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AN APPLICATION OF MATRICIAL FIBONACCI IDENTITIES TO THE COMPUTATION OF SPECTRAL NORMS

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1. Introduction. Among the most intensively studied integer sequences are the Fibonacci and Lucas sequences. Both are instances of *second order recurrences* **[9]**, satisfying $s_{k-2} + s_{k-1} = s_k$ for all integers k, but where the Fibonacci sequence (f_i) begins with $f_0 = 0$ and $f_1 = 1$, the Lucas sequence (l_i) has $l_0 = 2$ and $l_1 = 1$. Several authors have recently been interested in the singular values of Toeplitz, circulant and Hankel matrices that are obtained from the Fibonacci and Lucas sequences (see **[1, 4, 5, 12, 13, 14]**), where the authors obtain bounds for the "spectral norms," i.e., the largest singular value. In **[1]** a formula is given for the exact value of the spectral norms of the Lucas and Fibonacci Hankel matrices. In this paper, we present the exact value for the spectral norms of Toeplitz matrices involving Fibonacci and Lucas numbers.

All matrix and vector spaces will be considered as ones over the complex numbers. If A is a complex matrix, we let A^* denote the adjoint of A. The *singular values* of a matrix $A = (a_{ij})$ are defined to be the non-zero eigenvalues of $|A| \equiv (A^*A)^{1/2}$, and they are traditionally enumerated in descending order,

$$s_1 \geq s_2 \geq \cdots \geq s_k > 0,$$

with k equal to the rank of the matrix A. The *Schatten norms* (**[7, 10, 11]**) are a family of norms, denoted by $\|\cdot\|_p$ $(1 \leq p \leq \infty)$, defined for matrices in terms of their singular values via

$$\|A\|_p = \left(\sum_{i=1}^{k} s_i^p\right)^{1/p},$$

for $1 \leq p < \infty$, and

$$\|A\|_\infty = s_1.$$

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These norms differ markedly from the $\ell^p_{n^2}$-norms one is tempted to endow on $n \times n$ matrices using the formula

$$\left(\sum_{ij} |a_{ij}|^p \right)^{1/p},$$

in that the Schatten p-norms have the important property of being *unitary invariant norms*, i.e., for every unitary matrix U, one has

$$\|A\|_p = \|AU\|_p = \|UA\|_p.$$

Unless a matrix has some special form, it is usually very difficult to see a correspondence between the matrix entries and the Schatten norms, with one notable exception: when $p = 2$, one has

$$\|A\|_2 = \left(\sum_{i=1}^{k} s_i^2 \right)^{1/2} = \left(\sum_{ij} |a_{ij}|^2 \right)^{1/2}.$$

This is because, among the $\ell^p_{n^2}$-norms, one can easily show that $\ell^2_{n^2}$ is the only unitary invariant norm, and consequently, writing A in its polar decomposition, we see that the $\ell^2_{n^2}$ norm must agree with the Schatten 2-norm.

There is a beautiful duality theory for Schatten norms that is a non-commutative analogue of the Banach space duality of ℓ^p spaces. In infinite dimensions, the Schatten norms give rise to interesting ideals of compact operators [**8**]. The Schatten norms are also related to natural norms endowed upon tensor products of Banach spaces [**11**].

The terminology introduced so far is standard for functional analysts and students of operator theory, but it is common to see different terminology describing the same norms in applied linear algebra and matrix theory. For example, when $p = 2$ the Schatten norm is also called the *Frobeneous* norm, and when $p = \infty$, where the corresponding Schatten norm returns the maximal singular value of A, the norm $\|\cdot\|_\infty$ is called the *spectral norm* of A. Moreover, it is common to see the symbol $\|A\|_2$ describing the Schatten-2 norm by analysts, with the same symbol denoting the spectral norm by matrix theorists. In this paper, we will be concerned with only two norms, and we will attempt to strike a compromise with regard to notation: we will let $\|A\|$ denote the spectral norm, i.e., the Schatten-∞ norm or Hilbert space operator norm, and we will let $\|A\|_{\mathfrak{F}}$ denote the Frobeneous norm, i.e., the Schatten-2 or Hilbert-Schmidt norm (see **[6, 7, 10]**).

