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MULTIPLICITY RESULTS ON PERIODIC SOLUTIONS TO HIGHER-DIMENSIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. This paper continues our study on the existence and multiplicity of periodic solutions to delay differential equations of the form

$$\dot{z}(t) = -f(z(t-1)) - f(z(t-2)) - \cdots - f(z(t-n+1)),$$

where $z \in \mathbf{R}^N$, $f \in C(\mathbf{R}^N, \mathbf{R}^N)$ and $n > 1$ is an odd number. By using the Galerkin approximation method and the S^1-index theory in the critical point theory, some known results for Kaplan-Yorke type differential delay equations are generalized to the higher-dimensional case. As a result, the Kaplan-Yorke conjecture is proved to be true in the case of higher-dimensional systems.

1. Introduction. In 1974, Kaplan and Yorke [**20**] studied the existence of periodic solutions to delay differential equations of the forms

$$x'(t) = -f(x(t-1)), \quad x \in \mathbf{R} \tag{1.1}$$

and

$$x'(t) = -f(x(t-1)) - f(x(t-2)), \quad x \in \mathbf{R}, \tag{1.2}$$

when f is an odd function. They reduced equations (1.1) and (1.2) to coupled ordinary differential equations and derived some precise conditions under which equations (1.1) and (1.2) have periodic solutions of periods 4 and 6, respectively. For more general equations of the form

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n+1)), \quad x \in \mathbf{R}, \tag{1.3}$$

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Kaplan and Yorke conjectured that when f is a suitable odd function, the existence of periodic solutions of (1.3) with period $2n$ can be obtained by the existence of $2n$-periodic solutions of a coupled ordinary differential system.

In 1978, Nussbaum [**28**] obtained the existence of $2n$-periodic solutions for a class of delay differential equations which include (1.1)–(1.3) as special cases. He formulated a more general delay equation whose solutions correspond to fixed points of a differentiable, asymptotically linear map in Banach space. Under appropriate conditions, made to accommodate the more general equation, he was able to apply a fixed point theorem of Krasnosel'skii to prove the existence of periodic solutions with period $2n$. Thus, the Kaplan-Yorke conjecture was confirmed. But the method used in [**28**] is quite different from that in [**20**].

Afterwards, by developing the technique of Kaplan and Yorke, numerous results have been established on the existence of periodic solutions for some delay differential equations; see, for example, [**1, 2, 5, 6, 8, 7, 10, 13, 15, 16, 18, 26, 32, 33**]. In particular, in 1998, Li and He [**21**] firstly studied the periodic solutions of (1.3) by using critical point theory. They reduced the problem of finding periodic solutions of (1.3) to that of finding periodic solutions of an associated Hamiltonian system. Then, by applying the Morse-Ekeland index theory of critical point theory to the corresponding functional, they gave some interesting existence results. For related results, see also [**22, 23**].

In 2006, by using S^1-pseudo index theory, Fei [**11, 12**] obtained some important results on the existence and multiplicity of periodic solutions of (1.3). Due to great efforts by Nussbaum, Li, He, Fei and many other scholars, the correctness of Kaplan-Yorke's conjecture has been completely proved.

It is natural for us to ask whether Kaplan-Yorke's conjecture is true or not in the case of higher-dimensional systems? What about the multiplicity results on periodic solutions of (1.3) when $x \in \mathbf{R}^N$ with $N > 1$? Even in the case of a single delay ($n = 1$), one cannot study the orbits of the equation

$$z'(t) = -f(z(t-1)), \quad z \in \mathbf{R}^N \tag{1.4}$$

in the phase plane when $N > 1$. Thus, the technique used by Kaplan and Yorke is no longer available when $N > 1$. On the other hand, although the methods of Nussbaum can be extended in a straightforward way to the vector-valued case, it is difficult to obtain the multiplicity results by using fixed point theorems.

Very recently, the second author and Yu [**17**] studied the existence and multiplicity of periodic solutions of the following delay differential equations with $2n-1$ delays

$$\dot{z}(t) = -f(z(t-1)) - f(z(t-2)) - \cdots - f(z(t-2n+1), \quad z \in \mathbf{R}^N, \tag{1.5}$$

where $f \in C(\mathbf{R}^N, \mathbf{R}^N)$ and $N > 1$ is an integer. By using the S^1 pseudo geometrical index theory in critical point theory, the authors proved that Kaplan and Yorke's conjecture is true in the case of higher-dimensional systems. More precisely, if the $4n$-periodic solution $y(t) = (y_1(t), y_2(t), \ldots, y_{2n}(t))^T$ of the following system:

$$\dot{y}(t) = A_{2nN} \nabla W(y(t)), \quad \text{where} \tag{1.6}$$

$$A_{2nN} = \begin{pmatrix} 0 & -I & -I & \cdots & -I \\ I & 0 & -I & \cdots & -I \\ I & I & 0 & \cdots & -I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & I & I & \cdots & 0 \end{pmatrix}_{2nN \times 2nN}$$

satisfies

$$y(t-1) = Ty(t) \quad \text{with} \quad T = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ -I & 0 & 0 & \cdots & 0 \end{pmatrix}_{2nN \times 2nN}, \tag{1.7}$$

then $z(t) = y_1(t)$ is a $4n$-periodic solution of (1.7) with $z(t-2n) = -z(t)$. Here $\nabla W(y) = (f(y_1), f(y_2), \ldots, f(y_{2n}))^T$ and I denotes the $N \times N$ identity matrix.

The method used in [**17**] is variational. They established a new variational framework for (1.7), in which a delay variable is involved in the corresponding functional J over a Hilbert space H. For (1.7), the function J is invariant and J' is equivariant about a compact group action related to (1.7). This allows the authors to find critical points of J on a subspace of H which is invariant under the previously mentioned group action. Then, by using an existence theorem due to Degiovanni and Fannio **[9]**, some sufficient conditions for the existence and multiplicity results of periodic solutions of (1.7) were obtained.

This paper is devoted to the existence and multiplicity of periodic solutions for delay differential systems with $2n$ delays, namely, we shall study equations of the form:

$$\dot{z}(t) = -f(z(t-1)) - f(z(t-2)) - \cdots - f(z(t-2n)), \quad z \in \mathbf{R}^N, \tag{1.8}$$

where $f \in C(\mathbf{R}^N, \mathbf{R}^N)$. We are interested in finding periodic solutions of (1.8) with period $2(2n+1)$ and satisfying $z(t-(2n+1)) = -z(t)$ for all $t \in \mathbf{R}$.

For (1.8), the function J is still invariant about a similar compact group action. But J' is not equivariant about this compact group action anymore. Therefore, we cannot directly apply the same idea as in **[17]**. To overcome this difficulty, we combine the Galerkin approximation method with S^1-index theory **[27]** to obtain critical points of J in a subspace of H.

As in **[17]**, we make the following assumptions on f.

(f_1) $f \in C(\mathbf{R}^N, \mathbf{R}^N)$ and, for any $z \in \mathbf{R}^N$, $f(-z) = -f(z)$.

(f_2) There exists a continuously differentiable function F satisfying $F(0) = 0$ and $\nabla F(z) = f(z)$ for any $z \in \mathbf{R}^N$.

(f_3)

$$f(z) = A_\infty z + o(|z|) \quad \text{as} \quad |z| \to +\infty, \tag{1.9}$$

and

$$f(z) = A_0 z + o(|z|) \quad \text{as} \quad |z| \to 0, \tag{1.10}$$

where A_∞ and A_0 are symmetric $N \times N$ matrices.

Some notations are needed now in order to introduce our main results.

