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ON HARMONIC FUNCTIONS ON SURFACES WITH POSITIVE GAUSS CURVATURE AND THE SCHWARZ LEMMA

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ABSTRACT. We prove some versions of the Schwarz lemma for real harmonic functions in certain Riemann surfaces with positive Gauss curvature.

1. Introduction and statement of the main result. We first recall that the hyperbolic metric $d_h(z,w)$ of the unit disk $\mathbf{U} := \{z : |z| < 1\}$ with constant Gauss curvature -4 is defined by

$$\tanh d_h(z,w) = \frac{|z-w|}{|1-z\overline{w}|}.$$

The classical Schwarz (Schwarz-Pick) lemma says that an analytic function f of the unit disk into itself is a contraction in the hyperbolic metric, i.e., for $z, w \in \mathbf{U}$,

$$\frac{|f(z)-f(w)|}{|1-f(z)\overline{f(w)}|} \leq \frac{|z-w|}{|1-z\overline{w}|}. \tag{1.1}$$

Letting $z \to w$, we get

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}. \tag{1.2}$$

The interested reader can find some extensions of the Schwarz lemma for harmonic and analytic functions in **[1, 2, 3, 4, 8]**.

The starting point of this paper is the following recent extension of Schwarz lemma for harmonic functions:

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Theorem 1.1. **[6]** *Let f be a real harmonic function with respect to the Euclidean metric $\rho = 1$ of the unit disk $\mathbf{U} := \{z : |z| < 1\}$ into $(-1,1)$. Then the following sharp inequality holds*

$$|\nabla f(z)| \leq \frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}, \quad |z|<1. \tag{1.3}$$

1.1. Harmonic mappings between Riemann surfaces. Let (M,σ) and (N,ρ) be Riemann surfaces with metrics σ and ρ, respectively. If a mapping $f : (M,\sigma) \to (N,\rho)$ is C^2, then f is said to be harmonic (to avoid confusion, we will sometimes sayρ-harmonic) if

$$f_{z\bar{z}} + (\log\rho)_w \circ f f_z\, f_{\bar{z}} = 0, \tag{1.4}$$

where z and w are the local parameters on M and N, respectively (see **[9]**). Also f satisfies (1.4) if and only if its Hopf differential defined as

$$\Psi = \rho \circ f f_z \overline{f_{\bar{z}}} \tag{1.5}$$

is a holomorphic quadratic differential on M.

In this paper, we will extend Theorem 1.1 for real harmonic functions with respect to certain metrics. Let D be a domain in $\mathbf{C}$ and ρ a conformal metric in D. The Gaussian curvature on the domain is given by

$$K_D = -\frac{1}{2}\frac{\Delta\log\rho}{\rho}.$$

Consider the radial metric

$$\rho(z) = \varrho(|z|),$$

$|z| < 1$, where ϱ is a non-negative function. It can be easily proved that, if the Gauss curvature of ρ is positive (negative), then the function $r \to \varrho(r)$ is a decreasing (increasing) function for $r \in [0,1]$.

Definition 1.2. We will say that a metric ρ is *admissible* if $\rho(z) = \varphi(|z|)$, where $\varphi : \mathbf{U} \to \mathbf{C}\setminus\mathbf{R}_0^-$ is some analytic function defined in the unit disk satisfying the following properties:

(1) $\varphi(|z|) \leq |\varphi(z)|$, and φ is nonincreasing in $[0,1]$,
(2) $\varphi(-1,1) \subset \mathbf{R}$ and $\int_0^1 (\sqrt{\varphi(x)} - \sqrt{\varphi(-x)})\,dx = 0$.

By using the fact that Hopf differential of a harmonic mapping is a holomorphic mapping, the following proposition can be proved.

Proposition 1.3. [**5**] *Let* $\varphi : \mathbf{U} \to \mathbf{C} \setminus \mathbf{R}_0^-$ *be an analytic function. If* $f : \Omega \to D$ *is* $\rho(z) = |\varphi(z)|$*-harmonic and*

$$\phi = \int_0^z \sqrt{\varphi(z)}\, dz,$$

then the mapping $F = \phi \circ f$ *is harmonic with respect to the Euclidean metric.*

Notice that the parameter $w = \phi(z)$ is called the distinguished parameter ([**10**]). Proposition 1.3 implies

Lemma 1.4. *Let* φ *be an analytic function defined in the unit disk such that* $\varphi(r) \geq 0$ *for* $-1 < r < 1$. *Then the real function* $f : \Omega \to \mathbf{R}$ *is* $\rho(z) = \varphi(|z|)-$ *harmonic if and only if* $\phi \circ f$ *is harmonic with respect to the Euclidean metric, where*

$$\phi(s) = \int_0^s \sqrt{\varphi(t)}\, dt.$$

With the help of Lemma 1.4, we can construct the family of all real harmonic mappings with respect to the Poincaré and Riemann metrics.