2. Fibonacci and Lucas matrices. We are primarily concerned with the matrices

$$F = (f_{i-j})_{i,j=0}^{n-1} \quad \text{and} \quad L = (l_{i-j})_{i,j=0}^{n-1},$$

with f and l denoting the Fibonacci and Lucas sequences, and we refer to F and L as the Fibonacci and Lucas matrices. An $n \times n$ matrix $A = (a_{ij})$ is called *Toeplitz* when there is a sequence $(\alpha_k)_{k=1}^{2n-1}$ such that $a_{ij} = \alpha_{i-j}$ for all i, j. This is a precise way of saying that the matrix A is constant along all its upper left to lower right diagonals. Both F and L are Toeplitz.

Let U be the $n \times n$ unitary matrix, all of whose matrix entries are zero, except those along the main cross diagonal, where the matrix entries are one. We are describing the unitary for which UA reverses the order of the rows of A and AU reverses the order of the columns of A. We intend to use the following fact repeatedly, so we isolate it for reference (the proof is trivial).

Lemma 2.1. *If A is a square Toeplitz matrix with real entries, then*

1. $UAU = A^*$ *and* $UA^*U = A$,
2. $U(A^*A)U = AA^*$ *and* $U(AA^*)U = A^*A$.

The spectral norm of the Fibonacci matrix is visible from the matrix entries because the spectral norm, while not equal to the Frobeneous norm, turns out to be a constant multiple of the Frobeneous norm. It is well known that these matrices are all rank two, but what appears to have been overlooked is that the *two non-zero singular values are equal.* Once this is proved, then, letting α denote this common value, we see that

$$\|F\|^2 = \alpha^2 = \frac{1}{2}(\alpha^2 + \alpha^2) = \frac{1}{2}\|F\|_{\mathfrak{F}}^2 = \frac{1}{2}\sum_{i,j=0}^{n-1} f_{i-j}^2.$$

Because the matrix is Toeplitz, the expression on the far right simplifies further: we might as well only sum the squares of the lower triangular part of F to obtain its spectral norm, resulting in the simple expression

$$\|F\|^2 = \sum_{k=0}^{n-1} \sum_{i=0}^{k} f_i^2 = \sum_{k=0}^{n-1} f_k f_{k+1},$$

which is the famous sequence of the sum of areas of Fibonacci rectangles (OEIS sequence A064831). Finally, as was noted in **[12]**, this last formula reduces to simply f_n^2 when n is even and $f_n^2 - 1$ when n is odd. We record a slight generalization of the pertinent observation for reference.

Lemma 2.2. *Assume A is a rank k matrix for which $s_1 = \cdots = s_k$. We have*

$$\|A\| = \frac{1}{\sqrt{k}} \|A\|_{\mathfrak{F}}.$$

In **[3]**, a formula is given for the spectral norms of the matrix

$$(\alpha_i - \alpha_j)$$

where α_i is a non-constant real sequence. We obtain a simple proof of this result using the previous lemma.[1]

Corollary 2.3 (**[3]**)**.** *Assume $(\alpha_i)_{i=1}^n$ is a non-constant finite sequence of real numbers, and $A = (a_{ij})$ is the $n \times n$ matrix with $a_{ij} = \alpha_i - \alpha_j$. We then have*

$$\|A\| = \frac{1}{\sqrt{2}} \|A\|_{\mathfrak{F}} = \sqrt{\sum_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2}.$$

Proof. As noted in **[3]**, this matrix is rank two and skew symmetric. Since the trace is zero, the two non-zero eigenvalues have the same moduli (they are negatives of one another), and, being skew symmetric, the moduli of these eigenvalues are the singular values. The lemma then applies, and we have

$$\|A\| = \frac{1}{\sqrt{2}} \|A\|_{\mathfrak{F}}.$$

By skew symmetry, the expression above may be calculated by summing the squares of just the lower triangular part of A, and that is what

$$\sqrt{\sum_{1\le i<j\le n} (\alpha_i-\alpha_j)^2}$$

is intended to represent. $\square$

In **[9]**, we are treated to an elegant exposition of *generalized* Fibonacci and Lucas sequences. The general recurrence relation is

