ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2014

COEFFICIENT CONDITIONS FOR HARMONIC UNIVALENT MAPPINGS AND HYPERGEOMETRIC MAPPINGS

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ABSTRACT. In this paper, we obtain coefficient criteria for a normalized harmonic function defined in the unit disk to be close-to-convex and fully starlike, respectively. Using these coefficient conditions, we present different classes of harmonic close-to-convex (respectively, fully starlike) functions involving Gaussian hypergeometric functions. In addition, we present a convolution characterization for a class of univalent harmonic functions discussed recently by Mocanu, and later by Bshouty and Lyzzaik in 2010. Our approach provides examples of harmonic polynomials that are close-to-convex and starlike, respectively.

1. Introduction and two lemmas. One of the basic coefficient inequalities states that if a normalized power series $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ satisfies the condition

$$\sum_{n=2}^{\infty} n|a_n| \leq 1, \tag{1}$$

then f is analytic in the unit disk $\mathbf{D} = \{z : |z| < 1\}$ and $\operatorname{Re} f'(z) > 0$ in $\mathbf{D}$, and hence the range $f(\mathbf{D})$ is a close-to-convex domain. We recall that a domain D is *close-to-convex* if the complement of D can be written as a union of non-intersecting half-lines. Moreover, it is also well known that each f satisfying the condition (1) implies that $|zf'(z)/f(z) - 1| < 1$ for $z \in \mathbf{D}$ and, in particular, $f \in \mathcal{S}^*$, the class of

2010 AMS *Mathematics subject classification.* Primary 30C45.
Keywords and phrases. Harmonic, univalent, close-to-convex, starlike and convex mappings, coefficient estimates, Gaussian hypergeometric functions, Hadamard product (convolution).
The research of the first author was supported by a fellowship of the University Grants Commission, India.
The second author is on leave from the Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India.
Received by the editors on August 31, 2011, and in revised form on February 22, 2012.
DOI:10.1216/RMJ-2014-44-3-753

starlike univalent functions in $\mathbf{D}$. One of the most natural questions is therefore to discuss its analog coefficient conditions for complex-valued harmonic functions to be close-to-convex or starlike in $\mathbf{D}$.

A complex-valued harmonic function $f = u + iv$ in $\mathbf{D}$ admits the decomposition $f = h + \overline{g}$, where both g and h are analytic in $\mathbf{D}$ (see [**7**]). Here g and h are referred to as analytic and co-analytic parts of f. A complex-valued harmonic function $z \mapsto f(z) = h(z) + \overline{g(z)}$ is *locally univalent* if and only if the Jacobian J_f is non-vanishing in $\mathbf{D}$, where $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. For convenience, we let $f(0) = 0$ and $f_z(0) = 1$ so that every harmonic function f in $\mathbf{D}$ can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} := h + \overline{g}. \tag{2}$$

We denote by $\mathcal{H}$ the class of all normalized harmonic functions f in $\mathbf{D}$ of this form. The class of functions $f \in \mathcal{H}$ that are sense-preserving and univalent in $\mathbf{D}$ is denoted by $\mathcal{S}_H$. Two interesting subsets of $\mathcal{S}_H$ are

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : \, b_1 = f_{\overline{z}}(0) = 0\}$$

and

$$\mathcal{S} = \{f \in \mathcal{S}_H : \, g(z) \equiv 0\}.$$

In recent years, properties of the class $\mathcal{S}_H$ together with its interesting geometric subclasses have been the subject of investigations. We refer to the pioneering works of Clunie and Sheil-Small [**7**], the book of Duren [**8**] and the recent survey articles of Ponnusamy and Rasila [**15**] and Bshouty and Hengartner [**4**]. Let $\mathcal{C}$, $\mathcal{C}_H$, and $\mathcal{C}_H^0$ denote the subclasses of $\mathcal{S}$, $\mathcal{S}_H$, and $\mathcal{S}_H^0$, respectively, with close-to-convex images. In [**17**], the following result has been proved.

Lemma A. *Suppose that $f = h + \overline{g}$, where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ in a neighborhood of the origin and $|b_1| < 1$. If*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \le 1, \tag{3}$$

then $f \in \mathcal{C}_H^1$, where $\mathcal{C}_H^1 = \{f \in \mathcal{S}_H : \, \mathrm{Re}\, f_z(z) > |f_{\overline{z}}(z)| \text{ in } \mathbf{D}\}$.

Condition (3) is easily seen to be sufficient for $f \in \mathcal{C}_H^1$ if a_n and b_n are non-positive for all $n \geq 1$ $(a_1 = 1)$. Since the proof is routine as in the analytic case, we omit the detail.

In [**11**] (see also [**17**] for a slightly more general result), Mocanu has shown that functions in $\mathcal{C}_H^1$ are univalent in $\mathbf{D}$. On the other hand, in [**17**], the authors have shown that each $f \in \mathcal{C}_H^1$ is indeed close-to-convex in $\mathbf{D}$. In view of the information known for the class of analytic functions, it is natural to ask whether the coefficient condition (3) is sufficient for f to belong to $\mathcal{S}_H^*$, where

$$\mathcal{S}_H^* = \{f \in \mathcal{S}_H : f(\mathbf{D}) \text{ is a starlike domain with respect to the origin}\}.$$

Functions in $\mathcal{S}_H^*$ are called starlike functions. In the sequel, we also need

$$\mathcal{S}_H^{*0} = \{f \in \mathcal{S}_H^* : f_{\overline{z}}(0) = 0\}.$$

Harmonic starlikeness is not a hereditary property because it is possible that, for $f \in \mathcal{S}_H^*$, $f(|z| < r)$ is not necessarily starlike for each $r < 1$ (see [**8**]).

Definition 1. A harmonic mapping $f \in \mathcal{H}$ is said to be *fully starlike* (respectively, fully convex) if each $|z| < r$ is mapped onto a starlike (respectively, convex) domain (see [**6**]).

Fully convex mappings are known to be fully starlike but not the converse as the function $f(z) = z + (1/n)\overline{z}^n$ $(n \geq 2)$ shows. It is easy to see that the harmonic Koebe function K with the dilation $\omega(z) = z$ is not fully starlike, although $K = H + \overline{G} \in \mathcal{S}_H^{*0}$, where

$$H(z) = \frac{z - (1/2)z^2 + (1/6)z^3}{(1-z)^3} \quad \text{and} \quad G(z) = \frac{(1/2)z^2 + (1/6)z^3}{(1-z)^3}.$$

For further details, we refer to [**6**].

