ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2014

SEARCHING FOR CUTKOSKY'S EXAMPLE

FRANCESCA DI GIOVANNANTONIO, ANNA GUERRIERI AND IRENA SWANSON

ABSTRACT. We provide a concrete class of rings in which there exists a primary ideal with respect to the maximal ideal that has only one Rees valuation.

1. Motivation. In this work all rings are commutative with identity, and most are Noetherian domains. If $I \subseteq R$ is an ideal, $\overline{I}$ denotes the integral closure of I, namely,

$$\overline{I} = \{r \in R : r^n + a_1 r^{n-1} + \cdots + a_n = 0 \quad \text{for some } a_j \in I^j\}.$$

It is well known that $\overline{I} = \bigcap_V IV \cap R$, where V varies over all valuation rings between R/P and $Q(R/P)$, and P varies over the minimal primes. If R is Noetherian, there exist finitely many valuation rings that determine not just the integral closure of I but also the integral closure of all its powers. A minimal set of these valuation rings is called the set of *Rees valuation rings* of I. We give the necessary background on Rees valuations in the next section.

In **[2]**, Cutkosky proved the existence of a two-dimensional complete integrally closed local domain (R, m) in which every m-primary ideal has more than one Rees valuation. However, no explicit example of such a ring has been found. Our work narrows the classes of rings in which Cutkosky's example can be found. In Section 3, we prove that if R is a power series ring modulo a quasi-homogeneous prime ideal, there always exists in R an m-primary ideal with only one Rees valuation (see Theorem 3.3). In Section 4, we identify an additional concrete class of polynomial and power series rings in which there exists a zero-dimensional primary ideal that has only one Rees valuation.

Received by the editors on October 11, 2011, and in revised form on March 4, 2012.
DOI:10.1216/RMJ-2014-44-3-865

2. Background.

Definition 2.1. ([**9**]). The Rees valuation rings of I are valuation rings $V_1, \ldots, V_s$ such that:

1. Each V_i is Noetherian and is not a field;
2. For each $i = 1, \ldots, s$, there exists a minimal prime P_i in R such that V_i is a ring between R/P_i and $Q(R/P_i)$;
3. $\overline{I^n} = \cap_{i=1}^s (I^n V_i) \cap R$, for all $n \in \mathbf{N}$;
4. The set $\{V_1, \ldots, V_s\}$ satisfying the above conditions is minimal.

Let $\mathcal{RV}(I)$ denote the set of Rees valuation rings of I. In [**9**], Rees proved the existence and uniqueness of Rees valuations for any ideal in a Noetherian ring R. If R is a domain, there are different ways of constructing the Rees valuations (see [**4,** Chapter 10] for more details), we use the following.

Let $\mathcal{U}$ be a generating set of I. Assume $0 \notin \mathcal{U}$. For each $a \in \mathcal{U}$, we find all prime ideals $K_{a_1}, \ldots, K_{a_l}$ of $\overline{R\left[\frac{I}{a}\right]}$ that are minimal over $a\overline{R\left[\frac{I}{a}\right]}$. Then

$$\mathcal{RV}(I) = \bigcup_{a \in \mathcal{U}} \left\{ \overline{R\left[\frac{I}{a}\right]}_{K_{a_i}} : i = 1, \ldots, l \right\}.$$

Observation 2.2. Let I be an ideal. By the construction above, it is straightforward to prove that, for all positive integers n,

$$\mathcal{RV}(\overline{I}) = \mathcal{RV}(I) = \mathcal{RV}(I^n),$$

and that if R' is an integral extension of R with the same field of fractions, then

$$\mathcal{RV}(I) = \mathcal{RV}(IR').$$

Consequently, if $I = (x_1, \ldots, x_n)$ is a finitely generated ideal, for any positive integer m, one has $\mathcal{RV}(I) = \mathcal{RV}((x_1, \ldots, x_n)^m) = \mathcal{RV}((x_1{}^m, \ldots, x_n{}^m))$. This holds because