$$s_n = as_{n-1} + bs_{n-2},$$

the Fibonacci sequence has the same initial values $f_0 = 0$ and $f_1 = 1$, while the Lucas sequence begins $l_0 = 2$ and $l_1 = a$. Once two successive terms of a sequence are dictated, one then uses the recurrence to determine all other terms of a bilateral sequence that satisfy the relation for all $n \in \mathcal{Z}$. The set of all bilateral sequences that satisfy this relation is denoted $\mathcal{B}(a, b)$, and it is observed that, for each a and non-zero b, the set $\mathcal{B}(a, b)$ is a two-dimensional vector space. We might also regard $\mathcal{B}(a, b)$ as a two-dimensional subspace of $\mathbf{C}^n$ for any $n \geq 2$. We will not attempt to distinguish between the $\mathcal{B}(a, b)$ for various values of n but will rely on context. We will only be concerned with $\mathcal{B}(1, 1)$ and $\mathcal{B}(-1, 1)$, but we have a few general comments to make.

If A is an $n \times m$ matrix whose columns all belong to $\mathcal{B}(a, b)$, and B is any $m \times k$ matrix $(n, m, k \geq 2)$, then the columns of AB also belong to $\mathcal{B}(a, b)$. This follows from that fact that $\mathcal{B}(a, b)$ is a vector space. Dually, if the rows of A belong to $\mathcal{B}(a, b)$ and B is $k \times n$, then the rows of BA also belong to $\mathcal{B}(a, b)$. We will say that two matrices of the same size, A and B, have the same recursion pattern if the columns of both matrices lie in $\mathcal{B}(a, b)$ and the rows of both matrices all lie in $\mathcal{B}(c, d)$, for some a, b, c, d. If A and B have the same recursion pattern, and if

$$\begin{pmatrix} a_{ij} & a_{i(j+1)} \\ a_{(i+1)j} & a_{(i+1)(j+1)} \end{pmatrix} = \alpha \begin{pmatrix} b_{ij} & b_{i(j+1)} \\ b_{(i+1)j} & b_{(i+1)(j+1)} \end{pmatrix}$$

for some 2×2 sub-matrix of A and B, then $A = \alpha B$. This follows from the fact that the recursion formulas will generate the rest of the matrices, once the values of such a 2×2 sub-matrix with adjacent entries are known. We record this observation for reference.

Lemma 2.4. *If A and B enjoy the same recursion pattern and*

$$\begin{pmatrix} a_{ij} & a_{i(j+1)} \\ a_{(i+1)j} & a_{(i+1)(j+1)} \end{pmatrix} = \alpha \begin{pmatrix} b_{ij} & b_{i(j+1)} \\ b_{(i+1)j} & b_{(i+1)(j+1)} \end{pmatrix}$$

for some 2×2 sub-matrix of A and B, then $A = \alpha B$.

In [**12**], the author works with the Fibonacci and Lucas sequences in $\mathcal{B}(k, 1)$, where many of the scalar identities look quite similar to the ones seen in $\mathcal{B}(1, 1)$ (see [**9**]). As it happens, the k-Fibonacci and k-Lucas matrices, obtained by generating the corresponding Toeplitz matrices from the Fibonacci and Lucas sequences of $\mathcal{B}(k, 1)$, enjoy the same fundamental property mentioned above: all, except the Lucas matrices of odd dimension, have both equal non-zero singular values. Thus, the spectral norms of these matrices can be computed as in Lemma 2.2. The generality does not enhance the exposition, however, which is on the verge of becoming burdensome enough, with an onslaught of scalar identities. We choose now to return to $\mathcal{B}(1, 1)$ and $\mathcal{B}(-1, 1)$, where we remain until the end of the paper.

3. Scalar Fibonacci identities. In the process of obtaining matricial Fibonacci identities, dozens of scalar identities are generated. Many of them are well known, but the literature is so vast, and the quantity of identities so large, finding and referencing a particular one is like finding a needle in a haystack. On the other hand, once an identity is written down, it is almost always quite easy to prove, using induction. We intend to list the identities we intend to use, and leave all but one of the elementary induction exercises to the reader.

Our list begins with standard identities that can be found in [**15**] and quickly progress to the ones we need to compute matrix products. A proof of identity (4) can be found in [**12**]. Often the identities bounce between a form for even n and a second expression for odd n, and for this reason it is useful to incorporate the characteristic function of the even integers, which we will denote with δ, so $\delta_n = 1$ if n is even, and otherwise $\delta_n = 0$. We also include some simple identities that we found useful for proving more complicated ones to follow.