For an $N \times N$ real symmetric matrix A, we set

$$M(A) = \{\text{numbers of negative eigenvalues of } A\},$$
$$\overline{M}(A) = \{\text{numbers of non-positive eigenvalues of } A\}.$$

Let

$$\Gamma = \{(2n+1)k + n + 1 : k = 0, 1, 2, \ldots\}, \qquad r = \frac{2n+1}{\pi}.$$

For two given real $N \times N$ symmetric matrices A, B and any $j \in \mathbf{N} \setminus \Gamma$, define

$$\begin{aligned}
\psi_j(A, B) = M\Big(&(2j-1)\tan\frac{2j-1}{4n+2}\pi - rA\Big) \\
&- \overline{M}\Big((2j-1)\tan\frac{2j-1}{4n+2}\pi - rB\Big), \\
\psi_j^{(0)}(A) = \overline{M}\Big(&(2j-1)\tan\frac{2j-1}{4n+2}\pi - rA\Big) \\
&- M\Big((2j-1)\tan\frac{2j-1}{4n+2}\pi - rA\Big),
\end{aligned}$$

and

$$\psi(A, B) = \sum_{j=1, j \notin \Gamma}^{+\infty} \psi_j(A, B),$$

$$\psi^{(0)}(A) = \sum_{j=1, j \notin \Gamma}^{+\infty} \psi_j^{(0)}(A).$$

It is easy to see that $\psi(A, B)$ and $\psi^{(0)}(A)$ are well defined due to the fact that $\psi_j(A, B) = \psi_j^{(0)}(A) = 0$ for j large enough.

Now, define two $2nN \times 2nN$ matrices M and Λ_∞ as

$$M = (t_{ij} I) \quad \text{with } t_{ij} = \begin{cases} (-1)^{i+j}, & i \neq j, \\ 2, & i = j, \end{cases} \tag{1.11}$$

$$\Lambda_\infty = \mathrm{diag}\underbrace{\{A_\infty, A_\infty, \ldots, A_\infty\}}_{2n}. \tag{1.12}$$

For $j \geq 1$, denote

$$D_j(A_\infty) = \begin{pmatrix} -\frac{1}{j}\Lambda_\infty M & \frac{\pi}{2n+1} A_{2nN}^{-1} \\ -\frac{\pi}{2n+1} A_{2nN}^{-1} & -\frac{1}{j}\Lambda_\infty M \end{pmatrix}_{4nN \times 4nN}$$

with A_{2nN} in (1.6). Now

$$n_\infty = \sum_{j=1}^{+\infty} \dim \ker D_j(A_\infty)$$

is well defined since $D_j(A_\infty)$ is invertible for sufficiently large j.

Note that, if $z(t)$ is a $2(2n+1)$-periodic solution of (1.8), so is $z(t+\theta)$ for each $\theta \in S^1 = \mathbf{R}/2(2n+1)\mathbf{Z}$. We say that two $2(2n+1)$-periodic solutions $z_1(t)$ and $z_2(t)$ are geometrically different, if there is no $\theta \in S^1$ such that $z_1(t+\theta) = z_2(t)$ for all $t \in \mathbf{R}$.

Our main results are the following theorems.

Theorem 1.1. *Suppose that f satisfies (f_1)–(f_3) and*
(f_4) $n_\infty = 0$.
Then (1.8) *possesses at least* $\psi(A_0, A_\infty)$ *geometrically different nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$ *whenever* $\psi(A_0, A_\infty) > 0$.

Remark 1. By the definitions of ψ and $\psi^{(0)}$, it is easy to get

$$\psi(A_0, A_\infty) + \psi(A_\infty, A_0) + \psi^{(0)}(A_0) + \psi^{(0)}(A_\infty) = 0.$$

Hence, if $\psi(A_0, A_\infty) \geq 0$, then $\psi(A_\infty, A_0) \leq 0$. But the inverse may not be true. However, in the case of $\psi(A_0, A_\infty) < 0$, one can expect that (1.8) possesses at least $\psi(A_\infty, A_0) > 0$ geometrically different nonconstant $2(2n+1)$-periodic solutions with $z(t-(2n+1)) = -z(t)$, provided $\psi^{(0)}(A_0) = \psi^{(0)}(A_\infty) = 0$. Generally, we have:

Corollary 1.2. *Suppose that f satisfies (f_1)–(f_4). Then* (1.8) *possesses at least* $\psi(A_\infty, A_0)$ *geometrically different nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$ *whenever* $\psi(A_\infty, A_0) > 0$.

Remark 2. As we will see in Section 3, $n_\infty = 0$ implies that $\psi^{(0)}(A_\infty) = 0$. Therefore, the nonlinearity is nonresonant at infinity. In this case, for the corresponding functional, the compactness condition in critical point theory is easily satisfied. If (f_4) does not hold, the nonlinearity of (1.8) is so-called resonant at infinity. In order to overcome the difficulty in verifying the compactness condition, some growth conditions have to be imposed upon. The following theorem and corollary are along this line.

Theorem 1.3. *Suppose that* f *satisfies* $(f_1)-(f_3)$ *and* $n_\infty \neq 0$. *Assume that:*
(f_5) $|f(z)-A_\infty z| \leq M_0$ *for any* $z \in \mathbf{R}^N$, *where* M_0 *is a positive constant,*
(f_6) $F(z)-(A_\infty z, z)/2 \to -\infty$ *as* $|z| \to +\infty$.
Then (1.8) *possesses at least* $\psi(A_0, A_\infty)+\psi^{(0)}(A_\infty)$ *geometrically different nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$ *whenever* $\psi(A_0, A_\infty)+\psi^{(0)}(A_\infty) > 0$.

Corollary 1.4. *Under the conditions of Theorem* 1.3, (1.8) *possesses at least* $\psi(A_\infty, A_0)+\psi^{(0)}(A_\infty)$ *geometrically different nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$ *whenever* $\psi(A_\infty, A_0)+\psi^{(0)}(A_\infty) > 0$.

Remark 3. There are still many other types of assumptions imposed on the nonlinearity so as to overcome the lack of compactness of the associated functional. These assumptions are often called resonant conditions. Once the original system is resonant at infinity, the associated functional still satisfies the compactness condition provided some kind of resonant conditions are imposed. For example, (f_5) and (f_6) are suitable conditions to ensure that the corresponding functional satisfies the compactness condition. One can find more such conditions and related results in **[3, 11, 12, 31]**, etc.

Remark 4. Our results extend the related results in **[12]** to the higher-dimensional case. In fact, in the case of $N=1$, (f_3) reads as
(f_3') $\lim_{z\to 0} f(z)/z = \alpha_0, \quad \lim_{z\to\infty} f(z)/z = \alpha_\infty;$
where α_0 and α_∞ are real numbers. And

$$D_j(\alpha_\infty) = \begin{pmatrix} -\frac{1}{j}\alpha_\infty M & \frac{\pi}{2n+1}A_{2n}^{-1} \\ -\frac{\pi}{2n+1}A_{2n}^{-1} & -\frac{1}{j}\alpha_\infty M \end{pmatrix}_{4n\times 4n.}$$

By the definitions in **[12]**, for any two real numbers α, β, it is easy to get

$$\psi(\alpha,\beta) = i(\alpha) - i(\beta) - \nu(\beta) \quad \text{if } \alpha > \beta;$$
$$\psi(\beta,\alpha) = i(\beta) - i(\alpha) - \nu(\alpha) \quad \text{if } \alpha < \beta;$$
$$\psi(\alpha,\alpha) = -\nu(\alpha); \quad \psi^{(0)}(\alpha) = \nu(\alpha).$$

So, by Theorem 1.1, Corollary 1.2 and Remark 2, we have:

Corollary 1.5. *In the case of* $N = 1$, *suppose that* f *satisfies* (f_1), (f_2), (f_3') *and*
(f_4') *for any* $j \in \mathbf{N}$, $D_j(\alpha_\infty)$ *is invertible.*
If $\alpha_0 > \alpha_\infty$, *then* (1.8) *possesses at least* $\psi(\alpha_0, \alpha_\infty) = i_0 - i_\infty$ *nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$.
If $\alpha_0 < \alpha_\infty$, *then* (1.8) *possesses at least* $\psi(\alpha_\infty, \alpha_0) = i_\infty - i_0 - \nu_0$ *geometrically different nonconstant* $2(2n+1)$*-periodic solutions with* $z(t-(2n+1)) = -z(t)$.

Hence, the conclusion of Corollary 1.5 is just the same as that of Theorem 1.2 (i) with $n_\infty = 0$ in **[12]**. The other cases can be discussed similarly, and we do not repeat them here. One can also compare our results with the main results in **[19]**, where by using a completely different approach, they obtained a better result than Corollary 1.5 under more restrictive assumptions on f, that is, f is an orientation preserving homeomorphism from $\mathbf{R}$ to $\mathbf{R}$.

