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WHEN THE COMAXIMAL AND ZERO-DIVISOR GRAPHS ARE RING GRAPHS AND OUTERPLANAR

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ABSTRACT. In this paper, we characterize the finite commutative rings such that their comaximal graph (or zero-divisor graph) are ring graphs, and we also study the case where they are outerplanar.

1. Introduction. Suppose that G is a graph with n vertices and q edges. Also assume that C is a cycle of G. A chord in G is any edge joining two nonadjacent vertices in C. A primitive cycle is a cycle without chords. Moreover, we say that a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The free rank of G, denoted by $\operatorname{frank}(G)$, is the number of primitive cycles of G. Also, the number $\operatorname{rank}(G) := q-n+r$, where r is the number of connected components of G, is called the cycle rank of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. These two numbers satisfy the inequality $\operatorname{rank}(G) \leq \operatorname{frank}(G)$, as is seen in [**7,** Proposition 2.2]. In the second section of [**7**], the authors provided a characterization of graphs such that the equality occurs.

The precise definition of a ring graph can be found in [**7,** Section 2]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. They showed that, for the graph G, the following conditions are equivalent:

(i) G is a ring graph,
(ii) $\operatorname{rank}(G) = \operatorname{frank}(G)$,

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(iii) G satisfies PCP and G does not contain a subdivision of K_4 as a subgraph.

Thus, every ring graph is planar. Moreover, in [**7**], the authors also showed that every outerplanar graph is a ring graph. Recently, in [**1**], the present authors investigated when the unit, unitary and total graphs are ring graphs, and also studied when these graphs are outerplanar. In this paper, we answer these questions for comaximal and zero-divisor graphs.

Now, we review some background of graph theory from [**6**]. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. Clearly, every outerplanar graph is planar.

Throughout the paper, R is a finite commutative ring with nonzero identity. Also, we denote the set of all unit elements and zero-divisor elements of R by $U(R)$ and $Z(R)$, respectively. For simplicity of notation, in the quotient ring $K[x]/I$, we denote the coset $x+I$ by X.

2. Ring graphs and outerplanar Comaximal graphs. In [**9**], Sharma and Bhatwadekar defined the comaximal graph of a commutative ring R, denoted by $\Gamma'(R)$, with vertices all elements of R and two distinct vertices a and b are adjacent if and only if $aR+bR=R$. In [**8, 10, 12**], a subgraph of the comaximal graph, denoted by $\Gamma_2(R)$, with non-unit elements of R as vertices, was studied. By [**12,** Corollary 5.3], we have that the comaximal graph $\Gamma'(R)$ is planar if and only if R is isomorphic to one of the following rings:

(i) $R \cong \mathbb{Z}_2$,
(ii) $R \cong \mathbb{Z}_3$,
(iii) $R \cong \mathbb{Z}_4$,
(iv) $R \cong \mathbb{Z}_2[x]/(x^2)$,
(v) $R \cong \mathbb{F}_4$,
(vi) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,
(vii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$,
(viii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In order to characterize all finite rings R such that $\Gamma'(R)$ is a ring graph, we need only check the planar comaximal graphs.

Theorem 2.1. *The graph $\Gamma'(R)$ is a ring graph if and only if R is isomorphic to one of the following rings:*

(i) $R \cong \mathbb{Z}_2$,
(ii) $R \cong \mathbb{Z}_3$,
(iii) $R \cong \mathbb{Z}_4$,
(iv) $R \cong \mathbb{Z}_2[x]/(x^2)$,
(v) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. At first, we assume that $\Gamma'(R)$ is a ring graph. Since every ring graph is planar, we have $\Gamma'(R)$ is planar. Thus we have the following cases:

Case 1. $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$. It is easy to see that $\Gamma'(\mathbb{Z}_2)$ and $\Gamma'(\mathbb{Z}_3)$ are complete graphs with 2 and 3 vertices, respectively. Thus, they are ring graphs.