$$\mathcal{RV}(\overline{(x_1, \ldots, x_n)^m}) = \mathcal{RV}(\overline{(x_1{}^m, \ldots, x_n{}^m)}),$$

since

$$\overline{(x_1, \ldots, x_n)^m} = \overline{(x_1{}^m, \ldots, x_n{}^m)}.$$

A ring R is called *equidimensional* if $\dim R = \dim R/P$ for all minimal primes P in R. A Noetherian local ring is a *formally equidimensional ring* (or alternately, *quasi-unmixed ring*) if its completion in the topology defined by the maximal ideal is equidimensional.

A fundamental tool in streamlining the construction of Rees valuations is the following theorem of Sally **[10]**.

Theorem 2.3. *Let (R, m) be a Noetherian formally equidimensional local domain of dimension $d > 0$, and let $I = (a_1, \ldots, a_d)$ be an m-primary ideal generated by a system of parameters. Then, for every Rees valuation ring V of I and every $i = 1, \ldots, d$, V is the localization of $\overline{R\left[\frac{I}{a_i}\right]}$ at a suitable height one prime ideal minimal over a_i.*

A consequence of this is that, given $V \in \mathcal{RV}(I)$, where I is a parameter ideal, one may take any minimal generator a of I to obtain V as a localization of the integral closure of $R\left[\frac{I}{a}\right]$ at a prime ideal minimal over a. More generally, we get the following useful corollary:

Corollary 2.4. *Let (R, m) be a Noetherian formally equidimensional local domain of dimension $d > 0$, let I be an m-primary ideal and let J be a parameter ideal that is a reduction of I. Then, for every Rees valuation ring V of I and every minimal generator a of J, V is the localization of $\overline{R\left[\frac{I}{a}\right]}$ at a suitable height one prime ideal minimal over a.*

Proof. As in Observation 2.2, $\mathcal{RV}(I) = \mathcal{RV}(\overline{I}) = \mathcal{RV}(\overline{J}) = \mathcal{RV}(J)$. Also, $\overline{R\left[\frac{I}{a}\right]} = \overline{R\left[\frac{J}{a}\right]}$. Thus, the conclusion follows from Theorem 2.3. $\square$

Observation 2.5. Let (R, m) be a Noetherian local ring, and let X be a variable over R. Let $R(X)$ denote $R[X]_{mR[X]}$. Then $R \subseteq R(X)$ is a faithfully flat extension of Noetherian local rings. The residue field of $R(X)$ is

$$\frac{R(X)}{mR(X)} \cong \frac{R[X]}{mR[X]}_{mR[X]},$$

which is the field of fractions of $(R/m)[X]$, therefore infinite. By **[1,** Proposition 5.13], if R is an integrally closed domain so is $R(X)$.

Moreover, for any ideal I of R, if $IR(X)$ has only one Rees valuation in $R(X)$ so I has only one Rees valuation in R. This follows in a straightforward way from [**4,** Proposition 1.6.2].

Observation 2.6. Let R be a homomorphic image of a finitely generated polynomial or power series ring over a field K. Let $\overline{K}$ be the algebraic closure of K. Let S denote the faithfully flat R-algebra $R \otimes_K \overline{K}$. By [**4,** Proposition 1.6.2], for any ideal I of R, $\overline{IS} \cap R = \overline{I}$. This implies that $\mathcal{RV}(I) \subseteq \{V \cap Q(R) : V \in \mathcal{RV}(IS)\}$, where $Q(R)$ is the field of fraction of R. In particular, if IS has only one Rees valuation, so does I.

In the constructions we will need to determine the integral closure of various rings. We recall here the main methods by which one decides if a ring is an integrally closed domain.