2. Reduction of the existence problem. In this section, we will reduce the existence of periodic solutions of (1.8) with period $2(2n+1)$ satisfying $z(t-(2n+1)) = -z(t)$ to the existence of critical points of an associated functional defined on a Hilbert space.

Suppose that $z(t)$ is a $2(2n+1)$-periodic solution of (1.8) with $z(t-(2n+1)) = -z(t)$. Let $y_1(t) = z(t)$, $y_2(t) = z(t-1), \dots, y_{2n}(t) = z(t-2n+1)$, $y_{2n+1}(t) = z(t-2n)$. Clearly,

$$y_1(t) = -y_{2n+1}(t-1), y_2(t) = y_1(t-1), y_3(t) = y_2(t-1), \dots, \quad y_{2n+1}(t) = y_{2n}(t-1). \tag{2.1}$$

Then (1.8) is reduced to the following generalized Hamiltonian systems

$$\dot{y}^*(t) = A^* \nabla W^*(y^*(t)), \tag{2.2}$$

where $y^* = (y_1, y_2, \ldots, y_{2n+1})^T$, $W^*(y_1, y_2, \ldots, y_{2n+1}) = F(y_1) + F(y_2) + \cdots + F(y_{2n+1})$,

$$A^* = \begin{pmatrix} 0 & -I & -I & \cdots & -I & -I \\ I & 0 & -I & \cdots & -I & -I \\ I & I & 0 & \cdots & -I & -I \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I & I & I & \cdots & 0 & -I \\ I & I & I & \cdots & I & 0 \end{pmatrix}_{(2n+1)N\times(2n+1)N},$$

where $\nabla F = f$. It follows that y^* is a $2(2n+1)$-periodic solution of (2.2) satisfying (2.1).

Conversely, assume that $y^* = (y_1, y_2, \ldots, y_{2n+1})^T$ is a $2(2n+1)$-periodic solution of (2.2) satisfying (2.1). Note that (2.1) implies that

$$y_i(t) = y_1(t - i + 1), \quad i = 2, 3, \ldots, 2n+1. \tag{2.3}$$

Hence, by the first equation of (2.2), we have

$$\dot{y}_1(t) = -f(y_1(t-1)) - f(y_1(t-2)) - \cdots - f(y_1(t-2n)),$$

and by (2.3) and (2.1),

$$\begin{aligned} y_1(t-(2n+1)) = y_1(t-2n-1) &= y_1((t-1)-(2n+1)+1) \\ &= y_{2n+1}(t-1) = -y_1(t). \end{aligned}$$

It follows that $z(t) = y_1(t)$ is a $2(2n+1)$-periodic solution of (1.8) with $z(t-(2n+1)) = -z(t)$.

Thus, we have

Lemma 2.1. *There is a one-to-one correspondence between* $2(2n+1)$*-periodic solutions of* (2.2) *satisfying* (2.1) *and* $2(2n+1)$*-periodic solutions of* (1.8) *with* $z(t-(2n+1)) = -z(t)$.

For the generalized Hamiltonian systems (2.2), there is an invariant function

$$C(y^*) \equiv C(y_1, y_2, \ldots, y_{2n+1}) = y_1 + \sum_{i=1}^{n} (y_{2i+1} - y_{2i}).$$

$C(y)$ is called a Casimir function **[25, 29]** of (2.2). The level set defined by $C(y) = c$ is called a symplectic leaf. According to the theory of generalized Hamiltonian systems, (2.2) can be reduced to a $2n$-dimensional Hamiltonian system on the symplectic leaf $y_1 + \sum_{i=1}^n (y_{2i+1} - y_{2i}) = c$. In fact, by choosing $c = 0$, the symplectic leaf is given by

$$y_{2n+1} = \sum_{i=1}^{n} (y_{2i} - y_{2i-1}),$$

and on this leaf we have

$$\begin{aligned} W(y) &\equiv W(y_1, y_2, \ldots, y_{2n}) \\ &= F(y_1) + F(y_2) + \cdots + F(y_{2n}) + F\left(\sum_{i=1}^{n} (y_{2i} - y_{2i-1}) \right), \end{aligned} \tag{2.4}$$

and

$$\dot{y}(t) = A \nabla W(y), \tag{2.5}$$

where

$$A = \begin{pmatrix} 0 & -I & -I & \cdots & -I & -I \\ I & 0 & -I & \cdots & -I & -I \\ I & I & 0 & \cdots & -I & -I \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I & I & I & \cdots & 0 & -I \\ I & I & I & \cdots & I & 0 \end{pmatrix}_{2nN \times 2nN.}$$

Now, let

$$T = \begin{pmatrix} I & -I & I & \cdots & I & -I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix}_{2nN \times 2nN.}$$

Then we claim that, if $y = (y_1, y_2, \ldots, y_{2n})^T$ is a solution of (2.5) satisfying $y(t) = Ty(t-1)$, then $(y_1, y_2, \ldots, y_{2n+1})^T$ with $y_{2n+1} = \sum_{i=1}^n (y_{2i} - y_{2i-1})$ is a solution of (2.5) satisfying (2.1). In fact, it is easy to check that any solution of (2.5) must satisfy (2.5) with $y_{2n+1} = \sum_{i=1}^n (y_{2i} - y_{2i-1})$. And $y(t) = Ty(t-1)$ implies (2.1) obviously hold except for the final equality. However, if $y(t) = Ty(t-1)$ holds, then we have

$$\begin{aligned}
y_{2n+1}(t) &= \sum_{i=1}^{n}\Big(y_{2i}(t)-y_{2i-1}(t)\Big)\\
&= y_2(t)-y_1(t)+\sum_{i=2}^{n}\Big(y_{2i}(t)-y_{2i-1}(t)\Big)\\
&= y_1(t-1)-y_1(t)+\sum_{i=2}^{n}\Big(y_{2i}(t)-y_{2i-1}(t)\Big)\\
&= \sum_{i=1}^{n-1}\Big(y_{2i}(t-1)-y_{2i+1}(t-1)\Big)\\
&\quad +y_{2n}(t-1)+\sum_{i=2}^{n}\Big(y_{2i}(t)-y_{2i-1}(t)\Big)\\
&= \sum_{i=1}^{n-1}\Big(y_{2i+1}(t)-y_{2i+2}(t)\Big)\\
&\quad +y_{2n}(t-1)+\sum_{i=2}^{n}\Big(y_{2i}(t)-y_{2i-1}(t)\Big)\\
&= y_{2n}(t-1).
\end{aligned}$$

Hence, the claim holds. Thus, so far, we only need to focus on the existence and multiplicity of periodic solutions of (2.5) satisfying $y(t)=Ty(t-1)$.

Let

$$r=\frac{2n+1}{\pi},\ s=\frac{\pi}{2n+1}t.$$

Then (2.5) transforms to

$$\dot{x}(t)=rA\nabla W(x(t)). \tag{2.6}$$

We will seek 2π periodic solutions of (2.6) with $x(t)=Tx(t-\frac{\pi}{2n+1})$, which of course correspond to $2(2n+1)$ periodic solutions of (2.6) with $y(t)=Ty(t-1)$.

The Hilbert space $H=H^{1/2}(S^1,\mathbf{R}^{2nN})$ for (2.6) is the space of 2π-periodic vector-valued functions with dimensions $2nN$, which possesses square integrable derivative of order $1/2$.

Similar to the treatment in [**3, 4**], we can introduce this space as follows. Let $C^\infty(S^1, \mathbf{R}^{2nN})$ be the space of 2π-periodic C^∞ vector-valued functions with dimension $2nN$. For any $x \in C^\infty(S^1, \mathbf{R}^{2nN})$, it has the following Fourier expansion in the sense that it is convergent in the space $L^2(S^1, \mathbf{R}^{2nN})$,

$$x(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ijt}, \tag{2.7}$$

where $a_j \in \mathbf{C}^{2nN}$, $j \in \mathbf{Z}$ and $a_{-j} = \overline{a_j}$.