Case 2. $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$. For these rings, by Figures 1 and 2, we have $\operatorname{rank}(\Gamma'(R)) = \operatorname{frank}(\Gamma'(R)) = 2$.

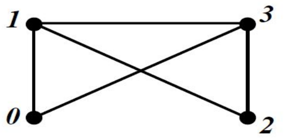

FIGURE 1. $\Gamma'(\mathbb{Z}_4)$

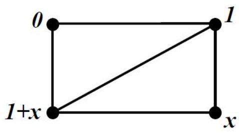

FIGURE 2. $\Gamma'(\mathbb{Z}_2[x]/(x^2))$.

Case 3. $R \cong \mathbb{F}_4$. In this case, $\Gamma'(\mathbb{F}_4)$ is a complete graph with 4 vertices, and so it is isomorphic to K_4, which implies that $\Gamma'(\mathbb{F}_4)$ is not a ring graph.

Case 4. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Figure 3, we have $\operatorname{rank}(\Gamma'(R)) = \operatorname{frank}(\Gamma'(R)) = 1$.

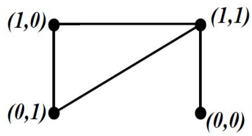

FIGURE 3. $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Case 5. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. The induced subgraph of $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_3)$ on the set $\{(1,0),(0,1),(1,1),(1,2)\}$ is a complete graph, and so $\Gamma'(R)$ is not a ring graph.

Case 6. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The induced subgraph of $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ on the set $\{(1,0,1),(0,1,1),(1,1,1),(1,1,0)\}$ is isomorphic to K_4, and so it is not a ring graph.

The converse statement follows easily. $\square$

Theorem 2.2. $\Gamma'(R)$ *is outerplanar if and only if it is a ring graph.*

Proof. Suppose that $\Gamma'(R)$ is outerplanar. Since outerplanar graphs are ring graphs, by Theorem 2.1, R is one of the following rings:

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Now, by Figures 1, 2 and 3, one can easily see that $\Gamma'(R)$ is outerplanar.

Conversely, if R is one of the rings $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2$, then one can easily check that $\Gamma'(R)$ is outerplanar. $\square$

In the rest of this section, we study the case where $\Gamma_2(R)$ is a ring graph and outerplanar. It is easy to see that, if R is local, then $\Gamma_2(R)$ is a totally disconnected graph. Hence, without loss of generality, we may assume that R is not local. In [**12,** Corollary 6.3], it was proved that $\Gamma_2(R)$ is planar if and only if R is isomorphic to one of the following rings:

(i) $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$,
(ii) $R \cong \mathbb{Z}_3 \times \mathbb{F}_q$,
(iii) $R \cong \mathbb{Z}_4 \times \mathbb{F}_q$,
(iv) $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_q$,
(v) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$,

(vi) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$,
(vii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In the following theorem, we determine all non-local finite rings R such that $\Gamma_2(R)$ are ring graphs.

Theorem 2.3. *The graph $\Gamma_2(R)$ is a ring graph if and only if R is isomorphic to one of the following rings*:

(i) $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$,
(ii) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$,
(iii) $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$,
(iv) $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$,
(v) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since R is finite, we have $R = U(R) \cup Z(R)$. So, the vertex set of $\Gamma_2(R)$ is $Z(R)$. Also, it is clear that 0 is an isolated vertex in $\Gamma_2(R)$. Thus, we focus on the induced subgraph of $\Gamma_2(R)$ with vertices $Z(R) \setminus \{0\}$.

First, assume that $\Gamma_2(R)$ is a ring graph. Since every ring graph is planar, we have $\Gamma_2(R)$ is planar. Thus we have the following cases:

Case 1. $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$. Suppose that $\mathbb{F}_q = \{0\} \cup \{a^i; 1 \leq i \leq q-1\}$, where a is a non zero element in $\mathbb{F}_q$. Note that $Z(R) \setminus \{(0,0)\} = \{(1,0)\} \cup \{(0,a^i); 1 \leq i \leq q-1\}$. Put $V_1 := \{(1,0)\}$ and $V_2 := \{(0,a^i); 1 \leq i \leq q-1\}$. Then the induced subgraph $\Gamma_2(R)$ on $Z(R) \setminus \{(0,0)\}$ is isomorphic to $K_{1,q-1}$, and so it is a star graph. Hence, $\Gamma_2(R)$ is a ring graph.