(1) $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$;
(2) $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$;
(3) $f_n f_{n-3} - f_{n-2} f_{n-1} = (-1)^n$;

(4) $\sum_{i=1}^{n-1} f_i f_{i+1} = f_n^2 - \delta_{n+1}$;
(5) $f_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $l_n = \alpha^n + \beta^n$ ($\alpha > \beta$ the roots of $x^2 - x - 1$);
(6) $l_i^2 + l_{i+1}^2 = 5(f_i^2 + f_{i+1}^2)$;
(7) $l_i^2 = 5f_i^2 + 4(-1)^i$;
(8) $\sum_{i=0}^{n-1} l_i^2 = 5\sum_{i=0}^{n-1} f_i^2 + 4\delta_{n-1}$;
(9) $\sum_{i=0}^{n-1} l_{i-1}l_i = 5f_n f_{n-2} - 7\delta_{n-1}$;
(10) $\sum_{i=0}^{n-1} f_{-i} f_{n-i-1} = \sum_{i=1}^{n-2}(-1)^{i+1} f_i f_{n-i-1} = \delta_{n-1} f_{n-1}$ (equals f_{n-1} for odd n and it equals 0 for even n);
(11) $\sum_{i=0}^{n-2} f_{-i} f_{n-i-3} = \delta_{n-1} f_{n-1} - \delta_n f_{n-2}$ (equals f_{n-1} for odd n and it equals $-f_{n-2}$ for even n);
(12) for $n > 2$, $\sum_{i=0}^{n-2}(-1)^i f_i f_{n-i-2} = -f_{n-2}\delta_n$ (equals 0 for odd $n > 2$);
(13) $\sum_{i=1}^{n} f_{-i} f_{n-i} = f_n \delta_n$ (equals 0 for odd n);
(14) $\sum_{i=1}^{n} f_{-i} f_{n-i-1} = f_{n+1}\delta_{n+1} - f_n\delta_n$ (equals f_{n+1} for odd n and $-f_n$ for even n);
(15) $\sum_{i=-1}^{n-2} f_{-i} f_{n-i-2} = f_{n-1}\delta_{n-1} + f_n\delta_n$ (equals f_{n-1} for odd n and f_n for even n);
(16) $\sum_{i=0}^{n-1} f_{-i} f_{n-i-2} = f_n\delta_n - f_{n-1}\delta_{n-1}$ (equals $-f_{n-1}$ for odd n and f_n for even n);
(17) $\sum_{i=-k}^{k} f_i l_i = 0$;
(18) $\sum_{i=-k}^{k} f_i l_{i+1} = 5f_k f_{k+1}$;
(19) $\sum_{i=-k}^{k} l_i^2 = 2\sum_{i=-k}^{k} l_{i+1} l_i$;
(20) $\sum_{i=-k}^{k} l_{i+1} l_i = l_k l_{k+1}$;
(21) $l_k l_{k+1} - 5f_k f_{k+1} = 2(-1)^k$;
(22) $\sum_{i=0}^{n-1} l_{-i} f_{n-1-i} = 2f_n\delta_n + f_{n-1}\delta_{n-1}$;
(23) $\sum_{i=0}^{n-2} l_{-i} f_{n-3-i} + f_{n-2} = 2f_n\delta_n - f_{n-3}\delta_{n-1}$;
(24) $f_{n-1} + \sum_{i=0}^{n-2} l_{-i} f_{n-2-i} = f_n\delta_n + 3f_{n-1}\delta_{n-1}$;
(25) $-\sum_{i=0}^{n-1} l_{-i} f_{n-2-i} = f_n\delta_n - f_{n+2}\delta_{n-1}$.