Definition 2. We say that a continuously differentiable function f in $\mathbf{D}$ is *starlike* in $\mathbf{D}$ if it is sense-preserving, $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbf{D} \setminus \{0\}$ and

$$\operatorname{Re}\left(\frac{Df(z)}{f(z)}\right) > 0 \quad \text{for all } z \in \mathbf{D} \setminus \{0\},$$

where $Df = zf_z - \overline{z}f_{\overline{z}}$.

The last condition gives that $(z = re^{i\theta})$

$$\frac{\partial}{\partial\theta}\big(\arg f(re^{i\theta})\big) = \mathrm{Re}\left(\frac{Df(z)}{f(z)}\right) > 0$$
$$\text{for all } z \in \mathbf{D}\setminus\{0\},$$

showing that the curve $C_r = \{f(re^{i\theta}) : 0 \le \theta < 2\pi\}$ is starlike with respect to the origin for each $r \in (0,1)$ (see [**11,** Theorem 1]). In this case, the last condition implies that f is indeed fully starlike in $\mathbf{D}$. At this point, it is also important to observe that Dg for C^1-functions behaves much like zg' for analytic functions, for example in the sense that for g univalent and analytic in $\mathbf{D}$, g is starlike if and only if $\mathrm{Re}\,(zg'(z)/g(z)) > 0$ in $\mathbf{D}$. A similar characterization has also been obtained by Mocanu [**11**] for convex (C^2) functions. It is worth pointing out that, in the case of analytic functions, fully starlike (respectively, fully convex) is the same as starlike (respectively, convex) in $\mathbf{D}$. Lately, interesting distortion theorems and coefficient estimates for convex and close-to-convex harmonic mappings were given by Clunie and Sheil-Small [**7**].

As a consequence of convolution theorem [**2,** Theorem 2.6, p. 908] (see also the proof of Theorem 1 in [**9**]) these authors obtained a sufficient coefficient condition for harmonic starlike mappings. Unfortunately, there is a minor error in the main theorem, and we would like to point this out as we use this for our applications.

Lemma 1. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$. Then f is fully starlike in $\mathbf{D}$ if and only if*

$$(4)\qquad h(z) * A(z) - \overline{g(z)} * \overline{B(z)} \neq 0 \quad \text{for } |\zeta| = 1,\ 0 < |z| < 1,$$

where

$$A(z) = \frac{z + ((\zeta - 1)/2)z^2}{(1-z)^2}$$

and

$$B(z) = \frac{\overline{\zeta}z - ((\overline{\zeta} - 1)/2)z^2}{(1-z)^2}.$$

Proof. A necessary and sufficient condition for a function f to be starlike in $|z| < r$ for each $r < 1$ is that

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \mathrm{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) > 0 \tag{5}$$
$$\text{for all } z \in \mathbf{D} \setminus \{0\}.$$

We remind the reader that, if $f = h + \overline{g} \in \mathcal{S}_H$ with $g'(0) = b_1 \neq 0$, then the limit

$$\lim_{z\to 0} \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}$$

does not exist, but the limit does exist which is 1 when $b_1 = 0$. This observation is crucial in the remaining part of our proof. Thus, by (5), f is fully starlike in $\mathbf{D}$ if and only if

$$\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \neq \frac{\zeta - 1}{\zeta + 1}, \quad |\zeta| = 1,\ \zeta \neq -1,\ 0 < |z| < 1$$

and, as in the proof of Theorem 2.6 [**2**], a simple computation shows that the last condition is equivalent to (4). The proof is complete. $\square$

In view of Lemma 1, the hypothesis that $f = h + \overline{g} \in \mathcal{S}_H$ in [**2,** Corollary 2.7, p. 908] can be relaxed as the condition (3) implies that $f \in \mathcal{S}_H$. So we may now reformulate [**2,** Corollary 2.7, p. 908] in the following improved form (see also [**18**]).

Lemma 2. *Let $f = h + \overline{g}$ be a harmonic function of the form* (2) *with $b_1 = g'(0) = 0$. If*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=2}^{\infty} n|b_n| \leq 1, \tag{6}$$

then $f \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}$. Moreover, f is fully starlike in $\mathbf{D}$.

Proof. By Lemma A, coefficient condition (6) ensures the univalency of f and, moreover, $f \in \mathcal{C}_H^1$. Now, in order to show that (6) implies $f \in \mathcal{S}_H^{*0}$, we apply Lemma 1. As in the proof of [**2,** Corollary 2.7], it suffices to show that condition (4) holds. Indeed, we easily have

$$\begin{aligned}
&\left|h(z) * A(z) - \overline{g(z)} * \overline{B(z)}\right| \\
&= \left|z + \sum_{n=2}^{\infty}\left(n + \frac{(n-1)(\zeta-1)}{2}\right)a_n z^n \right.\\
&\qquad \left. - \sum_{n=2}^{\infty}\left(n\zeta - \frac{(n-1)(\zeta-1)}{2}\right)\overline{b_n}\,\overline{z^n}\right| \\
&> |z|\left[1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} n|b_n|\right] \geq 0,
\end{aligned}$$

and so Lemma 1 gives that f is fully starlike in $\mathbf{D}$ and hence, $f \in \mathcal{S}_H^{*0}$. ☐

For instance, according to Lemma 2, it follows that if $\alpha \in \mathbf{C}$ is such that $|\alpha| \leq 1/n$ for some $n \geq 2$, then the function f defined by

$$f(z) = z + \alpha \overline{z}^n$$

belongs to $\mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}$. Later in Section 4, we present a number of interesting applications of Lemma 2.

2. Conjecture of Mocanu on harmonic mappings. According to our notation, the conjecture of Mocanu [**12**] may be reformulated in the following form.

Conjecture B. *If*

$$\mathcal{M} = \left\{f = h + \overline{g} \in \mathcal{H} : g' = zh', \operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2} \text{ for } z \in \mathbf{D}\right\},$$

then $f \in \mathcal{S}_H^0$.

In [**5**], Bshouty and Lyzzaik have solved the conjecture of Mocanu by establishing the following stronger result.

Theorem C. $\mathcal{M} \subset \mathcal{C}_H^0$.