Observation 2.7. Let K be a field, R an equidimensional finitely generated K-algebra, and P a prime ideal in R. Then, if $J_{R/K}$, the Jacobian ideal of R over K, is not contained in P, it follows that R_P is a regular ring. Conversely, if R_P is a regular ring and $Q(R/P)$ is separable over K (say if K is a perfect field), then $J_{R/K}$ is not contained in P. This statement is called the *Jacobian criterion* (see [**4,** Theorem 4.4.9]).

The following is a computationally useful consequence:

Proposition 2.8. ([**6,** Theorem 23.8] *or* [**4,** Corollary 4.5.8]). *If R is a finitely generated equidimensional reduced algebra over a perfect field K, then R_P is integrally closed for all prime ideals P of R if and only if the Jacobian ideal $J_{R/K}$ is R or it contains a regular sequence of length* 2.

3. Quasi-homogeneous complete local rings.

Let K be a field, let $X_1, \ldots, X_n$ be variables over K, let P be a prime ideal in $K[[X_1, \ldots, X_n]]$, and let $R = K[[X_1, \ldots, X_n]]/P$. In this section we prove that there exists a particular set of prime ideals $P \subseteq K[[X_1, \ldots, X_n]]$ such that R has an $(X_1, \ldots, X_n)$-primary ideal with only one Rees valuation.

First, we need some definitions.

Definition 3.1. A polynomial $f \in K[X_1, \ldots, X_n]$ is *quasi-homogeneous* if it is possible to assign positive degrees to the variables to make f homogeneous. An ideal $I \subseteq K[X_1, \ldots, X_n]$ is a *quasi-homogeneous ideal* if it is possible to assign positive degrees to the variables to make all elements of some generating set of I homogeneous.

Example 3.2. Let a, b, c be nonnegative integers. The polynomial $f = X^a + Y^b + Z^c \in K[X, Y, Z]$ is quasi-homogeneous.

Now we show that, when P is a quasi-homogeneous ideal, and thus also when P is a homogeneous ideal, there exists in $R = K[[X_1, \ldots, X_n]]/P$ an $(X_1, \ldots, X_n)$-primary ideal with only one Rees valuation.

Theorem 3.3. *Let R be a domain which is a homomorphic image of a power series ring $K[[X_1, \ldots, X_n]]$ in variables $X_1, \ldots, X_n$ over a field K. If there exists a positive integer grading $\deg X_i = a_i$ that makes R homogeneous, then $(X_1^{a_2 a_3 \cdots a_n}, X_2^{a_1 a_3 a_4 \cdots a_n}, \ldots, X_n^{a_1 a_2 \cdots a_{n-1}})R$ is primary to the unique maximal ideal of R and has only one Rees valuation.*

Proof. Write $R = K[[X_1, \ldots, X_n]]/P$, where $P = (f_1, \ldots, f_l)$. Let x_i denote the image of X_i in R. Thus, $R = K[[x_1, \ldots, x_n]]$. We denote by F the quotient field of R and by $\overline{F}$ the algebraic closure of F. For $i = 1, \ldots, n$, let $y_i \in \overline{F}$ be such that $y_i^{a_i} = x_i$. Then $R \subseteq S = K[[y_1, \ldots, y_n]]$ is an integral extension contained in $\overline{F}$. Note that S is a homomorphic image of a power series ring over a field K that is generated over the field by homogeneous elements y_i, each of which has degree 1 in $\overline{F}$. By Observation 2.2, if $e = a_1 a_2 \cdots a_n$, then

$$\begin{aligned}
\mathcal{RV}((y_1, \ldots, y_n)) &= \mathcal{RV}(({y_1}^e, \ldots, {y_n}^e)) \\
&= \mathcal{RV}((x_1^{a_2 a_3 \cdots a_n}, \ldots, {x_n}^{a_1 a_2 \cdots a_{n-1}})S) \\
&= \mathcal{RV}((x_1^{a_2 a_3 \cdots a_n}, \ldots, {x_n}^{a_1 a_2 \cdots a_{n-1}})R).
\end{aligned}$$

Thus, it suffices to prove the theorem for S, i.e., in the case where all a_i equal 1.