$H^{1/2}(S^1, \mathbf{R}^{2nN})$ is the closure of $C^\infty(S^1, \mathbf{R}^{2nN})$ with respect to the Hilbert norm

$$\|x\| = \left[2\pi|a_0|^2 + 4\pi \sum_{j=1}^{+\infty} j|a_j|^2\right]^{1/2},$$

where $|\cdot|$ denotes the classical norm in $\mathbf{C}^{2nN}$.

$H^{1/2}(S^1, \mathbf{R}^{2nN})$ can also be obtained by interpolation from the Sobolev space $H^1(S^1, \mathbf{R}^{2nN})$ and $L^2(S^1, \mathbf{R}^{2nN})$. More specifically, for any $x \in L^2(S^1, \mathbf{R}^{2nN})$, if x has a Fourier expansion with the convergence in the space $L^2(S^1, \mathbf{R}^{2nN})$, then x has a representation as in (2.7). Thus, $x \in H^{1/2}(S^1, \mathbf{R}^{2nN})$ if and only if $x \in L^2(S^1, \mathbf{R}^{2nN})$, and

$$2\pi|a_0|^2 + 4\pi \sum_{j=1}^{+\infty} j|a_j|^2 < +\infty.$$

For any $x_1, x_2 \in H^{1/2}(S^1, \mathbf{R}^{2nN})$, they have Fourier expansions as follows:

$$x_1(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ijt}, \qquad x_2(t) = \sum_{j=-\infty}^{+\infty} b_j e^{ijt},$$

where $a_j, b_j \in \mathbf{C}^{2nN}$, $a_{-j} = \overline{a_j}$ and $b_{-j} = \overline{b_j}$ for any $j \in \mathbf{Z}$. Then $\|x_1\|$

and $\langle x_1, x_2\rangle$ can be explicitly expressed by

$$\|x_1\| = \left[2\pi|a_0|^2 + 4\pi\sum_{j=1}^{+\infty} j|a_j|^2\right]^{1/2},$$

$$\langle x_1, x_2\rangle = 2\pi(a_0, b_0) + 2\pi\sum_{j=1}^{+\infty} j\Big[(a_j, b_j) + (a_{-j}, b_{-j})\Big],$$

where $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ denote the inner product in Hilbert space $H^{1/2}(S^1, \mathbf{R}^{2nN})$ and $\mathbf{C}^{2nN}$, respectively.

For our convenience, in the sequel, we denote by H the space $H^{1/2}(S^1, \mathbf{R}^{2nN})$.

Let $x, y \in L^2(S^1, \mathbf{R}^{2nN})$. If

$$\int_0^{2\pi} (x(t), z'(t))\,dt = -\int_0^{2\pi} (y(t), z(t))\,dt$$

holds for every $z \in C^\infty(S^1, \mathbf{R}^{2nN})$, then y is called a weak derivative of x.

Now consider a functional J defined on H

$$J(x) = \int_0^{2\pi}\left[\frac{1}{2}(A^{-1}\dot{x}(t), x(t)) - rW(x(t))\right]dt, \quad \forall x \in H, \tag{2.8}$$

where $\dot{x}$ denotes the weak derivative of x and the definition of a nonsingular matrix A can be found in (2.6). It is easy to check that J is well defined provided that f satisfies (f_1)-(f_3).

We define an operator $L_0 : H \to H^*$ as follows. For any $x \in H$, L_0x is defined by

$$L_0x(h) = \int_0^{2\pi} (A^{-1}\dot{x}(t), h(t))\,dt, \quad \forall h \in H,$$

where H^* denotes the dual space of H. Since $|L_0x(h)| \le C\|x\|\|h\|$ for some positive constant C, by the Riesz representation theorem, we can identify H with H^*. Then L_0x can also be viewed as a function belonging to H such that $\langle L_0x, h\rangle = L_0x(h)$ for any $x, h \in H$ and L_0 is a bounded, self-adjoint operator on H.

Define

$$\Phi_0(x) = r\int_0^{2\pi} W(x(t))dt, \quad \forall x \in H.$$

Then J can be rewritten as

$$J(x) = \frac{1}{2}\langle L_0 x, h\rangle - \Phi_0(x), \quad \forall x \in H.$$

By a standard argument as in **[3, 4, 30]**, we have

Lemma 2.2. *Let $x = (x_1, x_2, \ldots, x_{2n})^T$ with $x_i \in \mathbf{R}^N$, $i = 1, 2, \ldots, 2n$. Assume that $W \in C^1(\mathbf{R}^{2nN}, \mathbf{R})$, $\nabla W(x) = (f(x_1), f(x_2), \ldots, f(x_{2n}))^T$ satisfies (f_3). Then, the functional J is continuously differentiable on H and $J'(x)$ is defined by*

(2.9)
$$\langle J'(x), h\rangle = \int_0^{2\pi} \Big[(A^{-1}\dot{x}(t), h(t)) - r(\nabla W(x(t)), h(t))\Big]\, dt, \quad \forall h \in H.$$

And critical points of J in H are classic solutions of (2.6). *Moreover, $\Phi_0' : H \to H^*$ is a compact mapping defined as follows*:

$$\langle \Phi_0'(x), h\rangle = r\int_0^{2\pi} (\nabla W(x(t)), h(t))\, dt, \quad \forall h \in H.$$

Here we still view $\Phi_0'(x)$ as an element of H for any $x \in H$. As usual, we shall identify the equivalence class $x \in H$ and its continuous representative (see **[27]**).

Next, we set

$$X = \left\{x \in H | x(t) = Tx\left(t - \frac{\pi}{2n+1}\right)\right\}.$$

Then X is a closed subspace of H.

For any $x \in X$, let

$$x(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ijt},$$

where $a_j = (a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(2n)})^T$, $a_j^{(l)} \in \mathbf{C}^N$, $l = 1, 2, \ldots, 2n$. By $x(t) = Tx(t - \frac{\pi}{2n+1})$, we have

$$\sum_{j=-\infty}^{+\infty} a_j e^{ijt} = \sum_{j=-\infty}^{+\infty} T a_j e^{ijt} e^{-\pi j/(2n+1)i}.$$

Then

$$Ta_j = e^{\pi j/(2n+1)i} a_j, \quad \forall\, j \in \mathbf{Z}. \tag{2.10}$$

It is clear that $e^{\pi j/(2n+1)i}$ must be an eigenvalue of T and a_j is a corresponding eigenvector with respect to $e^{\pi j/(2n+1)i}$.

Note that T is similar to

$$\operatorname{diag} \underbrace{\{T', T', \ldots, T'\}}_{N},$$

where T' is a $2n \times 2n$ matrix given by

$$\begin{pmatrix} 1 & -1 & \cdots & 1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

And, by **[12]**,

$$\det(T' - \lambda I_{2n}) = \lambda^{2n} - \lambda^{2n-1} + \cdots - \lambda + 1,$$

Hence,

$$\det(T - \lambda I_{2nN}) = (\lambda^{2n} - \lambda^{2n-1} + \cdots - \lambda + 1)^N.$$

So, the eigenvalues of T are

$$\lambda = e^{\pm(2j-1)\pi/(2n+1)i}, \quad j = 1, 2, \ldots, n,$$

and each eigenvalue has multiplicity N.

Note that the eigenvalues of T can also be given by

$$\lambda = e^{(2j-1)\pi/(2n+1)i}, \quad j = 1, 2, \ldots, 2n+1,\ j \neq n+1.$$

So, in (2.10), j must be an odd number. Let $\lambda_j = e^{(2j-1)\pi/(2n+1)i}$. A direct computation shows that $x \in X$ if and only if x has a Fourier expansion

$$x(t) = \sum_{j=-\infty, j\notin\Gamma}^{+\infty} a_j e^{i(2j-1)t},$$

where $\Gamma = \{(2n+1)k + n + 1 : k \in \mathbf{Z}\}$, $a_j = (a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(2n)})^T$,

$$a_j^{(1)} = \lambda_j^{2n-1} a_j^{(2n)},\ a_j^{(2)} = \lambda_j^{2n-2} a_j^{(3)}, \ldots, a_j^{(2n-1)} = \lambda_j a_j^{(2n)},$$

where $a_j^{(2n)}$ is an arbitrary vector in $\mathbf{C}^N$, and, for any $j \geq 0$, $a_{-j} = \overline{a_{j+1}}$. Here we use the notation $\lambda_j = \lambda_{j \mod (2n+1)}$. So, it is easy to see that $a_j \in \operatorname{span}\{u_1, u_2, \ldots, u_N\}$ with $u_l = \{\lambda_j^{2n-1} c_l, \lambda_j^{2n-2} c_l, \ldots, \lambda_j c_l, c_l\}$, $l = 1, 2, \ldots, N$ where $\{c_1, c_2, \ldots, c_N\}$ forms a canonical basis of $\mathbf{C}^N$.