Case 2. $R \cong \mathbb{Z}_3 \times \mathbb{F}_q$. Again, assume that $\mathbb{F}_q = \{0\} \cup \{a^i; 1 \leq i \leq q-1\}$, where a is a non zero element in $\mathbb{F}_q$. Note that $Z(R) \setminus \{(0,0)\} = \{(1,0),(2,0)\} \cup \{(0,a^i); 1 \leq i \leq q-1\}$. Set $V_1 := \{(1,0),(2,0)\}$ and $V_2 := \{(0,a^i); 1 \leq i \leq q-1\}$. Clearly, the induced subgraph $\Gamma_2(R)$ on $Z(R) \setminus \{(0,0)\}$ is a complete bipartite graph which is isomorphic to $K_{2,q-1}$. Since the primitive cycles of this graph have length 4, we have that $\mathrm{frank}(\Gamma_2(R)) = (q-1)(q-2)/2$. On the other hand, $\mathrm{rank}(\Gamma_2(R)) = q-2$. Therefore, $\mathrm{rank}(\Gamma_2(R)) = \mathrm{frank}(\Gamma_2(R))$ if and only if $(q-1)/2 = 1$ or $q = 2$. So $\mathrm{rank}(\Gamma_2(R)) = \mathrm{frank}(\Gamma_2(R))$ if and only if $q = 2$ or $q = 3$. Hence, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Case 3. $R \cong \mathbb{Z}_4 \times \mathbb{F}_q$. By considering $\mathbb{F}_q = \{0\} \cup \{a^i; 1 \leq i \leq q-1\}$, we have $Z(R) \setminus \{(0,0)\} = \{(1,0),(3,0),(2,0)\} \cup \{(0,a^i),(2,a^i); 1 \leq i \leq q-1\}$. Clearly, $(2,0)$ is an isolated vertex in $\Gamma_2(R) \setminus \{(0,0)\}$. Put $V_1 := \{(1,0),(3,0)\}$ and $V_2 := \{(0,a^i),(2,a^i); 1 \leq i \leq q-1\}$. Now the induced subgraph $\Gamma_2(R) \setminus \{(0,0)\}$ on $Z(R) \setminus \{(0,0),(2,0)\}$ is a complete bipartite graph which is isomorphic to $K_{2,2q-2}$. Since the primitive cycles of this graph have length 4, $\operatorname{frank}(\Gamma_2(R)) = (q-1)(2q-3)$. Also, $\operatorname{rank}(\Gamma_2(R)) = 2q-3$. Hence, $\Gamma_2(R)$ is a ring graph if and only if $q-1=1$. Thus, $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Case 4. $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_q$. Let $\mathbb{F}_q$ be as above. Then we have

$$Z(R)\setminus\{(0,0)\} = \{(1,0),(1+X,0),(X,0)\} \cup \{(0,a^i),(X,a^i); 1 \leq i \leq q-1\}.$$

The vertex $(X,0)$ is an isolated vertex in $\Gamma_2(R) \setminus \{(0,0)\}$, and the induced subgraph of $\Gamma_2(R)$ with vertices $Z(R) \setminus \{(0,0),(X,0)\}$ is isomorphic to $K_{2,2q-2}$. Clearly, $\operatorname{frank}(\Gamma_2(R)) = (q-1)(2q-3)$. Also, $\operatorname{ran}(\Gamma_2(R)) = 2q-3$. Hence, $\Gamma_2(R)$ is a ring graph if and only if $q-1=1$. Thus, $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$.