As noted in Observation 2.5, one may pass to a faithfully flat extension $R'(t)$ (with the degree of t being 0) to assume that K is infinite. In that case, after a linear change of variables, we may assume that x_1 is part of a minimal generating set of a parameter ideal that is a reduction of $(x_1,\dots,x_n)$. The rest of this paragraph is a proof of this fact. Namely, by Northcott and Rees [**7**], there exist $z_1,\dots,z_d$ of R that generate a minimal reduction of $(x_1,\dots,x_n)$. Clearly, $z_i=\sum_{j=1}^n a_{ij}x_j$ where $a_{ij}\in R$. Write $a_{ij}=a_{ij0}+\sum_h a_{ijh}x_h$ for some $a_{ij0}\in K$ and $a_{ijh}\in(x_1,\dots,x_n)$. Let $u_i=\sum_{j=1}^n a_{ij0}x_j$. Then, $z_i-u_i=\sum_{j=1}^n(a_{ij0}+\sum_h a_{ijh}x_h)x_j-\sum_{j=1}^n a_{ij0}x_j=\sum_{j=1}^n(\sum_h a_{ijh}x_h)x_j\in(x_1,\dots,x_n)^2$, that is, $z_i\equiv u_i$ modulo $(x_1,\dots,x_n)^2$. By [**4,** Lemma 8.1.8], $(u_1,\dots,u_d)$ is a reduction of $(x_1,\dots,x_n)$. Thus, by possibly replacing the z_i with the u_i, we may assume that all $a_{ij}\in K$. Then all z_i are homogeneous of degree 1. By possibly relabeling and renaming the variables, we may assume that $x_1=z_1$.

By [**8,** Theorem 6.23] or [**5,** Theorem 5.3], it suffices to prove that the homomorphic image $R'=K[x_1,\dots,x_n]\subseteq R$ of the polynomial ring $K[X_1,\dots,X_n]$ has the property that the maximal ideal $(x_1,\dots,x_n)R'$ has only one Rees valuation. By Corollary 2.4, it suffices to prove that there is only one prime ideal of height one in the integral closure of $S_1=R'[(x_1,\dots,x_n)/x_1]$ that contains x_1. Write $R'=K[X_1,\dots,X_n]/(f_1,\dots,f_l)$ for some homogeneous polynomials $f_1,\dots,f_l$. If $\alpha_2,\dots,\alpha_n$ stands for $X_2/X_1,\dots,X_n/X_1$, then

$$S_1\cong K[X_1,\alpha_2,\dots,\alpha_n]$$

and X_1 is a variable over $T=K[\alpha_2,\dots,\alpha_n]$. Some obvious relations on the α_i are $f_1(1,\alpha_2,\dots,\alpha_n),\dots,f_l(1,\alpha_2,\dots,\alpha_n)$. Certainly, T is a domain, so also its integral closure $\overline{T}$ is a domain. Note that $S_1\cong T[X_1]$ and that the integral closure of S_1 is thus up to isomorphism $\overline{S_1}=\overline{T}[X_1]$. But, under this isomorphism, X_1 stands for x_1, so that any prime ideals in $\overline{S_1}$ that are minimal over x_1 correspond to the minimal prime ideals in

$$\frac{\overline{S_1}}{x_1\overline{S_1}}\cong\frac{\overline{T}[X_1]}{X_1\overline{T}[X_1]}\cong\overline{T},$$

which is a domain. Thus, there is only one such prime ideal, so $(x_1,\dots,x_n)R'$ has only one Rees valuation. This finishes the proof of the theorem. $\square$

We give an explicit example.