Proof of Identity (10). The first equality follows from $f_{-i} = (-1)^{i+1} f_i$. When $n = 3, 4,$ and 5 the second equality reads $f_1 = f_2$, $f_1 f_2 - f_2 f_1 = 0$, and

$$f_1 f_3 - f_2^2 + f_3 f_1 = f_4,$$

all true statements. Assume that

$$\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1} = f_{n-1}$$

for some odd n and

$$\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i} = 0.$$

Adding the two equations, we obtain

$$\begin{aligned}\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1} + \sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i} &= \sum_{i=1}^{n-2}(-1)^{i+1}f_i(f_{n-i-1}+f_{n-i}) - f_{n-1}\\ &= \sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1} - f_{n-1}\\ &= f_{n-1}.\end{aligned}$$

The last equality implies $\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1} = 2f_{n-1}$; hence,

$$\sum_{i=1}^{n}(-1)^{i+1}f_if_{n-i+1} = 2f_{n-1} - f_{n-1}f_2 + f_nf_1 = f_{n+1}.$$

Next, assume that

$$\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1} = 0$$

with n even, and

$$\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i} = f_n.$$

Adding the equations gives us

$$\begin{aligned}\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1}+\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i}&\\ &=\sum_{i=1}^{n-2}(-1)^{i+1}f_i(f_{n-i-1}+f_{n-i})+f_{n-1}\\ &=\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}+f_{n-1}\\ &=f_n.\end{aligned}$$

This time we have $\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}=f_n-f_{n-1}$, and

$$\sum_{i=1}^{n}(-1)^{i+1}f_if_{n-i+1}=(f_n-f_{n-1})+f_{n-1}f_2-f_nf_1=0.$$

4. Even dimensions. In all that follows, we let U denote the unitary in Lemma 2.1.

Theorem 4.1. *Let F denote the $n\times n$ Fibonacci matrix with n even. Then*

$$FUF=f_nF.$$

Proof. We compute (using identities (1)–(4)) the 2×2 lower left corner of the matrix $F(UF)$, understanding that the matrix of UF is obtained from the matrix of F by reversing the order of its rows, obtaining

$$\begin{aligned}\begin{pmatrix}\displaystyle\sum_{i=0}^{n-2}f_if_{i+1} & \displaystyle\sum_{i=0}^{n-2}f_i^2+1\\ \displaystyle\sum_{i=0}^{n-1}f_i^2 & \displaystyle\sum_{i=0}^{n-2}f_if_{i+1}\end{pmatrix}&=\begin{pmatrix}f_{n-1}^2-1 & f_{n-2}f_{n-1}+1\\ f_{n-1}f_n & f_{n-1}^2-1\end{pmatrix}\\ &=f_n\begin{pmatrix}f_{n-2} & f_{n-3}\\ f_{n-1} & f_{n-2}\end{pmatrix}\end{aligned}$$

Since FUF has the same recursion pattern as F, we obtain the conclusion from Lemma 2.4. $\square$

Theorem 4.2. *Let F denote the $n \times n$ Fibonacci matrix with n even. Then*
$$(F^*F)^2 = f_n^2(F^*F).$$

Proof. Assume the hypothesis of the theorem. Since $UA^*AU = AA^*$ is true whenever A is Toeplitz, and $F^*UF^* = f_nF^*$ follows from the previous theorem, we have
$$(F^*F)^2 = F^*(FF^*)F = (F^*UF^*)(FUF) = f_n^2F^*F. \qquad \square$$

Theorem 4.3. *Let F denote the $n \times n$ Fibonacci matrix with n even. Then*
$$\|F\| = \frac{1}{\sqrt{2}}\|F\|_{\mathfrak{F}} = \sqrt{\sum_{i=0}^{n-1} f_i f_{i+1}} = f_n.$$

Proof. It follows from Theorem 4.2 that F^*F is a non-zero multiple of a rank two orthogonal projection, so the two non-zero singular values of F are equal. We then draw upon Lemma 2.2 to justify the first equality of our conclusion, equate that to the Hilbert norm of the lower triangular part of F to justify the second equality, and point out that the third equality is identity (4) in Section 3. $\square$

Theorem 4.4. *Let F denote the $n \times n$ Fibonacci matrix, L the $n \times n$ Lucas matrix, and assume that n is even. Then*
$$LUL = 5f_nF \text{ and } L^*UL^* = 5f_nF^*.$$

