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A POWER MEAN INEQUALITY INVOLVING THE COMPLETE ELLIPTIC INTEGRALS

GENDI WANG, XIAOHUI ZHANG AND YUMING CHU

ABSTRACT. In this paper the authors investigate a power mean inequality for a special function which is defined by the complete elliptic integrals.

1. Introduction. Throughout this paper, we let $r' = \sqrt{1-r^2}$ for $0 < r < 1$. The well-known complete elliptic integrals of the first and second kind **[9, 11]** are, respectively, defined by

$$\begin{cases} \mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \\ \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty, \end{cases} \tag{1}$$

and

$$\begin{cases} \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2\theta}\, d\theta, \\ \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1. \end{cases} \tag{2}$$

In the sequel, we will use the symbols $\mathcal{K}$ and $\mathcal{E}$ for $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively. The complete elliptic integrals play a very important role in the study of conformal invariants **[7]**, quasiconformal mappings **[5, 6, 7, 12]** and Ramanujan's modular equations **[4]**. Numerous sharp inequalities and elementary approximations for the complete elliptic integrals have been proved in **[2, 3, 7, 8, 13, 14]**.

The special function $m(r)$ is defined as

$$m(r) = \frac{2}{\pi} r'^2 \mathcal{K}(r)\mathcal{K}'(r), \quad 0 < r < 1.$$

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This function is of importance in the research of distortion theory of quasiconformal mappings in the plane. Recently, various interesting inequalities of $m(r)$ have been obtained by several authors, see **[3, 5, 6, 7, 15]**.

The power mean is defined for $x, y > 0$ and real parameter λ by

$$M_\lambda(x,y) = \left(\frac{x^\lambda + y^\lambda}{2}\right)^{1/\lambda} \quad \text{for } \lambda \neq 0, \text{ and } M_0(x,y) = \sqrt{xy}.$$

It is well known that $M_\lambda(x, y)$ is continuous and increasing with respect to λ. Many interesting properties of power means are given in **[10]**. Power mean inequalities for some special functions can be found in **[1, 7, 16, 17, 18]**.

In this paper, we shall show a power mean inequality for the special function $m(r)$. Our main result is the following theorem:

Theorem 1.1. *Let λ be a real number. The inequality*

$$M_\lambda(m(x), m(y)) \leq m(M_\lambda(x,y)) \tag{3}$$

holds for all $x, y \in (0,1)$ if and only if $\lambda \leq 0$. The reverse of (3) *holds for all $x, y \in (0,1)$ if and only if $\lambda \geq C > 0$, where C is some constant. The sign of equality is valid in* (3) *if and only if $x = y$.*

2. Lemmas. In order to prove our main result we need some lemmas, which we present in this section. We establish some properties of certain functions, which are defined in terms of complete elliptic integrals of the first and second kinds, $\mathcal{K}$ and $\mathcal{E}$, respectively.

Now we list some derivative formulas [**7,** Appendix E, pages 474–475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$

and

$$\frac{d}{dr}m(r) = \frac{\pi - 4\mathcal{E}'\mathcal{K}}{\pi r},$$

where $0 < r < 1$. By the derivative of $m(r)$, it is easy to see that $m(r)$ is strictly decreasing from $(0, 1)$ onto $(0, \infty)$.

The following Lemma 2.1 is from [**4,** Lemma 5.2 (2)] and [**7,** Theorem 3.21 (1), (7) and Exercise 3.43 (32)].

Lemma 2.1.

(i) *The function* $f_1(r) = r'^2\mathcal{K}/\mathcal{E}$ *is decreasing from* $(0,1)$ *onto* $(0,1)$.
(ii) *The function* $f_2(r) = (\mathcal{E} - r'^2\mathcal{K})/r^2$ *is strictly increasing and convex from* $(0,1)$ *onto* $(\pi/4,1)$.
(iii) *For each* $c \in [1/2,\infty)$, *the function* $f_3(r) = r'^c\mathcal{K}$ *is decreasing from* $[0,1)$ *onto* $(0,\pi/2]$.
(iv) *The function* $f_4(r) = r^{-2}(\mathcal{K} - \mathcal{E})/\mathcal{K}$ *is increasing and convex from* $(0,1)$ *onto* $(1/2,1)$.

Lemma 2.2. *For* $0 < r < 1$, *let*

$$g(r) = \frac{\mathcal{K}\mathcal{K}'(\mathcal{E}\mathcal{E}' + r^2\mathcal{K}\mathcal{K}' - \mathcal{K}\mathcal{E}')}{(4\mathcal{E}'\mathcal{K} - \pi)^2}.$$

Then $g(r) > 0$ *for all* $r \in (0,1)$, *and* $g(0^+) = g(1^-) = 0$.