Example 3.4. Let K be a field. Let $R = K[X, Y, Z]/(f) = K[x, y, z]$, where $f = XY + YZ + XZ$ is homogeneous and irreducible in $K[X, Y, Z]$. By Theorem 3.3, (x, y, z) has only one Rees valuation in R. Instead of going through the construction as in the proof of Theorem 3.3 and first finding a two-generated reduction of (x, y, z), we simply rely on the original general construction of Rees valuations: if we can find a localization of the integral closure of

$$S = R\left[\frac{(x, y, z)}{x}\right] \cong \frac{K[X, Y, Z, \alpha, \beta]}{(X\alpha - Y, X\beta - Z, \alpha + \alpha\beta + \beta)} \cong \frac{K[X, \alpha, \beta]}{(\alpha + \alpha\beta + \beta)}$$

at a prime ideal minimal over x (respectively, X), then that will be the desired valuation. First of all, S is integrally closed by Proposition 2.8: $J_{S/K} = (1 + \beta, 1 + \alpha)S = S$, and then X generates a prime ideal in the integral closure of S, so that $V = K[X, \alpha, \beta]/(\alpha + \alpha\beta + \beta)_{(X)}$.

It is not true that the maximal ideal in a quasi-homogeneous homomorphic image of a power series ring over a field always has only one Rees valuation. Here is an example:

Example 3.5. Let $f = X^3 + Y^3 + Z^4 \in \mathbf{R}[X, Y, Z]$. This is a quasi-homogeneous and irreducible polynomial. Let us denote $R = \mathbf{R}[X, Y, Z]/(f) = \mathbf{R}[x, y, z]$. By Theorem 3.3, (x^{12}, y^{12}, z^9) has only one Rees valuation. We prove that the maximal ideal (x, y, z) has more than one Rees valuation. Clearly, (x, z) is a minimal reduction of (x, y, z). By Theorem 2.3, all Rees valuations of (x, y, z) are localizations of the integral closure of $S = R[(x, y, z)/x]$ at prime ideals minimal over x. Note that

$$S \cong \frac{\mathbf{R}[X, Y, Z, \alpha, \beta]}{(X\alpha - Y, X\beta - Z, 1 + \alpha^3 + \beta^4 X)} \cong \frac{\mathbf{R}[X, \alpha, \beta]}{(1 + \alpha^3 + \beta^4 X)}.$$

We prove that S is integrally closed. The Jacobian ideal of S over $\mathbf{R}$ is $J_{S/\mathbf{R}} = (\beta^4, 3\alpha^2, 4\beta^3 X)$. Any prime ideal in S containing $J_{S/\mathbf{R}}$ contains α and β. But, this prime ideal also contains $1 + \alpha^3 + \beta^4 X$, so $1 \in J_{S/\mathbf{R}}$. By Proposition 2.8, S is integrally closed. But then the only prime ideals in the integral closure S of S that are minimal over xS are the isomorphic images of

$$P = (X, 1 + \alpha) \quad \text{and} \quad P = (X, 1 - \alpha + \alpha^2).$$

Thus, the maximal ideal (x, y, z) of the quasi-homogeneous R has more than one Rees valuation. (Note that with the base field $\mathbf{C}$ instead there would be three Rees valuations.)

4. Classes of non-quasi-homogeneous rings. The Cohen structure theorem (see [**3,** Theorem 7.7]) states that a two-dimensional complete Noetherian local integrally closed domain (R, m) can be written as a power series ring $K[[X_1, \ldots, X_n]]$ modulo a prime ideal P, where K is a field or a complete Noetherian valuation domain and $X_1, \ldots, X_n$ are variables over K.

If we consider the case with $n = 3$, since R is two-dimensional, P must be principal and, since R is integrally closed, by Proposition 2.8, the Jacobian ideal of P, $J_{R/K}$, is (X_1, X_2, X_3)-primary. We write $P = (f)$. Then f is an irreducible power series. Samuel [**11**] showed that, whenever $P \equiv I$ modulo $m(J_{R/K})^2$, there exists a formal isomorphism of $K[[X_1, X_2, X_3]]$ which takes P onto I. Then, $K[[X_1, X_2, X_3]]/(f) \cong K[[X_1, X_2, X_3]]/(g)$, where g is a polynomial such that $f - g \in m(J_{R/K})^2$. Moreover, if $K[X_1, X_2, X_3]/(g)$ has an (X_1, X_2, X_3)-primary ideal with only one Rees valuation, then also $K[[X_1, X_2, X_3]]/(g)$ and R have a (X_1, X_2, X_3)-primary ideal with only one Rees valuation (see [**8,** Theorem 6.23] or [**5,** Theorem 5.3]). Thus, to study Rees valuations of ideals of a two-dimensional complete Noetherian local integrally closed domain that is a hypersurface, it suffices to take $R = K[X_1, X_2, X_3]/(f)$, with f being an irreducible polynomial.