Proof. We compute (using identities (3) and (7)–(9)) the 2×2 lower left corner of the matrix $L(UL)$, obtaining:
$$\begin{pmatrix} \sum_{i=0}^{n-1} l_{i-1}l_i & \sum_{i=0}^{n-2} l_i^2 + l_{-1}^2 \\ \sum_{i=0}^{n-1} l_i^2 & \sum_{i=0}^{n-1} l_{i-1}l_i \end{pmatrix} = \begin{pmatrix} 5f_nf_{n-2} & 5\sum_{i=0}^{n-2} f_i^2 + 4 + 1 \\ 5\sum_{i=0}^{n-1} f_i^2 & 5f_nf_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} 5f_nf_{n-2} & 5\Big(\sum_{i=0}^{n-2} f_i^2 + 1\Big) \\ 5f_nf_{n-1} & 5f_nf_{n-2} \end{pmatrix}$$
$$= 5f_n \begin{pmatrix} f_{n-2} & f_{n-3} \\ f_{n-1} & f_{n-2} \end{pmatrix}.$$

Since LUL has the same recursion pattern as F, Lemma 2.4 applies and $LUL = 5f_nF$. The second equality is obtained by taking adjoints. □

Theorem 4.5. *Let F denote the $n \times n$ Fibonacci matrix, L the $n \times n$ Lucas matrix, and assume that n is even. Then*

$$L^*L = 5F^*F.$$

Proof. By Lemma 2.2, Theorem 4.2 and Theorem 4.4, we have

$$(L^*L)^2 = L^*(LL^*)L = L^*UL^*LUL = 25f_n^2F^*F = 25(F^*F)^2.$$

Taking square roots of both sides finishes the proof. □

Corollary 4.6. *If L is the $n \times n$ Lucas matrix with n even, then*

$$\|L\| = \sqrt{5}\|F\| = \sqrt{5}f_n.$$

5. Odd dimensions. It is not true that $FUF = f_nF$ when n is odd. When n is even, we also have the identity

$$FUFUF = f_nFUF = f_n^2F,$$

and, as it happens, this identity almost holds for odd n too.

Theorem 5.1. *Assume F is a Fibonacci matrix. Then F satisfies*

$$FUFUF = \alpha F.$$

If F is $n \times n$ with n even, then $\alpha = f_n^2$, otherwise $\alpha = f_n^2 - 1$.

Proof. When n is even, the statement follows immediately from Theorem 4.1, so assume that n is odd. We compute the 2×2 northwest corner of $F(UF)$, obtaining

$$\begin{pmatrix} \displaystyle\sum_{i=0}^{n-1} f_{-i}f_{n-i-1} & \displaystyle\sum_{i=0}^{n-1} f_{-i}f_{n-i-2} \\ f_{n-1} + \displaystyle\sum_{i=0}^{n-2} f_{-i}f_{n-i-2} & \displaystyle\sum_{i=0}^{n-2} f_{-i}f_{n-i-3} + f_{n-2} \end{pmatrix},$$

which, using identities (10)–(12), equals

$$\begin{pmatrix} f_{n-1} & -f_{n-1} \\ f_{n-1} & f_n \end{pmatrix}.$$

Next, we use the recursion pattern to determine the entire first two rows. Note that the rows are elements of $\mathcal{B}(-1, 1)$, so knowing r_{i-2} and r_{i-1}, we obtain r_i as $r_i = r_{i-2} - r_{i-1}$. Thus, the first row becomes

$$f_{n-1}(f_{-1}, f_{-2}, f_{-3}, \ldots, f_{-n}),$$

and the second row equals

$$f_n(f_0, f_{-1}, \ldots, f_{-n+1}) + f_{n-1}(f_1, f_0, f_{-1}, \ldots, f_{-n+2}).$$

The last step is to use these two rows to compute the 2×2 northwest corner of $(FUF)(UF)$: the first row comes as

$$\begin{pmatrix} f_{n-1}\sum_{i=1}^{n} f_{-i}f_{n-i} & f_{n-1}\sum_{i=1}^{n} f_{-i}f_{n-i-1} \end{pmatrix},$$

which, using identities (2), (13) and (14) equals

$$\begin{pmatrix} 0 & f_n^2 - 1 \end{pmatrix}.$$

The $(2, 1)$ entry is

$$f_n \sum_{i=0}^{n-1} f_{-i}f_{n-i-1} + f_{n-1}\left(\sum_{i=-1}^{n-2} f_{-i}f_{n-i-2}\right)$$

which, using identities (14) and (15), equals

$$f_nf_{n-1} + f_{n-1}^2 = f_{n+1}f_{n-1} = f_n^2 - 1.$$

Finally, the $(2, 2)$ entry is

$$f_n \sum_{i=0}^{n-1} f_{-i}f_{n-i-2} + f_{n-1} \sum_{i=-1}^{n-2} f_{-i}f_{n-i-3}$$

which, using identities (13) and (16), equals

$$f_n(-f_n) + f_{n-1}(f_{n-1} + f_{n-2}) = f_{n-1}(-f_n + f_{n-1} + f_{n-2}) = 0.$$