Case 5. $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. It is easy to see that $Z(R) \setminus \{(0,0)\} = \{(0,1),(0,2),(0,3),(1,0),(2,0),(1,2),(2,2)\}$. The vertex $(0,2)$ is an isolated vertex in $\Gamma_2(R) \setminus \{(0,0)\}$, and the induced subgraph of $\Gamma_2(R)$ with vertex set $Z(R) \setminus \{(0,0),(0,2)\}$ is isomorphic to $K_{2,4}$. So, $\operatorname{frank}(\Gamma_2(R)) = 6$. Also, $\operatorname{rank}(\Gamma_2(R)) = 3$. Hence, $\Gamma_2(R)$ is not a ring graph.

Case 6. $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$. Clearly, $Z(R) \setminus \{(0,0)\} = \{(0,1),(0,X),(0,1+X),(1,0),(2,0),(1,X),(2,X)\}$. The vertex $(0,X)$ is an isolated point in graph $\Gamma_2(R) \setminus \{(0,0)\}$, and the induced subgraph $\Gamma_2(R) \setminus \{(0,0)\}$ on $Z(R) \setminus \{(0,0),(0,X)\}$ is a complete bipartite graph which is isomorphic to $K_{2,4}$. So, $\operatorname{frank}(\Gamma_2(R)) = 6$. Also, $\operatorname{rank}(\Gamma_2(R)) = 3$. Hence, $\Gamma_2(R)$ is not a ring graph.

Case 7. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Figure 6, we have $\operatorname{frank}(\Gamma_2(R)) = \operatorname{rank}(\Gamma_2(R)) = 1$. So $\Gamma_2(R)$ is a ring graph.

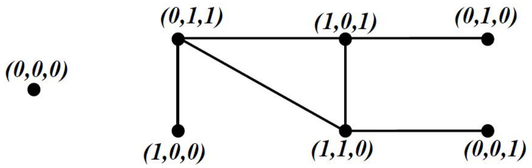

FIGURE 4. $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

The converse statement follows easily. $\square$

Theorem 2.4. *$\Gamma_2(R)$ is outerplanar if and only if it is a ring graph.*

Proof. Suppose that $\Gamma_2(R)$ is outerplanar. Since an outerplanar graph is a ring graph, by Theorem 2.3, R is one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{F}_q, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and one can easily check that $\Gamma_2(R)$ is outerplanar.

The converse statement is clear. $\square$

3. Ring graphs and outerplanar zero-divisor graphs. The zero-divisor graph $\Gamma(R)$ is a graph with vertex set $Z(R) \setminus \{0\}$ and two distinct vertices a and b are adjacent if and only if $ab = 0$. The planarity of $\Gamma(R)$ was studied in **[2, 3, 4, 11]**. In this section, we investigate all finite commutative rings R such that their zero-divisor graphs are ring graphs and also outerplanar.

Theorem 3.1. *Let R be a finite ring and $\mathbb{F}$ a finite field. Then the zero-divisor graph $\Gamma(R)$ is a ring graph if and only if R is isomorphic to one of the following rings*:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

(ii) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$,

(iii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2)$, $\mathbb{Z}_4[x]/(x^2+2x+2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^2,y^2)$, $\mathbb{Z}_4[x]/(2x, x^3-2)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$,

$\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$, $\mathbb{Z}_{27}$, $\mathbb{Z}_9[x]/(x^2-3, 3x)$, $\mathbb{Z}_9[x]/(x^2-6, 3x)$, $\mathbb{Z}_3[x]/(x^3)$.

Proof. Let R be a finite ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, for some $n \geqslant 1$, and each R_i is a local ring. Now, we consider the following cases:

Case 1. $n \geqslant 4$. In this case, as was shown in **[2, 3]**, $\Gamma(R)$ is not planar, which implies that $\Gamma(R)$ is not a ring graph.