Example 1.5. Let

$$\varphi(w) = \frac{1}{(1 - w^2)^2}.$$

Then f is $|\varphi|$-harmonic if and only if $f = \tanh g$, where g is harmonic (with respect to the Euclidean metric). Since, for real w, the function $\rho = |\varphi(w)|$ coincides with the density

$$\lambda = \frac{1}{(1 - |w|^2)^2}$$

of Poincaré (hyperbolic) metric, we obtain that f is real λ-harmonic if and only if $f = \tanh g$, where g is real Euclidean harmonic. In particular, the functions

$$f_n(z) = \tanh(nx), \quad z = x + iy$$

are real λ-harmonic functions of the unit disk into $(-1,1)$. Further

$$\nabla f_n(0) = (n,0).$$

Thus $|\nabla f(0)|$ is not uniformly bounded, and we do not have any version of Theorem 1.1 for hyperbolic harmonic functions. Notice that the Gauss curvature of the hyperbolic metric is equal to -4, and the hyperbolic metric is not an admissible metric.

Example 1.6. Let

$$\varphi(z) = \frac{1}{(1+w^2)^2}.$$

Then f is $|\varphi|$-harmonic if and only if $f = \tan g$, where g is harmonic. For real w, the metric density $\rho = |\varphi(w)|$ coincides with the density

$$\varrho = \frac{1}{(1+|w|^2)^2}$$

of the Riemann metric. It follows that f is real ϱ harmonic if and only if $f = \tan g$, where g is real harmonic. Observe that the Riemann metric is an *admissible metric* and its Gauss curvature is 4. Another admissible metric is the Hamilton cigar soliton known in physics as Witten's black hole. It is a Kähler metric defined on $\mathbf{C}$ by

$$\kappa = \frac{1}{1+|w|^2}.$$

The Gauss curvature of κ is given by

$$K = \frac{2}{1+|w|^2}.$$

In both these cases, we have $K > 0$. In the last case $\phi(z) = \operatorname{ArcSinh}(z)$ holds. Both these metrics are admissible, and we will show that the relation (1.3) holds for real harmonic functions with respect to these metrics.

Namely, we have the following theorem which is the main result of the paper.

Theorem 1.7. *Let $\rho(z)$ be an admissible metric and f a ρ-harmonic mapping of the unit disk into the interval $(-1,1)$. Then the following* *inequality*

$$|\nabla f(z)| \leq \frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}$$

holds.

Remark 1.8. As the Euclidean metric is an admissible metric, it follows that Theorem 1.7 is an extension of Theorem 1.1.

2. Some auxiliary results and the proof of the main result.

Lemma 2.1. *Let* $\chi : \mathbf{U} \to \mathbf{C}^*$ *be an analytic function defined in the unit disk satisfying the conditions*

(i) $|\chi(|z|)| \leq |\chi(z)|$ *and*
(ii) $|\chi(r)|$ *is nonincreasing for* $r \in [0,1]$.

Let

$$\psi(z) = \frac{\int_0^z \chi(z)\,dz}{\int_0^1 \chi(x)\,dx}$$

be an analytic function defined in the unit disk. Then $\mathbf{U} \subset \psi(\mathbf{U})$.

Proof. Let $0 < r < 1$ and $w \in \psi(r\mathbf{T})$, and let $\delta \subset \psi(\mathbf{U})$ be any curve with endpoints w and 0. Here $\mathbf{T} = \{z : |z| = 1\}$ is the unit circle. Because $\psi'(z) \neq 0$, it follows that ψ is a covering. Since ψ is a covering, it follows that $\psi^{-1}(\delta) = \bigcup_{i\in I}\gamma_i$, where $\{\gamma_i,\, i \in I\}$ is a set of mutually disjoint curves. Assume that 0 is the endpoint of γ_{i_0}, and that $l = l_r$ is the length of γ_{i_0}. Then ψ is univalent in γ_{i_0} and, because ψ is an open mapping, then $\lim_{r\to 1} l_r \geq 1$. Assume, in addition, that $g : [0,l] \to \gamma_{i_0}$ is an arc length parametrization of γ_{i_0}. Take $\chi(t) = \psi(g(t))$. Then