Remark 5. For $x \in H$, define

$$\delta x(t) = Tx\left(t - \frac{\pi}{2n+1}\right).$$

Then we have $\delta^{2n+1} x(t) = -x(t - (2n+1))$, $\delta^{2(2n+1)} x(t) = x(t)$, and $G = \{\delta, \delta^2, \ldots, \delta^{2(2n+1)}\}$ is a compact group action over H. Let J be given in (2.8) and $J|_X$ be the restriction of J on X. By (2.4) and direct computation, we have $J(\delta x) = J(x)$, i.e., J is G-invariant. However,

$$\nabla W(Tx) \neq T\nabla W(x), \quad TA^{-1} \neq A^{-1}T$$

which means that J' is not G-equivariant, and critical points of $J|_X$ over X may not be critical points of J over H. Hence, we cannot apply the same idea as in the case of odd number delays to get the critical points of J on X in **[17]**. However, critical points of J in X are indeed nonconstant classic $2(2n+1)$-periodic solutions of (2.6) satisfying (2.1), and hence they give solutions of (1.8) with the property $z(t-(2n+1)) = -z(t)$. Therefore, we will seek critical points of J in X in the following section.

3. Proofs of main results.

In this section, first we discuss the direct decomposition of the subspace X and study the behavior of J over this subspace. Then, we use Galerkin approximation method and the S^1 index theory to obtain critical points of J in X.

By (f_3) and (2.4), one can verify that

$$\nabla W(x) = \Lambda_\infty Mx + o(|x|) \quad \text{as } |x| \to \infty, \tag{3.1}$$

$$\nabla W(x) = \Lambda_0 Mx + o(|x|) \quad \text{as } |x| \to 0, \tag{3.2}$$

where Λ_0 is a $2nN \times 2nN$ symmetric matrix defined by

$$\Lambda_0 = \mathrm{diag}\,\underbrace{\{A_0, A_0, \cdots, A_0\}}_{2n},$$

and the definitions of M and Λ_∞ can be found in (1.11) and (1.12).

Denote

$$G_{A_\infty}(x) = W(x) - \frac{1}{2}(\Lambda_\infty Mx, x)$$

and

$$\Phi_{A_\infty}(x) = r\int_0^{2\pi} G_{A_\infty}(x(t))\,dt,$$

$$G_{A_0}(x) = W(x) - \frac{1}{2}(\Lambda_0 Mx, x)$$

and

$$\Phi_{A_0}(x) = r\int_0^{2\pi} G_{A_0}(x(t))\,dt.$$

Let L_{A_∞} and L_{A_0} be two bounded linear operators defined by the following forms:

$$\langle L_{A_\infty}x, h\rangle = \int_0^{2\pi} \Big(A^{-1}\dot{x}(t) - r\Lambda_\infty Mx(t), h(t)\Big)\,dt, \quad \text{for all } x, h \in X,$$

$$\langle L_{A_0}x, h\rangle = \int_0^{2\pi} \Big(A^{-1}\dot{x}(t) - r\Lambda_0 Mx(t), h(t)\Big)\,dt, \quad \text{for all } x, h \in X.$$

Then J can be reformulated by

$$J(x) = \frac{1}{2}\langle L_{A_\infty}x, x\rangle - \Phi_{A_\infty}(x)$$

or

$$J(x) = \frac{1}{2}\langle L_{A_0}x, x\rangle - \Phi_{A_0}(x).$$

Set

$$L^2_X(S^1, \mathbf{R}^{2nN}) = \left\{ x \in L^2(S^1, \mathbf{R}^{2nN}) \middle| x(t) = Tx\left(t - \frac{\pi}{2n+1}\right) \text{ a.e. for } t \in [0, 2\pi] \right\}$$

and

$$H^1_X(S^1, \mathbf{R}^{2nN}) = \left\{ x \in L^2_X(S^1, \mathbf{R}^{2nN}) \middle| \int_0^{2\pi} [x^2(t) + \dot{x}^2(t)]\, dt < \infty \right\}.$$

Note that $H^1_X(S^1, \mathbf{R}^{2nN})$ can be viewed as a subspace of $L^2_X(S^1, \mathbf{R}^{2nN})$ and

$$H^1_X(S^1, \mathbf{R}^{2nN}) \subset X \subset L^2_X(S^1, \mathbf{R}^{2nN}).$$

Now, we consider two differential operators

$$\mathfrak{L}_{A_\infty}, \mathfrak{L}_{A_0} : H^1_X(S^1, \mathbf{R}^{2nN}) \subset L^2_X(S^1, \mathbf{R}^{2nN}) \to L^2_X(S^1, \mathbf{R}^{2nN})$$

which are defined by

$$(\mathfrak{L}_{A_\infty}x)(t) = A^{-1}\dot{x}(t) - r\Lambda_\infty Mx(t), \quad \forall\, x \in H^1_X(S^1, \mathbf{R}^{2nN})$$

and

$$(\mathfrak{L}_{A_0}x)(t) = A^{-1}\dot{x}(t) - r\Lambda_0 Mx(t), \quad \forall\, x \in H^1_X(S^1, \mathbf{R}^{2nN}),$$

where $\dot{x}$ denotes the weak derivative of x.

For each $x \in H^1_X(S^1, \mathbf{R}^{2nN})$, it has a Fourier expansion

$$x(t) = \sum_{j=-\infty, j\notin\Gamma}^{+\infty} a_j e^{i(2j-1)t}.$$

And, hence,

$$
\begin{aligned}
\mathfrak{L}_{A_\infty} x(t) &= \sum_{j=-\infty, j\notin\Gamma}^{+\infty} \Big[i(2j-1)A^{-1}a_j - r\Lambda_\infty M a_j \Big] e^{i(2j-1)t} \\
&= \sum_{j=1, j\notin\Gamma}^{+\infty} \Big[i(2j-1)A^{-1}a_j - r\Lambda_\infty M a_j \Big] e^{i(2j-1)t} \\
&\quad + \Big[-i(2j-1)A^{-1}a_{-j+1} - r\Lambda_\infty M a_{-j+1} \Big] e^{-i(2j-1)t}.
\end{aligned}
\tag{3.3}
$$

Let

$$\mathfrak{L}_{A_\infty} x(t) = \nu x(t),$$

where ν is a constant. Then, for any $j \in \mathbf{Z} \setminus \Gamma$, we have

$$i(2j-1)A^{-1}a_j - r\Lambda_\infty M a_j = \nu a_j.$$

Since $T^{2n+1} = -E$ (here and below E denotes the $2nN \times 2nN$ identity matrix) and

$$
\begin{aligned}
AM &= T + T^2 + \cdots + T^{2n} \\
&= (E + T + T^2 + \cdots + T^{2n-1})T \\
&= (E-T)^{-1}(E - T^{2n})T = (E-T)^{-1}(E+T),
\end{aligned}
$$

the eigenvalue μ of AM and the eigenvalue λ of T satisfy the relation

$$\mu = (1-\lambda)^{-1}(1+\lambda).$$

Thus, letting $\beta_j = (2j-1)\pi/(2n+1)$, we have

$$\mu_j = i \cot \frac{\beta_j}{2}.$$

So all of the eigenvalues of AM can be enumerated by

$$\mu_j = i \cot \frac{\beta_j}{2}, \quad j = 1, 2, \ldots, 2n+1,\ j \neq n+1$$

each with multiplicity N.