Case 2. $n = 3$. In **[2, 3]**, it was proved that $\Gamma(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Also, it is easy to see that

$$\operatorname{rank}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = \operatorname{frank}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1,$$

and so $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is a ring graph. Consider the two cycles $(0,0,1) - (1,1,0) - (0,0,2) - (1,0,0) - (0,0,1)$ and $(0,0,1) - (0,1,0) - (0,0,2) - (1,0,0) - (0,0,1)$ in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ to deduce that the zero-divisor graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ does not satisfy PCP, and thus it is not a ring graph.

Case 3. $n = 2$. In **[2, 3]**, it was shown that $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2), \mathbb{Z}_2 \times R_2, \mathbb{Z}_3 \times R_2,$$

where $|Z(R_2)| \leqslant 3$. So, we have the following situations:

(i) R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2).$$

It is easy to see that

$$\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2)).$$

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_8$. Then, by Figure 5, we can easily find a subdivision of K_4. Hence, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3))$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2))$ are not ring graphs.

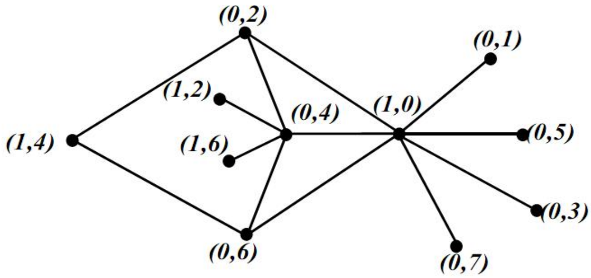

FIGURE 5. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8)$.

(ii) $R \cong \mathbb{Z}_2 \times R_2$ or $R \cong \mathbb{Z}_3 \times R_2$, where $|Z(R_2)| \leqslant 3$. If $|Z(R_2)| = 1$, then R_2 is a field. Suppose that $R_2 \cong \mathbb{F}$ with $|\mathbb{F}| = m$. Now, one can easy check that $\Gamma(\mathbb{Z}_2 \times \mathbb{F})$ is a star graph, which implies that $\Gamma(\mathbb{Z}_2 \times \mathbb{F})$ is a ring graph. Also, $\Gamma(\mathbb{Z}_3 \times \mathbb{F}) \cong K_{2,m-1}$. Thus, $\operatorname{rank}(\Gamma(\mathbb{Z}_3 \times \mathbb{F})) = m - 2$ and $\operatorname{frank}(\Gamma(\mathbb{Z}_3 \times \mathbb{F})) = (m-1)(m-2)/2$. Therefore, $\Gamma(\mathbb{Z}_3 \times \mathbb{F})$ is a ring graph if and only if $(m-1)(m-2)/2 = m-2$. So we must have $m = 2$ or $(m-1)/2 = 1$. Hence, $\Gamma(\mathbb{Z}_3 \times \mathbb{F})$ is a ring graph if and only if $m = 2$ or 3. If $|Z(R_2)| = 2$, then R_2 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. One can easily check that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))$. Now, by Figure 6, we have $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ are ring graphs. Also, by Figure 7, the two graphs $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ do not satisfy the PCP, and so they are not ring graphs.

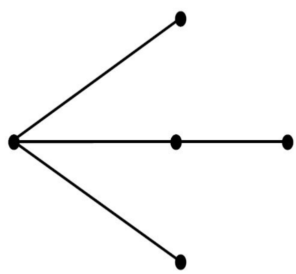

FIGURE 6. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$.

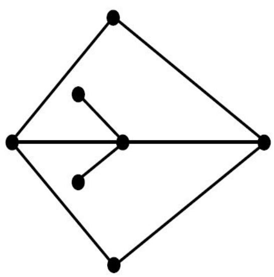

FIGURE 7. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))$.

If $|Z(R_2)| = 3$, then $R_2 \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$. It is easy to see that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2))$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2))$. So, by Figure 8, the graphs $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2))$ are ring graphs. Now, consider the two cycles $(1,0)-(0,1)-(2,0)-(0,2)-(0,1)$ and $(1,0)-(0,4)-(2,0)-(0,2)-(0,1)$ in the graphs $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, respectively, to deduce that these zero-divisor graphs do not satisfy PCP, and so they are not ring graphs.