It is worth reformulating this result in a general form.

Theorem 1. *Let $f = h+\overline{g}$ be a harmonic mapping of* $\mathbf{D}$, *with $h'(0) \neq 0$ that satisfies $g'(z) = e^{i\theta} z h'(z)$ and*

$$\operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2} \quad \text{for all } z \in \mathbf{D}.$$

Then f is a univalent close-to-convex mapping in $\mathbf{D}$.

Proof. This theorem is proved for $\theta = 0$ by Bshouty and Lyzzaik [**5**], i.e., $\mathcal{M} \subset \mathcal{C}_H^0$. However, it can be easily seen from their proof that the theorem continues to hold if the dilatation ω is chosen to be $\omega(z) = e^{i\theta} z$ instead of $\omega(z) = z$. So we omit the details. □

Using the method of extreme points, the authors in [**1**] presented an elegant and simple proof of Theorem 1.

Since the function $f \in \mathcal{M}$ satisfies the condition $f_{\overline{z}}(0) = 0$, it is natural to ask whether $\mathcal{M}$ is included in $\mathcal{S}_H^{*0}$ or in $\mathcal{C}_H^1$. First we construct a function $f \in \mathcal{M}$ such that $f \notin \mathcal{C}_H^1$.

Consider $f = h + \overline{g}$, where

$$h(z) = z - az^n \quad \text{and} \quad g(z) = \frac{z^2}{2} - \frac{n}{n+1} az^{n+1}$$

for $n \geq 2$ and $0 < a \leq 1/n$. It follows that $g'(z) = zh'(z)$ and

$$1 + z\frac{h''(z)}{h'(z)} = \frac{1 - n^2 az^{n-1}}{1 - naz^{n-1}}.$$

Also, it is a simple exercise to see that

$$w = \frac{1 - n^2 az^{n-1}}{1 - naz^{n-1}}$$

maps the unit disk $\mathbf{D}$ onto the disk

$$\left| w - \frac{1 - n^3 a^2}{1 - n^2 a^2} \right| < \frac{an(n-1)}{1 - n^2 a^2}$$

if $0 < a < 1/n$, and onto the half-plane $\operatorname{Re} w < (n+1)/2$ if $a = 1/n$. In particular, this disk lies in the half-plane

$$\operatorname{Re} w > \frac{1 - n^2 a}{1 - na}$$

and thus, $\mathrm{Re}\, w > -1/2$ if $(1-n^2a)/(1-na) \geq -1/2$, i.e., if $0 < a \leq 3/(n(1+2n))$. According to Theorem C, $f = h + \overline{g}$ is univalent close-to-convex mapping in $\mathbf{D}$ whenever a satisfies the condition

$$0 < a \leq \frac{3}{n(1+2n)}.$$

On the other hand, this function does not satisfy the coefficient condition (6). Moreover, it can be easily seen that $f \notin \mathcal{C}_H^1$. Indeed, if $a = 0.3$ and $n = 2$, then the corresponding function

$$f_0(z) = z - \frac{3}{10}z^2 + \overline{\frac{z^2}{2} - \frac{1}{5}z^3}$$

does not belong to $\mathcal{C}_H^1$. The graph of $f_0(z)$ is shown in Figure 1. This example shows that there are functions in $\mathcal{M}$ that do not necessarily belong to $\mathcal{C}_H^1$. Indeed, the above discussion gives

Theorem 2. $\mathcal{M} \not\subset \mathcal{C}_H^1$.

Moreover, the graph of

$$f(z) = z - \frac{3}{n(2n+1)}z^n + \overline{\frac{z^2}{2} - \frac{3}{(n+1)(2n+1)}z^{n+1}},$$

for various values of $n \geq 2$, shows that $f(z)$ is starlike in $\mathbf{D}$. This motivates us to state

Conjecture 1. $\mathcal{M} \subset \mathcal{S}_H^{*0}$.

Our next result gives a convolution characterization for functions $f \in \mathcal{M}$ to be starlike in $\mathbf{D}$.

Theorem 3. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ such that $g'(z) = zh'(z)$. Then f is fully starlike in $\mathbf{D}$ if and only if*

$$(7) \qquad h(z) * A(z) - \overline{z}\left(\overline{h(z)} * \overline{B(z)}\right) \neq 0 \quad \text{for } |\zeta| = 1,\ 0 < |z| < 1,$$

where

$$A(z) = \frac{2z + (\zeta - 1)z^2}{(1-z)^2}$$

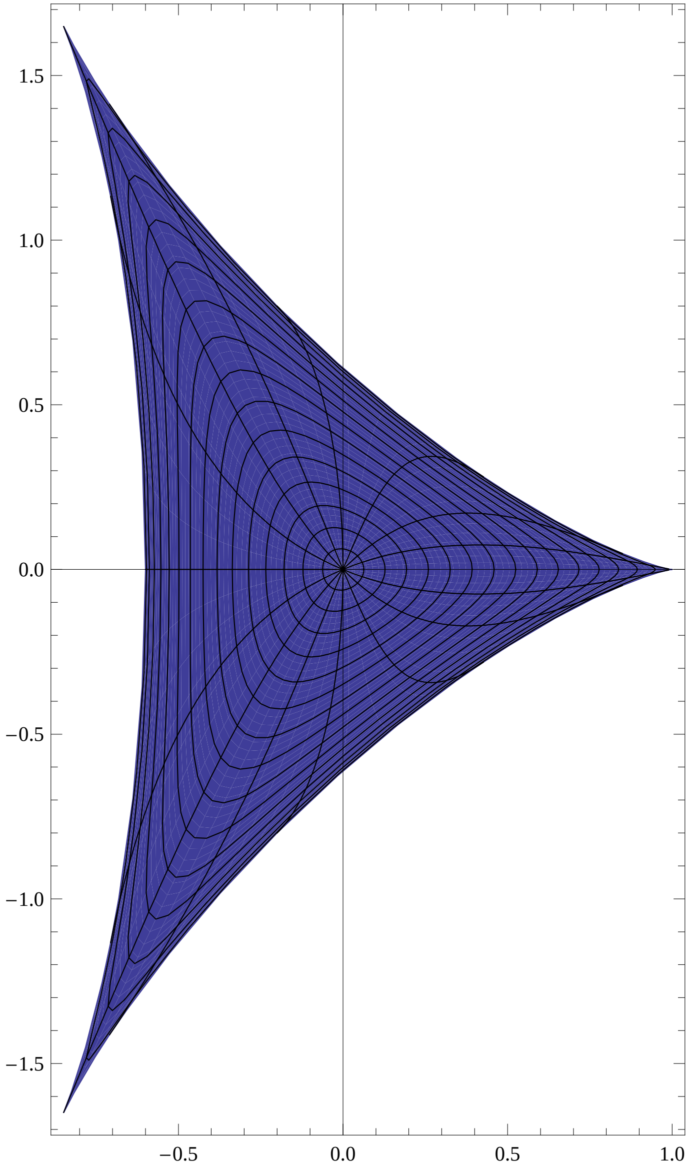

FIGURE 1. The graph of the function $f_0(z) = z - \frac{3}{10}z^2 + \overline{\frac{z^2}{2} - \frac{1}{5}z^3}$.