$$\begin{aligned}
|\delta| &= \int_0^l |\chi'(t)|\,dt = \int_0^l |\psi'(g(s))| \cdot |g'(s)|\,ds \\
&= \int_0^l |\psi'(g(s))|\,ds = \int_0^l |\chi(g(s))| ds \bigg/ \left|\int_0^1 \chi(x)\,dx\right| \\
&\geq \int_0^l |\chi(|g(s)|) ds \bigg/ \left|\int_0^1 \chi(x)\,dx\right|
\end{aligned}$$

$$\geq \int_0^{\min\{1,l\}} |\chi(s)|\, ds \Big/ \left| \int_0^1 \chi(x)\, dx \right|$$

It follows that $\limsup_{r\to 1} \operatorname{dist}(\psi(r\mathbf{T}), 0) \geq 1$, which implies that $\mathbf{U} \subset \psi(\mathbf{U})$. $\square$

By using Lemma 2.1, we can produce the following examples.

Example 2.2. Let $a(z) = \sum_{k=0}^{\infty} a_k z^k$ be a non-vanishing analytic function defined in the unit disk such that, for $k \in \mathbf{N}$, $a_k \geq 0$, $a_0 > 0$. Then, for $I = \int_0^1 (dx)/a(x)$, the function

$$\psi(z) = \frac{1}{I} \int_0^z \frac{dz}{a(z)}$$

satisfies the condition $\mathbf{U} \subset \psi(\mathbf{U})$. Figure 1 contains the graph of ψ for the special case $a(z) = 1 + z^3$.

Example 2.3. Let $a(z) = \sum_{k=0}^{\infty} a_k z^{2k} : \mathbf{U} \to \mathbf{C} \setminus \mathbf{R}_0^-$ be an analytic function defined in the unit disk such that for $k \in \mathbf{N}$, $a_k \geq 0$, $a_0 > 0$. Then, for every $p \geq 0$ and $I_p = \int_0^1 (dx)/a^p(x)$, the function

$$\psi(z) = \frac{1}{I_p} \int_0^z \frac{dz}{a^p(z)}$$

satisfies the conditions $\mathbf{U} \subset \psi(\mathbf{U})$ and $\psi(-1,1) = (-1,1)$, and the corresponding metric $\rho(z) = a^{-2p}(|z|)$ is admissible. See the graph of ψ in Figure 2 for special cases $p = 1$ and $a(z) = 1 + z^4$.

Lemma 2.4. *Let $\psi : \Omega \to D$ be a univalent conformal mapping between two Jordan domains Ω and D such that $\Omega \subseteq \mathbf{U} \subseteq D$. Then, for $z \in \psi^{-1}(\mathbf{U})$, the following inequality holds*

$$\frac{1 - |\psi(z)|^2}{1 - |z|^2} \leq |\psi'(z)|.$$

Proof. Since $\psi^{-1}(z) : \mathbf{U} \mapsto \mathbf{U}$, the previous inequality follows by the Schwarz-Pick lemma. $\square$

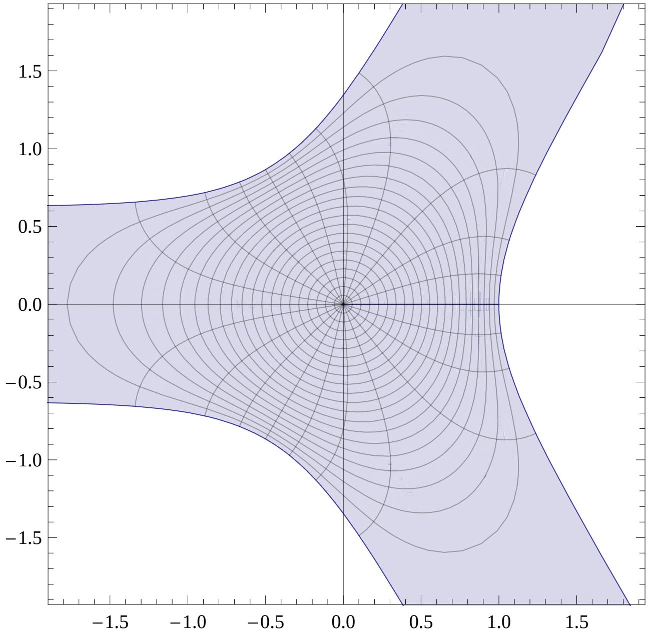

FIGURE 1. The graph of the function ψ for $a(z) = 1 + z^3$ contains the unit disk.