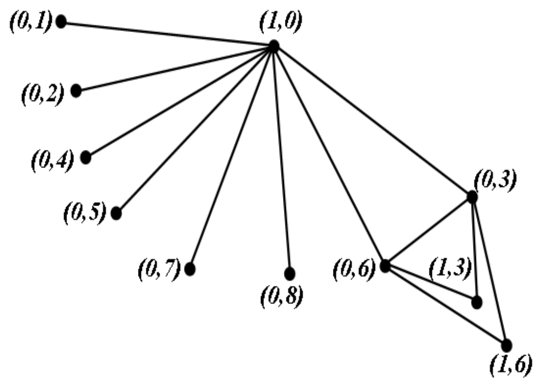

FIGURE 8. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9)$

Case 4. $n = 1$. In this case, R is a local ring. Note that, if R is a field, then $\Gamma(R)$ is an empty graph. Thus, without loss of generality, we may assume that R is not a field. In **[2,** Section 1], it was proved that, if $|R| > 32$, then $\Gamma(R)$ is not planar. Also, by **[4,** Proposition 5] and **[11]**, if $|R| = 32$, then $\Gamma(R)$ is not planar. Hence, if $|R| \geqslant 32$, then $\Gamma(R)$ is not a ring graph. Now, if R is a local ring with non-empty planar zero-divisor graph, then, by **[4,** Propositions 2,3,4] and **[11]**, R is isomorphic to one of the following rings of order 4, 8, 9, 16, 25 or 27:

$\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$,
$\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$,
$\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$,
$\mathbb{Z}_{16}$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2)$, $\mathbb{Z}_4[x]/(x^2+2x+2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$,
$\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(2x, x^3-2)$,
$\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$,
$\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$,
$\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$,
$\mathbb{Z}_{27}$, $\mathbb{Z}_9[x]/(x^2-3, 3x)$, $\mathbb{Z}_9[x]/(x^2-6, 3x)$, $\mathbb{Z}_3[x]/(x^3)$.

Now one can easily check the following graph isomorphisms:

$\Gamma(\mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_2[x]/(x^2)) \cong K_1$,
$\Gamma(\mathbb{Z}_8) \cong \Gamma(\mathbb{Z}_2[x]/(x^3)) \cong \Gamma(\mathbb{Z}_4[x]/(2x, x^2-2)) \cong K_{1,2}$,
$\Gamma(\mathbb{Z}_2[x,y]/(x,y)^2) \cong \Gamma(\mathbb{Z}_4[x]/(2x, x^2)) \cong K_3$,
$\Gamma(\mathbb{Z}_9) \cong \Gamma(\mathbb{Z}_3[x]/(x^2)) \cong K_2$,
$\Gamma(\mathbb{F}_4[x]/(x^2)) \cong \Gamma(\mathbb{Z}_4[x]/(x^2+x+1)) \cong K_3$.

Therefore, for all the rings,

$\mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$,
$\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$,
$\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$,

their zero-divisor graphs are ring graphs.

Also, the figures of the graphs $\Gamma(\mathbb{Z}_4[x]/(x^2-2))$ and $\Gamma(\mathbb{Z}_4[x]/(x^2+2x+2))$ are isomorphic to Figure 10, and are ring graphs.

Also, we have

$$\Gamma(\mathbb{Z}_{25}) \cong \Gamma(\mathbb{Z}_5[x]/(x^2)) \cong K_4,$$

which implies that they are not ring graphs.

It is not hard to see that

$$\Gamma(\mathbb{Z}_{27}) \cong \Gamma(\mathbb{Z}_9[x]/(x^2 \pm 3, 3x)) \cong \Gamma(\mathbb{Z}_3[x]/(x^3)).$$

Now, by Figure 9, $\mathrm{frank}\,(\Gamma(\mathbb{Z}_{27})) = \mathrm{rank}\,(\Gamma(\mathbb{Z}_{27})) = 6$. So $\Gamma(\mathbb{Z}_{27})$, $\Gamma(\mathbb{Z}_9[x]/(x^2 \pm 3, 3x))$ and $\Gamma(\mathbb{Z}_3[x]/(x^3))$ are ring graphs.