It follows that this 2×2 corner is $f_n^2 - 1$ times the corresponding 2×2 of F, and by Lemma 2.4, the theorem follows. □

Theorem 5.2. *Assume F is an $n \times n$ Fibonacci matrix. Then we have*

$$(F^*F)^3 = \beta F^*F.$$

If n is even, then $\beta = f_n^4$, otherwise $\beta = (f_n^2 - 1)^2$.

Proof. Using the facts that $UFU = F^*$ and $UF^*U = F$, we have

$$(F^*F)^3 = F^*(F)F^*F(F^*)F = F^*UF^*UF^*FUFUF = \alpha^2 F^*F. \quad \square$$

Theorem 5.3. *Assume F is a Fibonacci matrix. Then*

$$\|F\| = \frac{1}{\sqrt{2}}\|F\|_{\mathfrak{F}} = \sqrt{\sum_{i=0}^{n-1} f_i f_{i+1}}.$$

In the case where F is $n \times n$ with n odd, this expression simplifies to

$$\|F\| = \sqrt{f_n^2 - 1}.$$

Proof. Letting α_1 and α_2 denote the two positive eigenvalues of F^*F, we have by Theorem 5.2 that

$$\frac{\alpha_1^3}{\alpha_1} = \beta = \frac{\alpha_2^3}{\alpha_2},$$

and hence $\alpha_1 = \alpha_2$, and Lemma 2.2 applies again. □

We now turn to the Lucas matrices of odd dimension, and here we must abandon our strategy of proving the two non-zero singular values equal, and then appealing to Lemma 2.4 because the sobering fact is these singular values are not equal.

Theorem 5.4. *Assume L is the $n \times n$ Lucas matrix, $n = 2k + 1$. The two non-zero singular values of L are $5f_k f_{k+1}$ and $l_k l_{k+1}$. We have $l_k l_{k+1} \geq 5f_k f_{k+1}$ if and only if k is even, from which we conclude $\|L\| = l_k l_{k+1}$ for even k, and $\|L\| = 5f_k f_{k+1}$ when k is odd.*

Proof. Assume the hypothesis of the theorem. If s is a bilateral sequence, let v_s be the element of $\mathbf{C}^n$ defined by

$$v_s = (s_k, s_{k-1}, \dots, s_1, s_0, s_{-1}, \dots, s_{-k+1}, s_{-k})^*.$$

Let $H = UL$, so $\|H\| = \|L\|$, and H, being a real Hankel matrix, is symmetric. We will prove that v_f and v_l are two eigenvectors corresponding to the two distinct positive eigenvalues of H. Computing the middle $(k+1)$ coordinate of Hv_f we have

$$\sum_{i=-k}^{k} l_i f_i,$$

which equals 0 by identity (17). We use identity (18) to find that coordinate k of Hv_f is $5f_k f_{k+1}$. Since the vector Hv_f is an element of $\mathcal{B}(-1,1)$, the recursion rule determines the remaining coordinates and we see that

$$Hv_f = 5f_k f_{k+1} v_f.$$

In exactly the same manner, we see from identity (19) that the $k+1$ coordinate of Hv_l is twice the k- coordinate, and identity (20) says the k coordinate of Hv_l is $l_k l_{k+1}$. With these two coordinates established, and equal to $l_k l_{k+1}$ times the corresponding coordinates of v_l, the recursion rule then implies $Hv_l = l_k l_{k+1} v_l$. Concerning when $l_k l_{k+1} \geq 5f_k f_{k+1}$, this is the content of identity (21). □

When the two non-zero singular values are equal, the general Schatten p-norm is again some constant multiple of the Frobenous norm, but with n odd, our Lucas matrix, with its distinct non-zero singular values, has more interesting Schatten norms.