and

$$B(z)=\frac{2z^2+z(\overline{\zeta}-1)+(1-z)^2(\overline{\zeta}-1)\log(1-z)}{z(1-z)^2}.$$

Proof. As in the proof of Lemma 1, f is fully starlike if and only if

$$\mathrm{Re}\left(\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right)>0 \quad \text{for all } z\in \mathbf{D}\setminus\{0\}. \tag{8}$$

Since $g'(0)=0$ and $g'(z)=zh'(z)$, we obtain that

$$\lim_{z\to 0}\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}=1,$$

and therefore condition (8) holds if and only if

$$\frac{zh'(z)-\overline{z^2h'(z)}}{h(z)+\overline{\int_0^z th'(t)\,dt}}\neq\frac{\zeta-1}{\zeta+1}$$
$$\text{for } |\zeta|=1,\ \zeta\neq -1,\ 0<|z|<1.$$

The last condition is equivalent to

$$0\neq(\zeta+1)\left[zh'(z)-\overline{z^2h'(z)}\right]-(\zeta-1)\left[h(z)+\overline{\int_0^z th'(t)\,dt}\right],$$

which is the same as

$$0\neq h(z)*A(z)-\overline{\left[(\overline{\zeta}+1)z^2h'(z)+(\overline{\zeta}-1)\int_0^z th'(t)\,dt\right]}. \tag{9}$$

Finally, as

$$z^2h'(z)=z\left[h(z)*\frac{z}{(1-z)^2}\right]$$

and

$$g(z)=\int_0^z th'(t)\,dt=\frac{z}{2}\left[h(z)*\left(\frac{2}{1-z}+\frac{2}{z}\log(1-z)\right)\right],$$

condition (9) is easily seen to be equivalent to the required convolution condition (7). The proof is complete. □

Now, we consider the harmonic function $f = h + \overline{g}$, where

$$h(z) = z - \frac{z^2}{2} \quad \text{and} \quad g(z) = \frac{z^2}{2} - \frac{z^3}{3}$$

so that $g'(z) = zh'(z)$. It follows that

$$1 + z\frac{h''(z)}{h'(z)} = \frac{1-2z}{1-z},$$

and we see that

$$\text{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) < \frac{3}{2} \quad \text{for } z \in \mathbf{D}. \tag{10}$$

The function h satisfying condition (10) is known to satisfy the condition (see, e.g., [**14, 16**])

$$\left|\frac{zh'(z)}{h(z)} - \frac{2}{3}\right| < \frac{2}{3}, \quad z \in \mathbf{D},$$

and hence, h is starlike in $\mathbf{D}$. The graph of $f(z) = h(z) + \overline{g(z)}$ shown in Figure 2 shows that $f = h + \overline{g}$ is not univalent in $\mathbf{D}$. This example motivates raising the following:

Problem 1. For $\alpha \in (2/3, 1)$, define

$$\mathcal{P}(\alpha) = \Bigg\{ f = h + \overline{g} \in \mathcal{H} : \, g' = zh', \\ \text{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) < \frac{3\alpha}{2} \text{ for } z \in \mathbf{D} \Bigg\}.$$

Determine $\inf\{\alpha \in (2/3, 1) : \, \mathcal{P}(\alpha) \subset \mathcal{S}_H^0\}$.

3. Harmonic polynomials. One of the interesting problems in the class of harmonic mappings is to find a method of constructing sense-preserving harmonic polynomials that have some interesting geometric properties. In [**10, 19**], the authors discussed such polynomials with many interesting special cases. Prior to the work of Suffridge [**19**], few examples of such polynomials were known. In this section, we shall see that some of the results of [**10, 19**] have a closer link with our results in Section 1. Following the ideas from [**10, 19**], let $Q(z) = \sum_{k=1}^n c_k z^k$

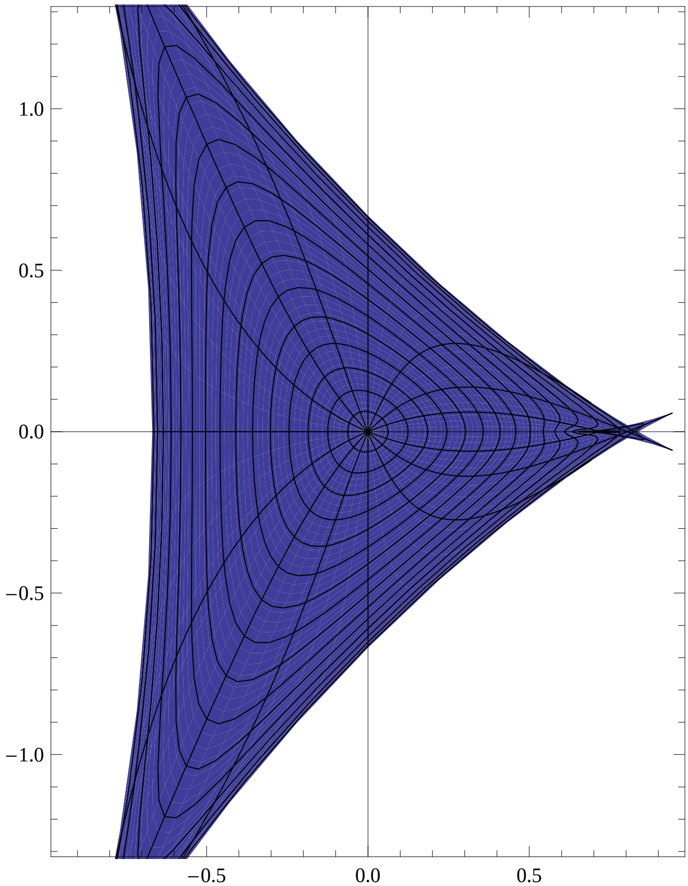

FIGURE 2. Graph of the function $f(z) = z - (1/2)z^2 + \overline{(1/2)z^2 - (1/3)z^3}$.

be a polynomial of degree n. Define

$$\widehat{Q}(z) = z^n \overline{Q(1/\overline{z})}.$$

Thus, if $Q(z) = c\prod_{j=1}^{n}(z - z_j)$, then $\widehat{Q}(z) = \overline{c}\prod_{j=1}^{n}(1 - z\overline{z_j})$, and it follows that the zeros of Q and $\widehat{Q}$ on the unit circle $|z| = 1$ are the same. In **[19]**, Suffridge proved the following theorem.