Since $Ta_j = \lambda_j a_j$, $AM = (E-T)^{-1}(E+T)$ and $M^{-1}A^{-1} = (E+T)^{-1}(E-T)$, we have $M^{-1}A^{-1}a_j = \mu_j^{-1}a_j$, where

$$u_j^{-1} = \frac{1-\lambda_j}{1+\lambda_j} = -i \tan \frac{\beta_j}{2}, \quad j = 1, 2, \ldots, 2n+1, j \neq n+1.$$

Thus,

$$M^{-1}A^{-1}a_j = -i\tan\frac{\beta_j}{2}a_j,$$

and

$$i(2j-1)A^{-1}a_j - r\Lambda_\infty Ma_j = \Big((2j-1)\tan\frac{\beta_j}{2} - r\Lambda_\infty\Big)Ma_j, \tag{3.4}$$

$$\begin{aligned} &-i(2j-1)A^{-1}a_{-j+1} - r\Lambda_\infty Ma_{-j+1} \\ &\qquad = \Big((2j-1)\tan\frac{\beta_j}{2} - r\Lambda_\infty\Big)Ma_{-j+1}. \end{aligned} \tag{3.5}$$

Hence,

$$\Big[(2j-1)\tan\frac{\beta_j}{2} - r\Lambda_\infty\Big]Ma_j = \nu a_j, \quad \forall j \in \mathbf{Z}\setminus\Gamma,$$

which implies that ν is an eigenvalue of operator $\mathfrak{L}_{A_\infty}$ if and only if ν is an eigenvalue of $((2j-1)\tan\beta_j/2 - r\Lambda_\infty)M$ for some $j \in \mathbf{Z}\setminus\Gamma$.

For the above reason, we consider for any $j \in \mathbf{Z}\setminus\Gamma$ the following eigenvalue problem

$$\Big((2j-1)\tan\frac{\beta_j}{2} - rA_\infty\Big)u = \theta u, \quad u \in \mathbf{R}^N. \tag{3.6}$$

Since A_∞ is a symmetric matrix, all the eigenvalues and the corresponding eigenvectors of $(2j-1)\tan\beta_j/2 - rA_\infty$ are real. Let $\theta_{j1}, \theta_{j2}, \ldots, \theta_{jN}$ be eigenvalues of $(2j-1)\tan\beta_j/2 - rA_\infty$, and let $u_{j1}, u_{j2}, \ldots, u_{jN}$ be the corresponding eigenvectors, which forms an orthogonal basis of $\mathbf{R}^N$.

Now, we set

$$\begin{aligned} e_{jk}^{(c)}(t) &= e^{i(2j-1)t}\xi_{jk} + e^{-i(2j-1)t}\overline{\xi}_{jk}, \\ e_{jk}^{(s)}(t) &= i[e^{i(2j-1)t}\xi_{jk} - e^{-i(2j-1)t}\overline{\xi}_{jk}], \end{aligned}$$

where $j \in \mathbf{N}\setminus\Gamma$, $k = 1, 2, \ldots, N$ and

$$\xi_{jk} = (\lambda_j^{2n-1}u_{jk}, \lambda_j^{2n-2}u_{jk}, \ldots, \lambda_j u_{jk}, u_{jk})^T.$$

It is easy to prove that

$$\{e_{jk}^{(c)}, e_{jk}^{(s)} | j \in \mathbf{N}\setminus\Gamma, k = 1, 2, \ldots, N\}$$

forms a complete orthogonal basis of X.

For

$$\begin{aligned} z_{jk} &= ae_{jk}^{(c)}(t) + be_{jk}^{(s)}(t) \\ &= (a\xi_{jk} + ib\xi_{jk})e^{i(2j-1)t} + (a\overline{\xi}_{jk} - ib\overline{\xi}_{jk})e^{-i(2j-1)t} \end{aligned}$$

with $a, b \in \mathbf{R}$, noting that $\Lambda_\infty M = M\Lambda_\infty$, by (3.3)–(3.5), we have

$$\begin{aligned} \langle L_{A_\infty} z_{jk}, z_{jk}\rangle &= 2\pi\Big[(i(2j-1)A^{-1}(a\xi_{jk} + ib\xi_{jk}) \\ &\qquad - r\Lambda_\infty M(a\xi_{jk} + ib\xi_{jk}), a\xi_{jk} + ib\xi_{jk}) \\ &\qquad + (-i(2j-1)A^{-1}(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}) \\ &\qquad - r\Lambda_\infty M(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}), a\overline{\xi}_{jk} - ib\overline{\xi}_{jk})\Big] \\ &= 2\pi\Big[(((2j-1)\tan\frac{\beta_j}{2} - r\Lambda_\infty) \\ &\qquad \times M(a\xi_{jk} + ib\xi_{jk}), a\xi_{jk} + ib\xi_{jk}) \\ &\qquad + (((2j-1)\tan\frac{\beta_j}{2} - r\Lambda_\infty) \\ &\qquad \times M(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}), a\overline{\xi}_{jk} - ib\overline{\xi}_{jk})\Big] \\ &= 2\pi\theta_{jk}\Big((M(a\xi_{jk} + ib\xi_{jk}), a\xi_{jk} + ib\xi_{jk}) \\ &\qquad + (M(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}), a\overline{\xi}_{jk} - ib\overline{\xi}_{jk})\Big). \end{aligned}$$

Note that

$$\begin{aligned} a\xi_{jk} + ib\xi_{jk} &= \frac{a(\xi_{jk} + \overline{\xi}_{jk}) + ib(\xi_{jk} - \overline{\xi}_{jk})}{2} \\ &\qquad + \frac{ia(\overline{\xi}_{jk} - \xi_{jk}) + b(\xi_{jk} + \overline{\xi}_{jk})}{2} i, \end{aligned}$$

so,

$$\begin{aligned} &(M(a\xi_{jk} + ib\xi_{jk}), a\xi_{jk} + ib\xi_{jk}) + (M(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}), a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}) \\ &\quad = \frac{1}{2}\Big[(M(a(\xi_{jk} + \overline{\xi}_{jk}) + ib(\xi_{jk} - \overline{\xi}_{jk})), a(\xi_{jk} + \overline{\xi}_{jk}) + ib(\xi_{jk} - \overline{\xi}_{jk})) \\ &\quad + (M(ia(\overline{\xi}_{jk} - \xi_{jk}) + b(\xi_{jk} + \overline{\xi}_{jk})), ia(\overline{\xi}_{jk} - \xi_{jk}) + b(\xi_{jk} + \overline{\xi}_{jk}))\Big]. \end{aligned}$$

Since M is positive definite, there exists $\gamma > 0$ such that

$$(Mu, u) \geq \gamma |u|^2, \quad \forall\, u \in \mathbf{R}^{2nN},$$

and hence,

$$\begin{aligned}
&(M(a\xi_{jk} + ib\xi_{jk}), a\xi_{jk} + ib\xi_{jk}) + (M(a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}), a\overline{\xi}_{jk} - ib\overline{\xi}_{jk}) \\
&\geq \frac{\gamma}{2}\Big(|a(\xi_{jk} + \overline{\xi}_{jk}) + ib(\xi_{jk} - \overline{\xi}_{jk})|^2 + |ia(\overline{\xi}_{jk} - \xi_{jk}) + b(\xi_{jk} + \overline{\xi}_{jk})|^2\Big) \\
&= \frac{\gamma}{4\pi(2j-1)}\|z_{jk}\|^2.
\end{aligned}$$

Note that

$$\sigma = \min\{|\theta_{jk}/2(2j-1)| : j \in \mathbf{N} \setminus \Gamma,$$
$$k = 1, 2, \ldots, N, \theta_{jk} \neq 0\} > 0.$$

Let $\delta = \sigma\gamma$. Then we have

$$\begin{aligned}
\langle L_{A_\infty} z_{jk}, z_{jk}\rangle &\geq \frac{2\theta_{jk}\gamma}{2j-1}\|z_{jk}\|^2 \\
&\geq \sigma\gamma\|z_{jk}\|^2 = \delta\|z_{jk}\|^2, \quad \text{if } \theta_{jk} > 0;
\end{aligned}$$

$$\begin{aligned}
\langle L_{A_\infty} z_{jk}, z_{jk}\rangle &\leq \frac{2\theta_{jk}\gamma}{2j-1}\|z_{jk}\|^2 \\
&\leq -\sigma\gamma\|z_{jk}\|^2 = -\delta\|z_{jk}\|^2, \quad \text{if } \theta_{jk} < 0;
\end{aligned}$$

$$\langle L_{A_\infty} z_{jk}, z_{jk}\rangle = 0, \quad \text{if } \theta_{jk} = 0.$$