Corollary 5.5. *Let $n = 2k+1$ be an odd integer, and let L denote our $n \times n$ Lucas matrix. Then for all $1 \leq p < \infty$ we have*

$$\|L\|_p = (l_k^p l_{k+1}^p + 5^p f_k^p f_{k+1}^p)^{1/p}.$$

6. The Fibonacci algebra. When n is even, the matricial identities obtained in Theorems 4.1 and 4.4 hint that, when a multiplication is defined appropriately, there is an algebra structure on the set of Toeplitz matrices generated by elements of $\mathcal{B}(1,1)$ in which F is the identity and $L^2 = 5F$. This is indeed the case, which we will be able to exhibit after producing one more matrix identity.

Theorem 6.1. *Assume n is even and F and L denote the $n \times n$ Fibonacci and Lucas matrices. We have*

$$LUF = FUL = f_n L.$$

Proof. Calculating the upper left 2×2 corner of $L(UF)$, we see the equality of $LUF = f_n L$ from identities 22 and 24. These same identities can be used to compute the lower right 2×2 corner of $F(UL)$. $\square$

We are now able to assemble all of our matricial identities for even n into one statement. Given two $n \times n$ matrices A and B, define a multiplication $\circ$ by

$$A \circ B \equiv \frac{1}{f_n} AUB.$$

Let $\mathcal{A}$ denote the two-dimensional linear span of F and L. If, instead of usual matrix multiplication, we endow upon $\mathcal{A}$ the newly defined multiplication $\circ$, then we have that

$$F \circ (aF + bL) = (aF + bL) \circ F = (aF + bL),$$

for all $a, b \in \mathbf{C}$, and

$$(aF + bL) \circ (cF + dL) = (ac + 5bd)F + (ad + bc)L,$$

which proves:

Theorem 6.2. *The space $\mathcal{A}$ with the multiplication $\circ$ is a commutative algebra with involutive element $(1/\sqrt{5})L$ and unit F, isomorphic to the set of all complex matrices of the form*

$$\begin{pmatrix} a & 5b \\ b & a \end{pmatrix}.$$

7. Notes and remarks. Recently, the spectral norms for Hankel Fibonacci and Lucas matrices were obtained in [**1**]. Their proof can be used to infer that, unlike F and L, there are elements of $\mathcal{A}$ whose non-zero singular values are not equal for any n. For example, their method reveals that the matrix

$$H = (f_{i+j})_{i,j=0}^{n-1}$$

has two distinct eigenvalues for all n, from which we see that

$$HU = \left(f_n - \frac{f_{n-1}}{2}\right)F + \frac{f_{n-1}}{2}L$$

has distinct non-zero singular values for every n. Working in the other direction, we find Hankel matrices whose singular norms equal those of F and L by considering UF, FU, LU and UL.

In [**14**], the author obtains bounds for the spectral norms of circulant matrices constructed from the Fibonacci and Lucas sequences. Today, the exact values of the spectral norms are a consequence of a nice theorem of Bani-Domi and Kittaneh [**2**].

Theorem 7.1 ([**2**]). *If $A_1, A_2, \ldots, A_n$ are $n \times n$ matrices and M is the $n^2 \times n^2$ circulant matrix built with these operators, then*

$$\|M\| = \max_{t=0}^{n-1} \left\| \sum_{j=1}^{n} \omega^{k(1-j)} A_j \right\|,$$

with ω a primitive nth root of unity.

Corollary 7.2. *If C is an $n \times n$ circulant matrix built upon the non-negative real numbers $a_1, \ldots a_n$, then*

$$\|C\| = \sum_{i=1}^{n} a_i.$$

Corollary 7.3. *If C_f and C_l are the $n \times n$ circulant matrices built upon the Fibonacci and Lucas sequences, then*

$$\|C_f\| = \sum_{i=0}^{n-1} f_i = f_{n+1} - 1 \quad \text{and} \quad \|C_l\| = \sum_{i=0}^{n-1} l_i = l_{n+1} - 1.$$

In [**13**], the authors obtain loose bounds for spectral norms of r-circulant matrices built with Fibonacci and Lucas numbers. Tight bounds are likely to involve the growth of the "triangular truncation" operator [**7**].

ENDNOTES

1. In our version of **[3]** there is a typo: the square root seems to be missing.

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