Proof of Theorem 1.7. Let $\phi(z) = \int_0^z \sqrt{\varphi(z)}\, dz$ be the distinguished parameter, and define

$$\psi(z) = \phi(z)/\phi(1).$$

According to Lemma 1.4, f is ρ-harmonic if and only if $\psi \circ f$ is real harmonic with respect to the Euclidean metric. By Definition 1.2, ψ satisfies the condition $\psi(-1) = -\psi(1) = -1$, and therefore $\psi(-1, 1) = (-1, 1)$. Observe next that, since $\varphi(z) \notin \mathbf{R}_0^-$, we have

$$\Re(\psi'(z)) = \Re(\sqrt{\varphi(z)}) \Big/ \int_0^1 \sqrt{\varphi(x)}\, dx > 0$$

and thus, by the Kaplan theorem **[7]**, ψ is a univalent mapping. By Lemma 2.1, $\mathbf{U} \subset \psi(\mathbf{U})$. By Lemma 2.4, for $z \in \psi^{-1}(\mathbf{U})$, we have

$$\frac{1}{|\psi'(z)|} \leq \frac{1 - |z|^2}{1 - |\psi(z)|^2}.$$

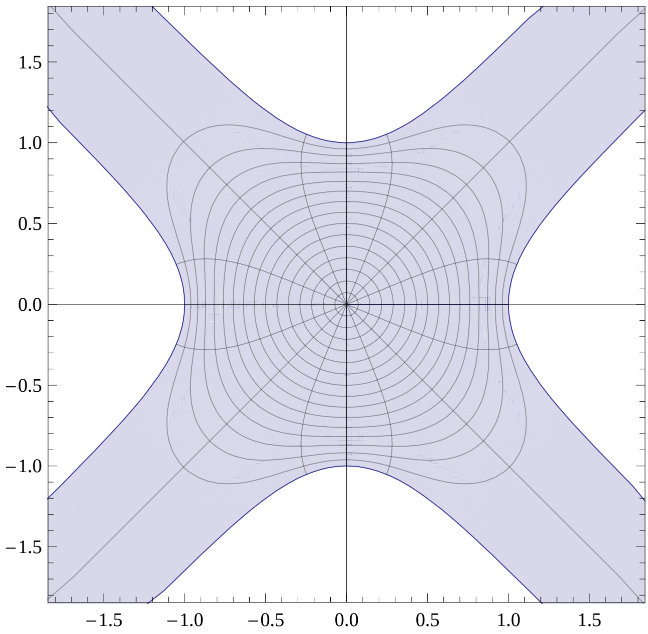

FIGURE 2. The graph of the function ψ for $a(z) = 1 + z^4$ contains the unit disk and $\psi(\mathbf{U}) \cap \mathbf{R} = (-1, 1)$. The corresponding metric $\rho(z) = 1/(1 + |z|^4)^2$ is admissible.

As $(-1, 1) \subset \psi^{-1}(\mathbf{U})$, for $z \in \mathbf{U}$, we obtain that

$$\frac{1}{|\psi'(f(z))|} \leq \frac{1 - |f(z)|^2}{1 - |\psi(f(z))|^2}. \tag{2.1}$$

According to Theorem 1.1, we have

$$|\nabla(\psi \circ f)(z)| \leq \frac{4}{\pi} \frac{1 - |(\psi \circ f)(z)|^2}{1 - |z|^2}.$$

Further,

$$\nabla(\psi \circ f) = \psi'(f(z)) \nabla f(z).$$

Thus, by using (2.1), for $w = f(z)$, we obtain

$$\begin{aligned} |\nabla f(z)| &= \frac{|\nabla(\psi \circ f)|}{|\psi'(f(z))|} \\ &\le \frac{1-|w|^2}{1-|\psi(w)|^2} \frac{4}{\pi} \frac{1-|(\psi\circ f)(z)|^2}{1-|z|^2} \\ &= \frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}. \end{aligned}$$

□

2.1. An open problem. Does every real harmonic function f of the unit disk into the unit interval with respect to some metric of a positive Gauss curvature satisfy the following inequality:

$$|\nabla f| \le \frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}?$$

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