Theorem D ([**19,** Theorem 1]). *Let $Q(z)$ be a polynomial of degree $q \leq n-2$ with $Q(0)=1$, and assume that $Q(z) \neq 0$ when $z \in \mathbf{D}$. Let g and h be defined by $h(0)=g(0)=0$, and*

$$g'(z) = e^{i\beta} t z \widehat{Q}(z), \qquad h'(z) = Q(z) + e^{i\phi}(1-t) z \widehat{Q}(z),$$

where ϕ, β and t are real, $0 \leq t \leq 1$. Then the harmonic polynomial $f = h + \overline{g}$ has degree n and is sense-preserving in $\mathbf{D}$.

With an additional condition on Q, we can improve this result by showing that the harmonic polynomial $f = h + \overline{g}$ described in Theorem D is indeed close-to-convex in $\mathbf{D}$. More precisely, we prove the following theorem.

Theorem 4. *Let Q, g, h, ϕ, β and t be defined as in Theorem* D. *If Q satisfies the condition $\mathrm{Re}\,\{Q(z)\} > |z\widehat{Q}(z)|$ for all $z \in \mathbf{D}$, then $f = h + \overline{g}$ belongs to $\mathcal{C}_H^1$, and hence, f is close-to-convex in $\mathbf{D}$.*

Proof. It follows from the hypotheses that

$$\begin{aligned}\mathrm{Re}\,(h'(z)) &= \mathrm{Re}\,(Q(z) + e^{i\phi}(1-t) z \widehat{Q}(z)) \\ &= \mathrm{Re}\,(Q(z)) + \mathrm{Re}\,(e^{i\phi}(1-t) z \widehat{Q}(z)) \\ &\geq \mathrm{Re}\,(Q(z)) - |e^{i\phi}(1-t) z \widehat{Q}(z)| \\ &> |z\widehat{Q}(z)| - (1-t)|z\widehat{Q}(z)| = |g'(z)|.\end{aligned}$$

Thus, the desired conclusion follows (see [**17**]). □

Example 1. Consider

$$f(z) = z + e^{i\phi}\frac{(1-t)}{n} z^n + e^{i\beta}\frac{t}{m}\overline{z}^m,$$

where $n \geq 2$, $m \geq 1$, $\phi \in \mathbf{R}$, $\beta \in \mathbf{R}$ and $0 \leq t \leq 1$. Then, according to Lemma 1, we have

$$\begin{aligned}n|a_n| + m|b_m| &= n\left|e^{i\phi}\frac{(1-t)}{n}\right| + m\left|e^{i\beta}\frac{t}{m}\right| \\ &= (1-t) + t = 1,\end{aligned}$$

showing that f is not only close-to-convex, but also in $\mathcal{C}_H^1$. On the other hand, by Lemma 2, f is also fully starlike in $\mathbf{D}$ whenever $m \geq 2$.

In particular, the function

$$f(z) = z + e^{i\phi}(1-t)(z^2/2) + e^{i\beta}t(\overline{z}^2/2)$$

is close-to-convex and fully starlike in $\mathbf{D}$. By a direct method, Suffridge [**19,** Example 1] showed that this function is univalent in $\mathbf{D}$.

Using Theorem 1, it is possible to give a new proof of the limit mapping theorem of Suffridge et al. [**10,** Theorem 3.1]. To do this, we assume that all the zeros of $Q(z)$ lie on the unit circle $|z| = 1$. Then, for $q = n-2$ and $t = 1$ in Theorem D, we have

$$h_n'(z) = Q(z) = \prod_{j=1}^{n-2}(1 - e^{-i\psi_j}z) = \frac{1 - z^{n+1}}{\prod_{j=1}^{3}(1 - ze^{i\psi_j})}$$

and

$$g_n'(z) = z\prod_{j=1}^{n-2}(z - e^{i\psi_j}).$$

It is clear that $h_n'(z)$ converges uniformly on the compact subsets of the unit disk to

$$h'(z) = \frac{1}{\prod_{j=1}^{3}(1 - ze^{i\psi_j})}. \tag{11}$$

Similarly, $g_n'(z)$ converges uniformly on the compact subsets of the unit disk to

$$g'(z) = ze^{i\theta}h'(z).$$

If we take the logarithmic derivative of (11), we see that

$$1 + z\frac{h''(z)}{h'(z)} = \sum_{j=1}^{3}\frac{ze^{i\psi_j}}{1 - ze^{i\psi_j}} + 1, \quad z \in \mathbf{D}.$$

Since $w(z) = z/(1-z)$ maps $\mathbf{D}$ onto the half plane $\operatorname{Re} w > -1/2$, the last formula clearly implies that

$$\operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbf{D}.$$

According to Theorem 1, f is univalent close-to-convex in $\mathbf{D}$. This provides an alternate proof of [**10,** Theorem 3.1].