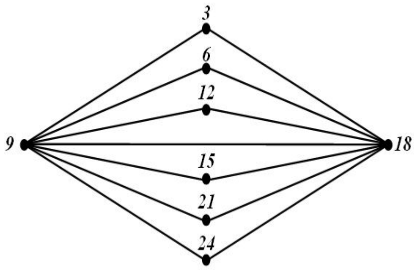

FIGURE 9. $\Gamma(\mathbb{Z}_{27})$.

It is easy to see that $\Gamma(\mathbb{Z}_{16}) \cong \Gamma(\mathbb{Z}_2[x]/(x^4)) \cong \Gamma(\mathbb{Z}_4[x]/(2x, x^3 - 2))$. Now, by Figure 10, we have $\mathrm{frank}\,(\Gamma(\mathbb{Z}_{16})) = \mathrm{rank}\,(\Gamma(\mathbb{Z}_{16}))$. Thus, $\Gamma(\mathbb{Z}_{16})$, $\Gamma(\mathbb{Z}_2[x]/(x^4))$ and $\Gamma(\mathbb{Z}_4[x]/(2x, x^3 - 2))$ are ring graphs.

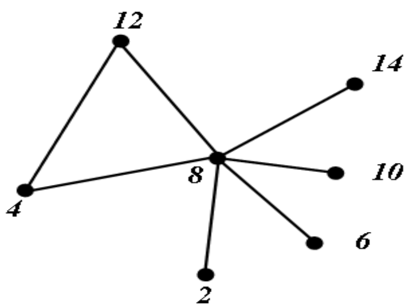

FIGURE 10. $\Gamma(\mathbb{Z}_{16})$.

By Figure 11, one can easily find a subdivision of K_4 in $\Gamma(\mathbb{Z}_2[x, y]/(x^2 - y^2, xy))$. Hence, $\Gamma(\mathbb{Z}_2[x, y]/(x^2 - y^2, xy))$ is not a ring graph.

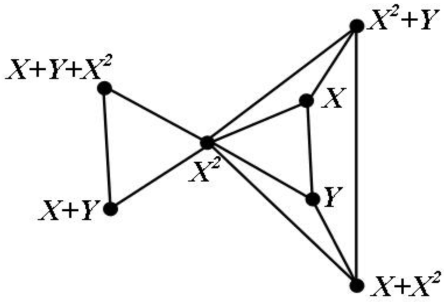

FIGURE 11. $\Gamma(\mathbb{Z}_2[x,y]/(x^2-y^2,xy))$

By Figure 12, we have

$$\operatorname{frank}(\Gamma(\mathbb{Z}_2[x,y]/(x^2,y^2))) = \operatorname{rank}(\Gamma(\mathbb{Z}_2[x,y]/(x^2,y^2))) = 3,$$

and so $\Gamma(\mathbb{Z}_2[x,y]/(x^2,y^2))$ is a ring graph.

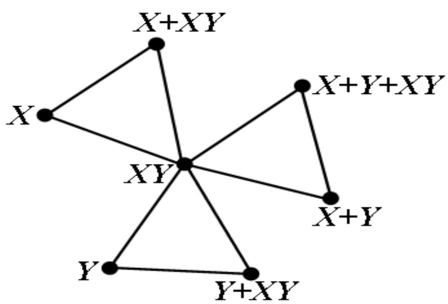

FIGURE 12. $\Gamma(\mathbb{Z}_2[x,y]/(x^2,y^2))$.

By Figure 13, the graph $\Gamma(\mathbb{Z}_4[x,y]/(x^2-2,xy,y^2-2,2x))$ has a subdivision of K_4, which implies that $\Gamma(\mathbb{Z}_4[x,y]/(x^2-2,xy,y^2-2,2x))$ is not a ring graph.