Proof. By formula (2) and parts (1) and (2) of Lemma 2.1, we have

$$\frac{(4\mathcal{E}'\mathcal{K} - \pi)^2}{\mathcal{K}\mathcal{K}'} g(r) = \mathcal{E}\left(\mathcal{E}' - \frac{r'^2\mathcal{K}}{\mathcal{E}}\,\frac{\mathcal{E}' - r^2\mathcal{K}'}{r'^2}\right) > 0,$$

and hence $g(r) > 0$ for all $r \in (0,1)$.

By Lemma 2.1 (3) and (4), we get

$$\begin{aligned} g(0^+) &= \lim_{r\to 0+} g(r) = \lim_{r\to 1-} \frac{\mathcal{K}\mathcal{K}'(\mathcal{E}\mathcal{E}' + r'^2\mathcal{K}\mathcal{K}' - \mathcal{K}'\mathcal{E})}{(4\mathcal{E}\mathcal{K}' - \pi)^2} \\ &= \lim_{r\to 1-} \frac{\mathcal{K}'}{(4\mathcal{E}\mathcal{K}' - \pi)^2}\left[(r'\mathcal{K})^2\mathcal{K}' - \frac{\mathcal{K}' - \mathcal{E}'}{r'^2\mathcal{K}'}(r'^2\mathcal{K})(\mathcal{K}'\mathcal{E})\right] \\ &= 0, \end{aligned}$$

and

$$g(1^-) = \lim_{r\to 1-} g(r) = \lim_{r\to 1-} \frac{\mathcal{K}'}{(4\mathcal{E}' - \pi/\mathcal{K})^2}\left(\frac{\mathcal{E}\mathcal{E}'}{\mathcal{K}} + r^2\mathcal{K}' - \mathcal{E}'\right) = 0. \quad \square$$

Lemma 2.3. *Let λ be a real number. The function*

$$h(r) = \frac{4\mathcal{E}'\mathcal{K} - \pi}{m(r)} \left(\frac{m(r)}{r}\right)^{\lambda}$$

is strictly increasing on $(0,1)$ if and only if $\lambda \le 0$.

Proof. By logarithmic differentiation,

$$\begin{aligned}
\frac{h'(r)}{h(r)} &= \frac{4\left[\mathcal{K}((\mathcal{E}'-\mathcal{K}')/r')(-r/r') + \mathcal{E}'(\mathcal{E}-r'^2\mathcal{K})/(rr'^2)\right]}{4\mathcal{E}'\mathcal{K}-\pi} \\
&\quad - \frac{\pi - 4\mathcal{E}'\mathcal{K}}{\pi r m(r)} + \lambda\left(\frac{\pi - 4\mathcal{E}'\mathcal{K}}{\pi r m(r)} - \frac{1}{r}\right) \\
&= \left(\frac{4\mathcal{E}'\mathcal{K}-\pi}{\pi r m(r)} + \frac{1}{r}\right) \\
&\quad \times \left[\frac{1}{1+(\pi m(r))/(4\mathcal{E}'\mathcal{K}-\pi)}\left(1+8\frac{\mathcal{K}\mathcal{K}'(\mathcal{E}\mathcal{E}'+r^2\mathcal{K}\mathcal{K}'-\mathcal{K}\mathcal{E}')}{(4\mathcal{E}'\mathcal{K}-\pi)^2}\right)\right. \\
&\qquad \left. -\lambda\right], \qquad (4)
\end{aligned}$$

which is positive for all $r \in (0,1)$ if and only if $\lambda \le 0$ by Lemma 2.2, since $m(r)/(4\mathcal{E}'\mathcal{K}-\pi)$ is clearly decreasing from $(0,1)$ onto $(0,\infty)$. $\square$

Remark 1. Let

$$H(r) = \frac{1}{1+(\pi m(r))/(4\mathcal{E}'\mathcal{K}-\pi)}\left(1+8\frac{\mathcal{K}\mathcal{K}'(\mathcal{E}\mathcal{E}'+r^2\mathcal{K}\mathcal{K}'-\mathcal{K}\mathcal{E}')}{(4\mathcal{E}'\mathcal{K}-\pi)^2}\right)$$

for $r \in (0,1)$ and $H(0)=0$, $H(1)=1$. Then H is continuous on $[0,1]$. Hence, there exists $r_0 \in (0,1]$ such that H obtains its maximum C at r_0. Thus, it is easy to conclude from (4) that h is strictly decreasing on $(0,1)$ if and only if $\lambda \ge C$.