4. Applications of Lemmas 1 and 2. Consider the Gaussian hypergeometric function

$$(12)\qquad {}_2F_1(a,b;c;z) := F(a,b;c;z) = \sum_{n=0}^{\infty} A_n z^n,$$

where

$$A_n = \frac{(a,n)(b,n)}{(c,n)(1,n)}.$$

Here a, b, c are complex numbers such that $c \neq -m$, $m = 0, 1, 2, 3, \ldots$, $(a, 0) = 1$ for $a \neq 0$ and, for each positive integer n, $(a, n) := a(a+1)\cdots(a+n-1)$, see for instance the recent book of Temme **[20]** and Anderson et al. **[3]**. We see that $(a,n) = \Gamma(a+n)/\Gamma(a)$. Often the Pochhammer notation $(a)_n$ is used instead of (a,n). In the exceptional case $c = -m$, $m = 0, 1, 2, 3, \ldots$, the function $F(a,b;c;z)$ is clearly defined even if $a = -j$ or $b = -j$, where $j = 0, 1, 2, \ldots$ and $j \leq m$. The following well-known Gauss formula **[20]** is crucial in the proof of our results of this section:

$$(13)\qquad F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty \quad \text{for } \mathrm{Re}\, c > \mathrm{Re}\,(a+b).$$

In order to generate nice examples of (fully) starlike and close-to-convex harmonic mappings, we consider mappings whose co-analytic part involves the Gaussian hypergeometric function.

Theorem 5. *Let either* $a, b \in (-1, \infty)$ *with* $ab > 0$*, or* $a, b \in \mathbf{C} \setminus \{0\}$ *with* $b = \overline{a}$*. Assume that* c *is a positive real number such that* $c > \mathrm{Re}\,(a+b) + 1$*,* $\alpha \in \mathbf{C}$*, and let*

$$f_k(z) = z + \overline{\alpha z^k F(a,b;c;z)} \quad \text{for } k = 1, 2.$$

(a) *If*

$$(14)\qquad \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[ab + 2(c-a-b-1)] \leq \frac{1}{|\alpha|},$$

where $0 < |\alpha| < 1/2$*, then* $f_2 \in \mathcal{S}_H^{*0} \cap \mathcal{C}_H^1$.

(b) *If*

$$\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[ab+c-a-b-1] \leq \frac{1+|\alpha|}{|\alpha|}, \tag{15}$$

where $2|\alpha|ab \leq c$, *then* $f(z) = z + \overline{\alpha z(F(a,b;c;z)-1)} \in \mathcal{S}_H^{*0} \cap \mathcal{C}_H^1$.

(c) *If*

$$\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[ab+c-a-b-1] \leq \frac{1}{|\alpha|},$$

where $0 < |\alpha| < 1$, *then* $f_1 \in \mathcal{C}_H^1$.

Proof. We present a proof of (a) and, since the proofs of the other two cases follow in a similar fashion, we only include a key step for (b).

(a) Set $h(z) = z$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n = \alpha z^2 F(a,b;c;z)$ so that

$$f_2(z) = z + \overline{\alpha z^2 F(a,b;c;z)}.$$

By (12), we have

$$b_n = \alpha A_{n-2} = \alpha \frac{(a,n-2)(b,n-2)}{(c,n-2)(1,n-2)} \quad \text{for } n \geq 2. \tag{16}$$

By Lemma 2, it suffices to show that $K := \sum_{n=2}^{\infty} n|b_n| \leq 1$. Using (16), it follows that

$$\begin{aligned} K &= |\alpha| \sum_{n=2}^{\infty} \frac{n(a,n-2)(b,n-2)}{(c,n-2)(1,n-2)} \\ &= |\alpha| \left(2 + \sum_{n=1}^{\infty} \frac{(n+2)(a,n)(b,n)}{(c,n)(1,n)} \right) \\ &= |\alpha| \left(\frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a+1,n-1)(b+1,n-1)}{(c+1,n-1)(1,n-1)} + 2 \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} \right). \end{aligned}$$

By the hypothesis we have $c > a+b+1$, and both the series in the last expression converge so using formula (13), we get

$$K = |\alpha|\left(\frac{ab}{c}\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\right] + 2\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)$$
$$= |\alpha|\left(\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[ab + 2(c-a-b-1)]\right).$$

Clearly, (14) is equivalent to $K \le 1$. Thus, $f_2 \in \mathcal{C}_H^1$ and is also fully starlike in $\mathbf{D}$. We have completed the proof of (a).

(b) For the proof of (b), we consider g defined by

$$g(z) = \alpha z(F(a,b;c;z) - 1) = \alpha \sum_{n=2}^{\infty} A_{n-1} z^n,$$

and it suffices to observe that

$$\alpha \sum_{n=2}^{\infty} n|A_{n-1}| = |\alpha|\left(\frac{ab}{c}\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\right] + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right).$$

□

The case $a = 1$ of Theorem 5 (a) and (b) gives

Corollary 1. *Let b and c be positive real numbers and α a complex number.*

(a) *If* $0 < |\alpha| < 1/2$ *and*

(17) $$c \ge \beta^+ = \frac{3 - 6|\alpha| + (2-|\alpha|)b}{2(1-2|\alpha|)} + \frac{\sqrt{|\alpha|^2(b^2+4b+4) + 1 + |\alpha|(4b^2 - 2b - 4)}}{2(1-2|\alpha|)},$$

then

$$f_2(z) = z + \overline{\alpha z^2 F(1,b;c;z)} \in \mathcal{S}_H^{*0} \cap \mathcal{C}_H^1.$$

(b) *If* $2|\alpha|b \le c$ *and*

(18) $$c \ge r^+ = \frac{3 + 2b(1+|\alpha|) + \sqrt{b^2(4|\alpha|^2 + 4|\alpha|) + 1}}{2},$$

then

$$f(z) = z + \overline{\alpha z(F(1,b;c;z) - 1)} \in \mathcal{S}_H^{*0} \cap \mathcal{C}_H^1.$$

Proof. (a) Let $f_2(z) = z + \overline{\alpha z^2 F(1,b;c;z)}$. It suffices to prove that, if $c \geq \beta^+$, then the inequality (14) is satisfied with $a = 1$.

It can be easily seen that $\beta^+ > b+2$, and so the condition $c \geq \beta^+$ implies that $c > b+2$. Next, the condition (14) for $a = 1$ reduces to

$$\frac{\Gamma(c)\Gamma(c-b-2)}{\Gamma(c-1)\Gamma(c-b)}[b + 2(c-b-2)] \leq \frac{1}{|\alpha|},$$

which is equivalent to

$$(1-2|\alpha|)c^2 + c[b(|\alpha|-2) - 3 + 6\alpha] + b^2 + (3-|\alpha|)b + 2 - 4|\alpha| \geq 0.$$

Simplifying this inequality gives

$$(1-2|\alpha|)(c-\beta^-)(c-\beta^+) \geq 0,$$

where β^+ is given by (17) and

$$\begin{aligned}\beta^- = & \frac{3 - 6|\alpha| + (2-|\alpha|)b}{2(1-2|\alpha|)} \\ & - \frac{\sqrt{|\alpha|^2(b^2+4b+4) + 1 + |\alpha|(4b^2 - 2b - 4)}}{2(1-2|\alpha|)}.\end{aligned}$$

Since $\beta^+ \geq \beta^-$ and, by hypothesis $c \geq \beta^+$, the inequality (14) holds. It follows from Theorem 5 (a) that $f_2 \in \mathcal{C}_H^1$ and f_2 is fully starlike.