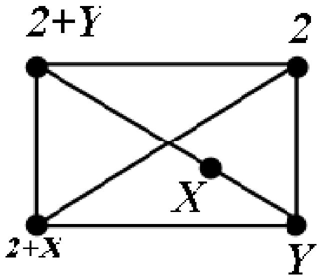

FIGURE 13.

It is easy to see that $\Gamma(\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)) \cong \Gamma(\mathbb{Z}_4[x]/(x^2))$. By Figure 14, we have

$$\text{frank}\,(\Gamma(\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2))) = \text{rank}\,(\Gamma(\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2))) = 3.$$

Thus, $\Gamma(\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2))$ and $\Gamma(\mathbb{Z}_4[x]/(x^2))$ are ring graphs.

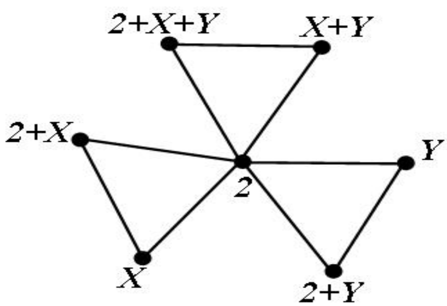

FIGURE 14. $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2))$.

By Figure 15, the graph $\Gamma(\mathbb{Z}_4[x]/(x^2-2x))$ has a subdivision of K_4, and so it is not a ring graph.

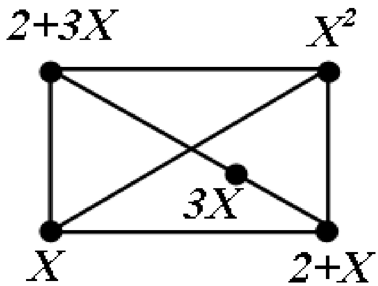

FIGURE 15.

By Figure 16, the graph $\Gamma(\mathbb{Z}_8[x]/(2x, x^2-4))$ contains a subdivision of K_4 with vertex set $\{2, 4, 6, 4+X, X\}$. Therefore, $\Gamma(\mathbb{Z}_8[x]/(2x, x^2-4))$ is not a ring graph.

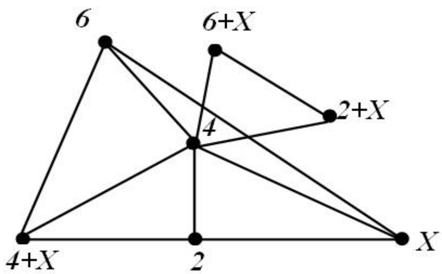

FIGURE 16. $\Gamma(\mathbb{Z}_8[x]/(2x, x^2-4))$.

The converse statement is provided straightforward. □

Now, since every outerplanar graph is a ring graph and we determined all finite commutative rings with ring graph zero-divisor graphs, one can establish a characterization for all finite commutative rings such that their zero-divisor graphs are outerplanar. Note that, in view of the proof of Theorem 3.1, one can easily check that among all zero-divisor graphs which are ring graphs, the zero-divisor graph of the rings: $\mathbb{Z}_2\times\mathbb{Z}_9$, $\mathbb{Z}_2\times\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$, $\mathbb{Z}_{27}$, $\mathbb{Z}_9[x]/(x^2-3, 3x)$, $\mathbb{Z}_9[x]/(x^2-6, 3x)$, $\mathbb{Z}_3[x]/(x^3)$, contain a copy of $K_{2,3}$, and so they are not outerplanar. Therefore, we have the following result.

Theorem 3.2. *Let R be a finite ring and $\mathbb{F}$ a finite field. The zero-divisor graph $\Gamma(R)$ is outerplanar if and only if R is isomorphic to one of the following rings*:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

(ii) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$,

(iii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2)$, $\mathbb{Z}_4[x]/(x^2+2x+2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(2x, x^3-2)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$.

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