By Theorem 3.3, it remains to analyze the non-quasi-homogeneous f for two-dimensional complete hypersurface rings $K[[X_1, X_2, X_3]]/(f)$. Our goal is to find (X_1, X_2, X_3)-primary ideals in $K[X_1, X_2, X_3]/(f)$ with only one Rees valuation. Below we provide a class of concrete locally integrally closed rings for which this holds. We first use Observation 2.7 to prove the integral closure of our class of rings.

Proposition 4.1. *Let a, b, c, d, e be positive integers such that $(a - 1 - d)b \neq ae \neq (a - d)b$. Let K be an algebraically closed field of characteristic not dividing c and e, and let $R = K[X, Y, Z]/(f) = K[x, y, z]$, where $f = X^a + Y^b + Z^c + X^d Y^e + X^{d+1} Y^e$ is a non quasi-homogenous polynomial. Then R is integrally closed.*

Proof. The Jacobian ideal of R over K,

$$J_{R/K} = (aX^{a-1} + X^{d-1}(d+(d+1)X)Y^e, bY^{b-1} + eX^d(1+X)Y^{e-1}, cZ^{c-1}).$$

By Proposition 2.8, it suffices to prove that $J_{R/K}$ contains a regular sequence of length 2, or equivalently, that every prime ideal P of R that contains $J_{R/K}$ contains a regular sequence of length 2. Since $\partial f/\partial Z = cZ^{c-1} \in P$, and since $c \neq 0$ in K, necessarily $c > 1$ and P contains Z. If $Y \in P$, then $X^a = f - Y^b - Z^c - Y^e(X^d + X^{d+1}) \in P$, so $X \in P$, whence $P = (X, Y, Z)R$ contains a regular sequence of length 2, and we are done. So we may assume that $Y \notin P$. Similarly we may assume that $X \notin P$.

We know that P contains $f - Z^c = X^a + Y^b + X^dY^e(1+X)$ and $Y(\partial f/\partial Y) = bY^b + eX^dY^e(1+X)$, whence it contains $eX^a + (e-b)Y^b$. Since $X \notin P$ and e is non-zero, it follows that $e \neq b$.

Also, P contains $\partial f/\partial X = aX^{a-1} + X^{d-1}(d+(d+1)X)Y^e$, so that modulo P,

$$\begin{aligned}(aX^{a-1})^b &\equiv (-X^{d-1}(d+(d+1)X)Y^e)^b \\ &= (-X^{d-1}(d+(d+1)X))^bY^{eb} \\ &\equiv (-X^{d-1}(d+(d+1)X))^b\left(\frac{e}{b-e}\right)^e X^{ae}.\end{aligned}$$

Thus, P contains the polynomial

$$g = (aX^{a-1})^b - (-X^{d-1}(d+(d+1)X))^b\left(\frac{e}{b-e}\right)^e X^{ae}.$$

We claim that this is a non-zero polynomial. If a is a multiple of the characteristic of K, as either d or $d+1$ is not zero in K, it follows that $g = -(-(d+(d+1)X))^b(e/(b-e))^eX^{b(d-1)+ae}$ is non-zero. So we may assume that a is not a multiple of the characteristic of K. By assumption $(a-1-d)b \neq ae$, so that if $d+1$ is not a multiple of the characteristic of K, then either $a^bX^{(a-1)b}$ or $-(-(d+1)X^d)^b(e/(e-b))^eX^{ae}$ is the leading term of g, so that g is non-trivial. So we may assume that $d+1$ is a multiple of the characteristic. Then $g = a^bX^{(a-1)b} - (-dX^{d-1})^b(e/(e-b))^eX^{ae}$, and since $(a-d)b \neq ae$, again it follows that g is a non-trivial polynomial. So in all cases g is non-trivial in $K[X] \cap P$.