Therefore, if we define

$$U^+ = \overline{\operatorname{span}\{e_{jk}^{(c)}, e_{jk}^{(s)} | \theta_{jk} > 0, j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\}},$$
$$U^- = \overline{\operatorname{span}\{e_{jk}^{(c)}, e_{jk}^{(s)} | \theta_{jk} < 0, j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\}},$$
$$U^0 = \operatorname{span}\{e_{jk}^{(c)}, e_{jk}^{(s)} | \theta_{jk} = 0, j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\},$$

then

$$X = U^+ \oplus U^- \oplus U^0.$$

Similarly, we can study the operator L_{A_0} and the eigenvalue problem

$$\Big((2j-1)\tan\frac{\beta_j}{2} - rA_0\Big)u = \theta u, \quad u \in \mathbf{R}^N.$$

In this case, we can find another orthogonal basis of X of the form

$$\{\widehat{e}_{jk}^{(c)}, \hat{e}_{jk}^{(s)} | j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\}$$

where

$$\widehat{e}_{jk}^{(c)}(t) = e^{i(2j-1)t}\widehat{\xi}_{jk} + e^{-i(2j-1)t}\overline{\widehat{\xi}}_{jk},$$
$$\widehat{e}_{jk}^{(s)}(t) = i[e^{i(2j-1)t}\hat{\xi}_{jk} - e^{-i(2j-1)t}\overline{\widehat{\xi}}_{jk}],$$

$j \in \mathbf{N} \setminus \Gamma$, $k = 1, 2, \ldots, N$, $\widehat{\xi}_{jk} = (\lambda_j^{2n-1}\widehat{u}_{jk}, \lambda_j^{2n-2}\widehat{u}_{jk}, \ldots, \lambda_j \widehat{u}_{jk}, \widehat{u}_{jk})^T$ and $\widehat{u}_{j1}, \widehat{u}_{j2}, \ldots, \widehat{u}_{jN}$, which are eigenvectors of $(2j-1)\tan(\beta_j/2) - rA_0$ corresponding to the eigenvalues $\widehat{\theta}_{j1}, \widehat{\theta}_{j2}, \ldots, \widehat{\theta}_{jN}$, respectively, form an orthogonal basis of $\mathbf{R}^N$.

Define

$$V^+ = \overline{\operatorname{span}\{\widehat{e}_{jk}^{(c)}, \widehat{e}_{jk}^{(s)} | \widehat{\theta}_{jk} > 0,\ j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\}},$$
$$V^- = \overline{\operatorname{span}\{\widehat{e}_{jk}^{(c)}, \widehat{e}_{jk}^{(s)} | \widehat{\theta}_{jk} < 0,\ j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\}},$$
$$V^0 = \operatorname{span}\{\widehat{e}_{jk}^{(c)}, \widehat{e}_{jk}^{(s)} | \widehat{\theta}_{jk} = 0,\ j \in \mathbf{N} \setminus \Gamma,\ k = 1, 2, \ldots, N\},$$

then

$$X = V^+ \oplus V^- \oplus V^0.$$

In the following, we will use the Galerkin approximation method and the S^1-index theory to get critical points of J in X.

For $m \geq 1$, let

$$H_m = \left\{ x(t) = \sum_{j=-m, j \notin \Gamma}^{m} a_j e^{ijt},\ a_j \in \mathbf{C}^{2nN},\ a_{-j} = \overline{a_j} \right\},$$

and $P_m : H \to H_m$ be the orthogonal projections from H to H_m. Then $\{P_m : m = 1, 2, \ldots\}$ is a Galerkin approximation scheme with respect to L_{A_∞} and L_{A_0}. By the above discussion, we have:

Lemma 3.1. *For m large enough,*

$$\ker(P_m L_{A_\infty} P_m) = \ker(L_{A_\infty}); \qquad \ker(P_m L_{A_0} P_m) = \ker(L_{A_0});$$
$$\dim \ker(L_{A_\infty}) = n_\infty,$$
$$U^0 \subset \ker(L_{A_\infty}); \qquad V^0 \subset \ker(L_{A_0}).$$

Lemma 3.2. *Under the condition* (f_1) *and either* $n_\infty = 0$ *or* (f_5)–(f_6), *we have*

(i) *J satisfies $(PS)_c$ condition over H for any $c \in \mathbf{R}$, i.e., every sequence $\{z_j\} \subset H$ with $J(z_j) \to c$ and $J'(z_j) \to 0$ possesses a convergent subsequence.*

(ii) *For m large enough, J_m satisfies $(PS)_c$ condition over H_m for any $c \in \mathbf{R}$.*

(iii) *J satisfies the $(PS)_c^*$ condition over H for any $c \in \mathbf{R}$, i.e., every sequence $\{z_j\} \subset H$ with $z_j \in H_j$, $J_j(z_j) \to c$ and $J_j'(z_j) \to 0$ possesses a convergent subsequence.*

Proof. By (3.1), for sufficiently small ϵ, there exists a constant $M_1 > 0$ such that, for any $x \in \mathbf{R}^{2nN}$,

$$|\nabla W(x) - \Lambda_\infty M x| \le \epsilon |x| + M_1.$$

Thus, there must exist $c_1, c_2 > 0$ such that

$$|W(x) - \frac{1}{2}(\Lambda_\infty M x, x)| \le \frac{1}{2}\epsilon |x|^2 + M_1|x| \le c_1|x|^2 + c_2.$$

Hence, we have

$$\langle \Phi'_{A_\infty}(x), y\rangle = r \int_0^{2\pi} (\nabla W(x) - \Lambda_\infty M x, y)\, dt,$$

and it is easy to show that Φ'_{A_∞} is compact and bounded (cf., **[30,** Proposition B.37]). In view of Lemma 3.1, the proof is just the same as the proof of Lemma 2.1 in **[24]** and Lemma 7.1 in **[31]**. □

Proof of Theorem 1.1. As we already mentioned above, critical points of $J|_X$ over X may not be critical points of J over H. However, critical points of J in X are indeed nonconstant classic $2(2n+1)$-periodic solutions of (2.6) satisfying (2.1), and hence they give solutions of (1.8) with the property $z(t-(2n+1)) = -z(t)$. Therefore, we will seek critical points of J in X and carry out the proof in several steps.

Step 1. (f_5) and (f_6) imply that:

$$|\nabla W(x) - \Lambda_\infty M x| \text{ is bounded and}$$
$$W(x) - \frac{1}{2}(\Lambda_\infty M x, x) \longrightarrow -\infty \text{ as } |x| \to \infty. \tag{3.7}$$

By Lemma 3.2, we know that J satisfies $(PS)_c$ condition and $(PS)_c^*$ for any $c \in \mathbf{R}$.

Recall that

$$J(x) = \frac{1}{2}\langle L_{A_\infty} x, x\rangle - \Phi_{A_\infty}(x), \qquad J(x) = \frac{1}{2}\langle L_{A_0} x, x\rangle - \Phi_{A_0}(x).$$

Step 2. Suppose $\psi(A_0, A_\infty) > 0$ for sufficiently large m. Let

$$U_m^+ = U^+ \cap H_m, \qquad V_m^- = V^- \cap H_m.$$

By (3.2) and similar discussion as in **[17]**, for any $\epsilon > 0$, there exists a constant $C_1 > 0$ such that

$$|\nabla W(x) - \Lambda_0 M x| \le \epsilon |x| + C_1 |x|^2, \quad \forall\, x \in \mathbf{R}^{2nN},$$

and hence

$$\begin{aligned} |W(x) - \frac{1}{2}(\Lambda_0 M x, x)| &= \left| \int_0^1 (\nabla W(sx) - \Lambda_0 M sx, x)\, ds \right| \\ &\le \frac{\epsilon}{2}|x|^2 + \frac{C_1}{3}|x|^3. \end{aligned} \tag{3.8}$$