Open problem 2.4. What is the exact expression for C?

3. Proof of the main result. We are now in a position to prove the main result.

Proof of Theorem 1.1. We first prove inequality (3) for $\lambda \neq 0$. We may assume that $x \leq y$. Define

$$F(x,y) = m\left(M_\lambda(x,y)\right)^\lambda - \frac{m(x)^\lambda + m(y)^\lambda}{2}, \quad \lambda \neq 0.$$

Let $t = M_\lambda(x,y)$. Then $\partial t/\partial x = (1/2)(x/t)^{\lambda-1}$. If $x < y$, we have that $t > x$. By differentiation,

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\lambda}{2} m(t)^{\lambda-1} \frac{\pi - 4\mathcal{E}'(t)\mathcal{K}(t)}{\pi t}\left(\frac{x}{t}\right)^{\lambda-1} \\
&\quad - \frac{\lambda}{2} m(x)^{\lambda-1} \frac{\pi - 4\mathcal{E}'(x)\mathcal{K}(x)}{\pi x} \\
&= \frac{\lambda x^{\lambda-1}}{2\pi}\left[\frac{4\mathcal{E}'(x)\mathcal{K}(x) - \pi}{m(x)}\left(\frac{m(x)}{x}\right)^\lambda - \frac{4\mathcal{E}'(t)\mathcal{K}(t) - \pi}{m(t)}\left(\frac{m(t)}{t}\right)^\lambda\right]
\end{aligned}$$

which is positive if and only if $\lambda < 0$ by Lemma 2.3. Hence, $F(x,y)$ is strictly increasing with respect to x and $F(x,y) \leq F(y,y) = 0$. We now obtain the inequality

$$m\left(M_\lambda(x,y)\right)^\lambda \leq \frac{m(x)^\lambda + m(y)^\lambda}{2},$$

that is, $m(M_\lambda(x,y)) \geq M_\lambda(m(x), m(y))$ if and only if $\lambda < 0$, with the equality if and only if $x = y$.

Similarly, by the statement in Remark 1, one can see that the reverse of (3) holds for all $x, y \in (0,1)$ if and only if $\lambda \geq C > 0$, and C is the same as Remark 1, with the equality if and only if $x = y$.

Now we prove inequality (3) for $\lambda = 0$. We may assume that $x \leq y$. Define

$$G(x,y) = \frac{m(\sqrt{xy})^2}{m(x)m(y)}.$$

Let $t = \sqrt{xy}$. Then $\partial t/\partial x = (1/2)(t/x)$. If $x < y$, we have that $t > x$. By logarithmic differentiation, we have

$$\begin{aligned}
\frac{1}{G(x,y)}\frac{\partial G}{\partial x} &= \frac{1}{m(t)} \frac{\pi - 4\mathcal{E}'(t)\mathcal{K}(t)}{\pi t} \frac{t}{x} \\
&\quad - \frac{1}{m(x)} \frac{\pi - 4\mathcal{E}'(x)\mathcal{K}(x)}{\pi x}
\end{aligned}$$

$$= \frac{1}{\pi x}\left(\frac{4\mathcal{E}'(x)\mathcal{K}(x)-\pi}{m(x)} - \frac{4\mathcal{E}'(t)\mathcal{K}(t)-\pi}{m(t)}\right),$$

which is negative. Hence, $G(x,y)$ is strictly decreasing with respect to x and $G(x,y) \geq G(y,y) = 1$. We now obtain the inequality

$$m(\sqrt{xy}) \geq \sqrt{m(x)m(y)},$$

that is, $m(M_0(x,y)) \geq M_0(m(x),m(y))$, with the equality if and only if $x = y$. This completes the proof. □

The following corollary is clear.

Corollary 3.1. *The function* $m(r)$ *is concave on* $(0,1)$ *with respect to power mean of order* λ *if and only if* $\lambda \leq 0$, *and convex on* $(0,1)$ *with respect to power mean of order* λ *if and only if* $\lambda \geq C$.

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College of Mathematics and Econometrics, Hunan University, Changsha 410082, China
Email address: wgdwang@gmail.com

Department of Physics and Mathematics, University of Eastern Finland, FI-80100 Joensuu, Finland
Email address: xiaohui.zhang@uef.fi

Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China
Email address: chuyuming@hutc.zj.cn