As K is algebraically closed, P must contain one of the factors $X-r$ for some $r \in K$. But then P contains $er^a + (e-b)Y^b$, which is a non-trivial polynomial in Y, so that P also contains a factor $Y-s$. Thus, P contains $X-r, Y-s$ and Z, so that P must have height 2, and since R is Cohen-Macaulay, P must contain a regular sequence of length 2. $\square$

Proposition 4.2. *Let a, b, c be positive integers such that $1 < a \le b$, let K be an algebraically closed field of characteristic not dividing $a-1$, and let $R = K[X,Y,Z]/(f) = K[x,y,z]$, where $f = X + X^{b-a+1}Y^b + Z^{a-1} + X^{c+b-a+1}Y^b + X^{c+b-a+2}Y^b$ is a non quasi-homogenous polynomial. Then R is integrally closed.*

Proof. The Jacobian ideal of R over K is

$$\begin{aligned}J_{R/K} = (1 + X^{b-a}Y^b[&(b-a+1) + (c+b-a+1)X^c \\ &+ (c+b-a+2)X^{c+1}], \\ &bY^{b-1}X^{b-a+1}(1 + X^c + X^{c+1}), (a-1)Z^{a-2}).\end{aligned}$$

By Proposition 2.8, it suffices to prove that $J_{R/K}$ contains a regular sequence of length 2, or equivalently, that every prime ideal P of $K[X,Y,Z]$ that contains f, $\partial f/\partial X$, $\partial f/\partial Y$, and $\partial f/\partial Z$, contains a regular sequence of length 3. Since $\partial f/\partial Z = (a-1)Z^{a-2} \in P$, since $a-1$ is not a multiple of the characteristic, and since $1 \notin P$, it follows that $a > 2$ and $Z \in P$. If $Y \in P$, then since $\partial f/\partial X \in P$, we have also that $1 \in P$, which is a contradiction. So $Y \notin P$. It follows that P contains the following simplifications of f and its partial derivatives (modulo the known elements of P):

$$\begin{aligned}&Z, X(1 + X^{b-a}Y^b(1 + X^c + X^{c+1})), \\ &1 + X^{b-a}Y^b[(b-a+1) + (c+b-a+1)X^c + (c+b-a+2)X^{c+1}], \\ &X^{b-a+1}(1 + X^c + X^{c+1}).\end{aligned}$$

If $X \in P$, then the following simplifications are also in P: $Z, X, 1 + X^{b-a}Y^b(b-a+1)$, which says P contains a regular sequence of length 3. Thus, we may assume that $X \notin P$. Then P contains in particular the elements $1 + X^{b-a}Y^b(1 + X^c + X^{c+1})$ and $1 + X^c + X^{c+1}$, whence it contains $1 = (1+X^{b-a}Y^b(1+X^c+X^{c+1})) - X^{b-a}Y^b(1+X^c+X^{c+1})$, which is a contradiction. $\square$

Now, we consider a non-quasi-homogeneous integrally closed two-dimensional ring as in Proposition 4.1, and we show that it cannot provide a concrete example for the context described by Cutkosky, namely, that it has a zero-dimensional ideal with only one Rees valuation. The zero-dimensional ideal is actually even the maximal ideal.

Example 4.3. Let $R = K[X,Y,Z]/(X^a + Y^b + Z^{a-1} + X^cY^b + X^{c+1}Y^b) = K[x,y,z]$, where a,b,c are positive integers, $1 < a \leq b$, and where K is a perfect field whose characteristic does not divide $a-1, b$. We prove that $I = (x,y,z)$ has only one Rees valuation.

