_{i,j}\frac{a_i(T)^q}{\|T\|^q}|(f_i|u_j)|^2+\sum_{i,j}\frac{a_j(W)^p}{\|W\|^p}|(f_i|u_j)|^2\right)^{1/2}$$
$$\le \frac{\|W\|\|T\|}{e}\left(\sum_i\frac{a_i(T)^q}{\|T\|^q}\sum_j|(e_i|v_j)|^2+\sum_j\frac{a_j(W)^p}{\|W\|^p}\sum_i|(e_i|v_j)|^2\right)^{1/2}$$

$$\left(\sum_i \frac{a_i(T)^q}{\|T\|^q}\sum_j |(f_i|u_j)|^2 + \sum_j \frac{a_j(W)^p}{\|W\|^p}\sum_i |(f_i|u_j)|^2\right)^{1/2}$$
$$\leq \frac{\|W\|\|T\|}{e}\left(\sum_i \frac{a_i(T)^q}{\|T\|^q} + \sum_j \frac{a_j(W)^p}{\|W\|^p}\right) = \frac{2}{e}\|W\|\|T\|.$$

This last result means that φ is bounded and can be extended to a bounded map. Clearly, φ makes the diagram of Theorem 3.2 commute so, by the 3-lemma, [**4,** page 3], it is an isomorphism. $\square$

Corollary 3.3. *Θ_p^* is isomorphic to Θ_q for $1/p + 1/q = 1$.*

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