The proof for case (b) follows if one adopts a similar approach. In fact, if we set $a = 1$ in Theorem 5 (b), then it is easy to see that the inequality (15) is equivalent to

$$c^2 + c(-2b(1+|\alpha|) - 3) + (b^2 + 3b)(1+|\alpha|) + 2 = (c - r^-)(c - r^+) \geq 0,$$

where r^+ is given by (18) and

$$r^- = \frac{3 + 2b(1+|\alpha|) - \sqrt{b^2(4|\alpha|^2 + 4|\alpha|) + 1}}{2}.$$

Since $r^+ \geq r^-$, the hypothesis that $c \geq r^+$ gives the desired conclusion. $\square$

As pointed out in Section 3, except for the work of [**10, 19**], a good technique does not seem to exist for generating univalent harmonic polynomials. In view of Theorem 5, we can obtain harmonic univalent polynomials that are close-to-convex and fully starlike in $\mathbf{D}$.

Corollary 2. *Let m be a positive integer, c a positive real number, $\alpha \in \mathbf{C}$, and let*

$$f_k(z) = z + \overline{\alpha z^k \sum_{n=0}^{m} \binom{m}{n} \frac{(m-n+1,n)}{(c,n)} z^n} \quad \textit{for } k = 1, 2.$$

(a) *If $0 < |\alpha| < 1/2$ and*

$$\frac{\Gamma(c)\Gamma(c+2m-1)}{(\Gamma(c+m))^2}[m^2 + 2(c+2m-1)] \leq \frac{1}{|\alpha|},$$

then $f_2 \in \mathcal{S}_H^{*0} \cap \mathcal{C}_H^1$, and f_2 is indeed fully starlike in $\mathbf{D}$.

(b) *If $0 < |\alpha| < 1$ and*

$$\frac{\Gamma(c)\Gamma(c+2m-1)}{(\Gamma(c+m))^2}[m^2 + c + 2m - 1] \leq \frac{1}{|\alpha|},$$

then $f_1 \in \mathcal{C}_H^1$.

Proof. The results follow if we set $a = b = -m$ in Theorem 5 (a) and (c), respectively. ☐

On the other hand, Theorem 5 (b) for $a = b = -m$ shows that, if m is a positive integer, c is a positive real number and $\alpha \in \mathbf{C}$ is such that $2|\alpha|m^2 \leq c$ and

$$\frac{\Gamma(c)\Gamma(c+2m-1)}{(\Gamma(c+m))^2}[m^2 + c + 2m - 1] \leq \frac{1+|\alpha|}{|\alpha|},$$

then

$$f(z) = z + \overline{\alpha z \sum_{n=1}^{m} \binom{m}{n} \frac{(m-n+1,n)}{(c,n)} z^n}$$

belongs to $\mathcal{S}_H^{*0} \cap \mathcal{C}_H^1$, and f is indeed fully starlike in $\mathbf{D}$.

Example 2. If we let $m = 3$ in Corollary 2 (a), then we have the following: if c is a positive real number such that $|\alpha|g(c) \leq 1$, where α

is a complex number with $0 < |\alpha| < 1/2$ and

$$g(c) = 2 + \frac{27}{c} + \frac{72}{c(c+1)} + \frac{30}{c(c+1)(c+2)},$$

then the harmonic function

$$f(z) = z + \overline{\alpha\left(z^2 + \frac{9}{c}z^3 + \frac{18}{c(c+1)}z^4 + \frac{6}{c(c+1)(c+2)}z^5\right)}$$

is fully starlike in $\mathbf{D}$.

Similarly, we see that if

$$c \geq \frac{14|\alpha| - 1 + \sqrt{36|\alpha|^2 + 52|\alpha| + 1}}{2(1 - 2|\alpha|)},$$

where α is a complex number with $0 < |\alpha| < 1/2$, then the harmonic function

$$f(z) = z + \overline{\alpha\left(z^2 + \frac{4}{c}z^3 + \frac{2}{c(c+1)}z^4\right)}$$

belongs to $\mathcal{S}_H^{*0} \cap \mathcal{C}_H^1$ and is indeed fully starlike in $\mathbf{D}$.

Example 3. The choice $m = 2$ in Corollary 2 (b) easily gives the following: if c is a positive real number such that

$$c \geq \frac{9|\alpha| - 1 + \sqrt{25|\alpha|^2 + 38|\alpha| + 1}}{2(1 - |\alpha|)},$$

where α is a complex number with $0 < |\alpha| < 1$, then

$$f(z) = z + \overline{\alpha\left(z + \frac{4}{c}z^2 + \frac{2}{c(c+1)}z^3\right)} \in \mathcal{C}_H^1.$$

Theorem 6. *Let $a, b \in (-1, \infty)$. Assume that c is a positive real number, $\alpha \in \mathbf{C}$ and, for $k = 0, 1$, define*

$$f_k(z) = z + \overline{\alpha z^k \int_0^z F(a, b; c; t)\, dt}.$$

(a) *Let $ab > 0$ or $a, b \in \mathbf{C} \setminus \{0\}$ with $b = \overline{a}$, where $c > \mathrm{Re}\,(a + b)$ and $0 < |\alpha| < 1$ such that*

$$\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \leq \frac{1}{|\alpha|}.$$

Then $f_0 \in \mathcal{C}_H^1$.

(b) *Let* $(a-1)(b-1) > 0$, *or* $a, b \in \mathbf{C} \setminus \{0,1\}$ *with* $b = \overline{a}$, *where* $c > \max\{1, \mathrm{Re}\,(a+b)\}$ *and* $0 < |\alpha| < 1/2$ *such that*

$$|\alpha| \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{c-a-b}{(a-1)(b-1)}\right) \le 1 + |\alpha| \frac{(c-1)}{(a-1)(b-1)}.$$

Then $f_1 \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}$. *Moreover,* f_1 *is fully starlike in* $\mathbf{D}$.