By Observation 2.6, it suffices to consider the case where K is algebraically closed. Now we use Proposition 4.1: the characteristic does not divide $a-1$ and c, and, in the notation of that proposition, $(a-1-d)b, ae, (a-d)b$ are in this example the quantities $(a-1-c)b, ab, (a-c)b$, and clearly the middle quantity does not equal the other two. Thus, by Proposition 4.1, R is integrally closed.

As $x^a+y^b+z^{a-1}+x^cy^b+x^{c+1}y^b = 0$ in R, $z^{a-1}+(x^a+y^b+x^cy^b+x^{c+1}y^b) = 0$, where $x^a+y^b+x^cy^b+x^{c+1}y^b \in (x,y)^{\min\{a,b\}} \subseteq (x,y)^{a-1}$. So, $z \in \overline{(x,y)}$. Since (x,y) is a minimal reduction of I, we may use Corollary 2.4 to get all the Rees valuations from localizations of the integral closure of $S = R[(x,y,z)/x]$ at all height one prime ideals containing x. Then

$$\begin{aligned} S &\cong \frac{K[X,Y,Z,\alpha,\beta]}{(X\alpha-Y, X\beta-Z, X+X^{b-a+1}\alpha^b+\beta^{a-1}+X^{c+b-a+1}\alpha^b+X^{c+b-a+2}\alpha^b)} \\ &\cong \frac{K[X,\alpha,\beta]}{(X+X^{b-a+1}\alpha^b+\beta^{a-1}+X^{c+b-a+1}\alpha^b+X^{c+b-a+2}\alpha^b)}. \end{aligned}$$

By Proposition 4.2, S is integrally closed. Then S/xS is isomorphic to $K[\alpha,\beta]/(\beta^{a-1})$, which has only one minimal prime ideal, so that (x,y,z) has only one Rees valuation.

REFERENCES

1. M.F. Atiyah and I.G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, Boston, 1969.

2. S.D. Cutkosky, *On unique and almost unique factorization of complete ideals* II, Invent. Math. **98** (1989), 59–74.

3. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Grad. Texts Math. **150**, Springer-Verlag, New York, 1995.

4. C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, Lond. Math. Soc. Lect. Note Ser. **336**. Cambridge University Press, Cambridge, 2006.

5. D. Katz and J. Validashti, *Multiplicities and Rees valuations*, Collect. Math. **61** (2010), 1–24.

6. H. Matsumura, *Commutative Ring Theory*, Cambr. Stud. Adv. Math. **8**, Cambridge University Press, Cambridge, 1986.

7. D. G. Northcott and D. Rees, *Reductions of ideals in local rings*. Proc. Cambr. Phil. Soc. **50** (1954), 145–158.

8. D. Rees, *Lectures on the asymptotic theory of ideals*, Lond. Math. Soc. Lect. Note Ser. **113**, Cambridge University Press, Cambridge, 1988.

9. ———, *Valuations associated with ideals* II, J. Lond. Math. Soc. **31** (1956), 221–228.

10. J. Sally, *One-fibered ideals*, in *Commutative algebra* Math. Sci. Res. Inst. Publ. **15** (1989), 437–442.

11. P. Samuel, *Singularities des varietes algebraiques*, Math. Bull. Soc. Math. France **79** (1951), 121–129.

Dipartimento di Matematica – Università degli Studi dell'Aquila – Via Vetoio snc, 67010 COPPITO (AQ) Italy
Email address: **fradigiovannantonio@gmail.com**

Dipartimento di Matematica – Università degli Studi dell'Aquila – Via Vetoio snc, 67010 COPPITO (AQ) Italy
Email address: **guerran@univaq.it**

Department of Mathematics, Reed College, 3203 SE Woodstock Blvd., Portland, OR 97202
Email address: **iswanson@reed.edu**