Then, for any $x \in V_m^-$, by (3.8), we have

$$\begin{aligned} J(x) &= \frac{1}{2}\langle L_{A_0} x, x\rangle - \Phi_{A_0}(x) \\ &\le -\frac{1}{2}\,\delta\|x\|^2 + r\int_0^{2\pi} \left|W(x(t)) - \frac{1}{2}(\Lambda_0 M x(t), x(t))\right| dt \\ &\le -\frac{1}{2}\,\delta\|x\|^2 + r\int_0^{2\pi} \left[\frac{\epsilon}{2}|x(t)|^2 + \frac{C_1}{3}|x(t)|^3\right] dt \\ &\le -\frac{1}{2}\,\delta\|x\|^2 + \frac{r\epsilon}{2}\|x\|^2 + \frac{rC_1}{3}\sqrt{2\pi}\|x\|_{L_6}^3 \\ &\le -\frac{1}{2}\delta\|x\|^2 + \frac{r\epsilon}{2}\|x\|^2 + \frac{rC_1C_2^3}{3}\sqrt{2\pi}\|x\|^3 \\ &= -\frac{1}{2}\left[\delta - \epsilon r - \frac{2rC_1C_2^3}{3}\sqrt{2\pi}\|x\|\right]\|x\|^2, \end{aligned}$$

where C_2 is a positive constant determined by the Sobolev inequality $\|x\|_{L_6} \le C_2\|x\|$ for any $x \in H$. Take

$$\epsilon = \frac{\delta}{2r}, \qquad \sigma = \frac{\epsilon r}{rC_1C_2^3\sqrt{2\pi}}.$$

Then, for any $x \in V_m^-$ with $\|x\| = \sigma$,

$$J(x) \leq -\frac{1}{12}\,\delta\sigma^2 := c_\infty < 0, \tag{3.9}$$

where σ and c_∞ are constants independent of m.

On the other hand, by (3.1), for any $\epsilon > 0$, there exists a constant $C_3 > 0$, such that for any $x \in \mathbf{R}^{2nN}$,

$$|\nabla W(x) - \Lambda_\infty Mx| \leq \epsilon|x| + C_3.$$

Thus,

$$\left| W(x) - \frac{1}{2}(\Lambda_\infty Mx, x) \right| \leq \frac{\epsilon}{2}|x|^2 + C_3|x|. \tag{3.10}$$

And, for any $x \in U_m^+$, by (3.10), we have

$$\begin{aligned} J(x) &= \frac{1}{2}\langle L_{A_\infty} x, x\rangle - \Phi_{A_\infty}(x) \\ &\geq \frac{1}{2}\,\delta\|x\|^2 - r\int_0^{2\pi} \left[W(x(t)) - \frac{1}{2}(\Lambda_\infty Mx(t), x(t)) \right] dt \\ &\geq \frac{1}{2}(\delta - r\epsilon)\|x\|^2 - rC_3\sqrt{2\pi}\|x\|. \end{aligned}$$

Choose $\epsilon < \delta/r$, then there exists a sufficiently large constant $C_4 > 0$, which is also independent of m, such that for any $x \in U_m^+$,

$$\inf_{x\in U_m^+} J(x) > -C_4 := c_0(< c_\infty) > -\infty. \tag{3.11}$$

Step 3. Let ind denote the S^1-index on X (cf., [**27**]). For $j \geq 0$ and sufficiently large m, denote

$$\Sigma_j(m) = \{D \subset X \cap H_m : D \text{ is compact, invariant for } S^1 \text{ and } \operatorname{ind}(D) \geq j\}.$$

Denote

$$c_{m,j} = \inf_{D\in\Sigma_j(m)} \sup_{u\in D} J_m(u).$$

Clearly, one has $\Sigma_j(m) \subset \Sigma_{j-1}(m)(j \geq 2)$, and hence

$$-\infty \leq c_{m,1} \leq c_{m,2} \leq \cdots < +\infty. \tag{3.12}$$

For $j \geq 1/2\text{co}\dim_{X\cap H_m} U_m^+ + 1$, by [**27,** Proposition 5.2], it is easy to get

$$D \cap U_m^+ \neq \emptyset, \quad \forall\ D \in \Sigma_j(m).$$

Hence,

$$\sup_D J_m \geq \sup_{D\cap U_m^+} J_m \geq \inf_{D\cap U_m^+} J_m \geq \inf_{U_m^+} J_m \geq c_0,$$

which implies that

$$c_{m,j} \geq c_0, \quad \text{if } j \geq \frac{1}{2}\text{co } \dim_{X\cap H_m} U_m^+ + 1. \tag{3.13}$$

On the other hand, for $j \leq 1/2 \dim V_m^-$, by [**27,** Proposition 5.3], we get

$$\text{ind}\,\partial D = \frac{1}{2}\dim V_m^-,$$

with $D = \{u \in V_m^-, \|u\| < r\}$. Since ∂D is compact, invariant for S^1 and $\text{ind}\,\partial D \geq j$, we have $\partial D \in \Sigma_j(m)$. Hence,

$$c_{m,j} \leq \max_{\partial D} J_m \leq c_\infty.$$

That is,

$$c_{m,j} \leq c_\infty, \quad \text{if } j \leq \frac{1}{2}\dim V_m^-. \tag{3.14}$$

For simplicity, denote

$$p = \frac{1}{2}\text{co } \dim_{X\cap H_m} U_m^+, \qquad q = \frac{1}{2}\dim V_m^-.$$

According to the results in [**17**], we have $q - p = \psi(A_0, A_\infty)$.

Now, (3.12)–(3.14) implies that

$$c_0 \leq c_{m,p+1} \leq c_{m,p+2} \leq \cdots \leq c_{m,q} \leq c_\infty.$$

Since c_0 and c_∞ are independent of m, passing to a subsequence if necessary, we have

$$c_0 \leq c_{\infty,p+1} \leq c_{\infty,p+2} \leq \cdots \leq c_{\infty,q} \leq c_\infty \quad \text{as } m \to \infty. \tag{3.15}$$

Hence, each $c_{\infty,j}(p+1 \leq j \leq q)$ is a critical value of J. If all of the $c_{\infty,j}$ are distinct, the proof is complete.

If $c_{\infty,j} = c_{\infty,k} = c$ for some $p+1 \leq j < k \leq q$, then, just following the proof of [**12**], we have

$$\text{ind}\,(K_c \cap X) \geq k - j + 1 \geq 2,$$

where $K_c = \{x \in H \mid J'(x) = 0, J(x) = c\}$. Combining this with (3.15) yields that J has at least $q-p$ different S^1-orbits in X. This completes the proof. $\square$

Proof of Corollary 1.2. In the case of $\psi(A_\infty, A_0) > 0$, we study the functional $-J$ instead of J. Let

$$\overline{U}_m^+ = U_m^- = U^- \cap H_m, \qquad \overline{V}_m^- = V_m^+ = V^+ \cap H_m.$$

One can prove Corollary 1.2 by simply repeating the proof of Theorem 1.1 with U_m^+ and V_m^- replaced by $\overline{U}_m^+$ and $\overline{V}_m^-$. $\square$

Proof of Theorem 1.3. If $n_\infty \neq 0$, for sufficiently large m, let

$$U_m^+ = (U^+ \oplus U^0) \cap H_m, \qquad V_m^- = V^- \cap H_m.$$

According to the results in **[17]**, we have $q - p = \psi(A_0, A_\infty) + \psi^{(0)}(A_\infty)$.

At the same time, by (3.7), $W(x) - \frac{1}{2}(\Lambda_\infty Mx, x)$ is bounded from above. Namely, there exists a constant $C_5 > 0$, such that for any $x \in \mathbf{R}^{2nN}$,

$$W(x) - \frac{1}{2}(\Lambda_\infty Mx, x) < C_5. \tag{3.16}$$

For $x = x_+ + x_0 \in U_m^+$, by (3.7) and (3.16), there exists a constant C_6 such that

$$J(x) \geq \frac{1}{2}\delta\|x_+\|^2 - \Phi_{A_\infty}(x_+ + x_0) \geq \frac{1}{2}\delta\|x_+\|^2 - C_6\|x_+\| - C_5,$$

which implies that (3.11) still holds in this case. The rest of the proof is very similar to that of Theorem 1.1, which will be omitted. The proof is finished. $\square$

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