Proof. We give the proof of (a) and, since the proof of (b) follows in a similar fashion, we omit the details.

(a) Set $f_0(z) := z + \overline{g(z)}$, where

$$g(z) = \alpha \int_0^z F(a,b;c;t)\,dt = \sum_{n=1}^{\infty} b_n z^n,$$
$$b_n = \alpha \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)n} \quad \text{for } n \ge 1.$$

Therefore,

$$\sum_{n=1}^{\infty} n|b_n| = |\alpha| \sum_{n=1}^{\infty} \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} = |\alpha| \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

as $c > \mathrm{Re}\,(a+b)$. The conclusion now follows from Lemma 2. □

For instance, the case $a = 1$ in Theorem 6 (a) shows that if b and c are positive real numbers such that

$$c \ge \frac{b+1-|\alpha|}{1-|\alpha|},$$

where α is a complex number satisfying $0 < |\alpha| < 1$, then

$$f(z) = z + \overline{\alpha \int_0^z F(1,b;c;t)\,dt} \in \mathcal{C}_H^1.$$

Corollary 3. *Assume that* c *is a positive real number and* α *is a complex number.*

(a) *Suppose that either* $a, b \in (-1, \infty)$ *with* $ab > 0$, *or* $a, b \in \mathbf{C} \setminus \{0\}$ *with* $b = \overline{a}$. *If* $c > \mathrm{Re}\,(a+b)+1$ *and* $0 < |\alpha| < 1$ *such that*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{1}{|\alpha|}, \tag{19}$$

then

$$f(z) = z + \overline{(\alpha c/(ab))[F(a,b;c;z)-1]} \in \mathcal{C}_H^1.$$

(b) *Suppose that either* $a, b \in (-1, \infty)$ *with* $ab > 0$, *or* $a, b \in \mathbf{C} \setminus \{0, 1\}$ *with* $b = \overline{a}$. *If* $c > \max\{1, \mathrm{Re}\,(a+b)+1\}$ *and* $0 < |\alpha| < 1/2$ *such that*

$$|\alpha| \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{c-a-b-1}{ab}\right) \leq 1 + |\alpha| \frac{c}{ab},$$

then

$$f(z) = z + \overline{[\alpha c/(ab)]z(F(a,b;c;z)-1)} \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}.$$

Moreover, f *is fully starlike in* $\mathbf{D}$.

Proof. (a) The proof follows as a consequence of the following simple identity for the first derivative of the hypergeometric function

$$abF(a+1,b+1;c+1;z) = cF'(a,b;c;z).$$

Since

$$\int_0^z F(a+1,b+1;c+1;t)\,dt = \frac{c}{ab}(F(a,b;c;z)-1),$$

the conclusion follows if we apply Theorem 6 (a) and replace a, b, c by $a+1, b+1, c+1$, respectively.

The proof of case (b) follows if we apply Theorem 6 (b) with $a+1$, $b+1$, and $c+1$ instead of a, b and c, respectively. □

Corollary 4. *Let* α *be a complex number such that* $0 < |\alpha| < 1$, b *and* c *positive real numbers such that*

$$c \geq r_1 = \frac{3 + 2b - |\alpha| + \sqrt{|\alpha|^2(4b+1) + |\alpha|(8b+2) + 1}}{2(1-|\alpha|)}. \tag{20}$$

Then $f(z) = z + \overline{(\alpha c/b)[F(1,b;c;z)-1]} \in \mathcal{C}_H^1$.

Proof. Let $f(z) = z + \overline{(\alpha c/b)[F(1,b;c;z)-1]}$. It suffices to prove that, if $c \geq r_1$, then inequality (19) is satisfied with $a = 1$. It is easily seen that $r_1 > b+2$ and, hence, $c > b+2$ holds. Now the inequality (19), with $a = 1$ and a simplification, is equivalent to

$$(1-|\alpha|)(c-r_1)(c-r_2) \geq 0, \tag{21}$$

where r_1 is given by (20) and

$$r_2 = \frac{3+2b-|\alpha|-\sqrt{|\alpha|^2(4b+1)+|\alpha|(8b+2)+1}}{2(1-|\alpha|)}.$$

Since $r_1 \geq r_2$ and, by hypothesis $c \geq r_1$, the inequality (21) holds and thus, by Corollary 3 (a), f belongs to $\mathcal{C}_H^1$, and hence f is close-to-convex in $\mathbf{D}$. □

Corollary 5. *Let m be a positive integer, c a positive real number and $\alpha \in \mathbf{C}$. For $k \in \{0,1\}$, let*

$$f_k(z) = z + \overline{\alpha z^k \sum_{n=0}^{m} \binom{m}{n} \frac{(m-n+1,n)}{(c,n)} \frac{z^{n+1}}{n+1}}$$

(a) *If $0 < |\alpha| < 1$ and*

$$\frac{\Gamma(c)\Gamma(c+2m)}{(\Gamma(c+m))^2} \leq \frac{1}{|\alpha|},$$

then $f_0 \in \mathcal{C}_H^1$.

(b) *If $0 < |\alpha| < 1/2$ and*

$$|\alpha|\frac{\Gamma(c)\Gamma(c+2m)}{(\Gamma(c+m))^2}\left(1+\frac{c+2m}{(m+1)^2}\right) \leq 1+|\alpha|\frac{(c-1)}{(m+1)^2},$$

then $f_1 \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}$.

Proof. Set $a = b = -m$ in Theorem 6 (a) and (b), respectively. □

Example 4. Corollary 5 (b) for $m = 2$ gives the following: if c is a positive real number such that

$$c \geq \frac{24|\alpha|-3+\sqrt{-48|\alpha|^2+168|\alpha|+9}}{6(1-2|\alpha|)},$$

where α is a complex number with $0 < |\alpha| < 1/2$, then

$$f(z) = z + \overline{\alpha\left(z^2 + \frac{2}{c}z^3 + \frac{2}{c(c+1)}\frac{z^4}{3}\right)} \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}.$$

Remark 1. After the manuscript was accepted, the authors in [**13**] disproved Conjecture 1.

Acknowledgments. The authors thank the referee for valuable comments which improved the presentation of the results.

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