

CLASSIFYING CAMINA GROUPS: A THEOREM OF DARK AND SCOPPOLA

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ABSTRACT. Recall that a group G is a Camina group if every nonlinear irreducible character of G vanishes on $G \setminus G'$. Dark and Scoppola classified the Camina groups that can occur. We present a different proof of this classification using Theorem 2, which strengthens a result of Isaacs on Camina pairs. Theorem 2 is of independent interest.

1. Introduction. Throughout this note, G will be a finite group. In this note, we focus on Camina groups. A nonabelian group G is a Camina group if the conjugacy class of every element $g \in G \setminus G'$ is gG' . It is clear from the earliest papers that the study of Camina groups and the more general objects Camina pairs was motivated by finding a common generalization of Frobenius groups and extra-special groups. It is not difficult to see that extra-special groups and Frobenius groups with an abelian Frobenius complement are Camina groups. Thus, questions that seem reasonable to ask are whether there are any other Camina groups and if we can classify the Camina groups.

In [2], Dark and Scoppola stated that they had completed the classification of all Camina groups. In particular, they have the following. G is a Camina group if and only if either:

- (1) G is a Camina p -group of nilpotence class 2 or 3,
- (2) G is a Frobenius group with a cyclic Frobenius complement, or
- (3) G is a Frobenius group whose Frobenius complement is isomorphic to the quaternions.

In fact, the work in [2] is the capstone of several results that combined leading to the classification. Unfortunately, there is a gap in the argument given in [2]. Their argument relies on [1, Theorem 3] which uses [10, Lemma 2.1] whose proof has a gap (ironically, this gap was pointed out in [2]). The referee of this paper has mentioned a way to

2010 AMS *Mathematics subject classification.* Primary 20D10, Secondary 20C15.
Keywords and phrases. Camina groups, Frobenius groups, extra-special p -groups.
Received by the editors on July 21, 2010, and in revised form on December 1, 2011.

fix the argument in [2] so that it does not rely on [10, Lemma 2.1] in any fashion.

Our purpose in this paper is to present a different proof of most of the classifications of Camina groups, and hence, entirely avoid the gaps in [2, 10]. In particular, we prove:

Theorem 1. *If G is Camina group, then one of the following holds:*

1. G is a p -group, or
2. G is a Frobenius group whose complement is either cyclic or quaternion.

In other words, we do not prove that Camina p -groups have nilpotence class at most 3. We refer the reader to Dark and Scoppola, [2], for the proof that Camina p -groups have nilpotence class 2 or 3, and in fact, we will use that result in our work here. Our proof of Theorem 1 is based on the work of Isaacs regarding Camina pairs in [7]. The key to our work is to generalize Lemma 3.1 of [7] where Isaacs proved that if P is a p -group with class at most 2 and P acts on a nontrivial p' -group Q so that $C_P(x) \leq P'$ for all $x \in Q \setminus \{1\}$, then the action is Frobenius and P is either cyclic or isomorphic to the quaternions. Theorem 2 builds on this result. We have replaced the hypothesis that the nilpotence class is at most 2 with the hypothesis that P is a Camina group. Essentially, what we prove is that P cannot be a Camina p -group with nilpotence class 3.

Theorem 2. *Let P be a Camina p -group that acts on a nontrivial p' -group Q so that $C_P(x) \leq P'$ for every $x \in Q \setminus \{1\}$. Then the action of P is Frobenius and P is the quaternions.*

If P and Q satisfy all the hypotheses of Theorem 2 except the first (i.e., we are not assuming that P is a Camina group), then P is called a Frobenius-Wielandt complement. For 2-groups, the generalized quaternion groups are such groups in their Frobenius action. An example of such a group is constructed for each odd prime in [5, Beispiel III.10.15]. These groups have been studied in a number of places (in particular, see [3, 13]). In any case, some additional hypothesis is needed on P for the conclusion Theorem 2 to be true.

Most of the work in proving Theorem 1 is included in the proof of Theorem 2. One motivation for publishing this note is that Herzog, Longobardi and Maj make use of Theorem 2 in their article [4].

2. Proof. In proving Theorem 2, we will strongly use facts regarding Camina p -groups that were proved in [2, 9, 10, 11].

Proof of Theorem 2. We will assume that the lemma is not true and work to find a contradiction. We take P to be a group that violates the conclusion with $|P|$ minimal.

If P has nilpotence class at most 2, then our conclusion is the conclusion of [7, Lemma 3.1], and so, P is not a counterexample. Thus, P has nilpotence class at least 3. MacDonald proved in [10] that a Camina 2-group has nilpotence class 2 and p must be odd. Dark and Scoppola proved in [2] that a Camina p -group has nilpotence class at most 3. Therefore, P must have nilpotence class 3.

Note that we may assume that Q has no proper nontrivial subgroups that are invariant under the action of P since we may replace Q by any such subgroup. Since Q has no proper nontrivial subgroups that are invariant under the action of P , it follows that Q is an elementary abelian q -group for some prime $q \neq p$. Let $Z = \mathbf{Z}(P)$. Suppose that there exists some element $x \in Q \setminus \{1\}$ with $\mathbf{C}_P(x) \cap Z > 1$. Let $Y = \mathbf{C}_P(x) \cap Z$, and let $C = \mathbf{C}_Q(Y)$. We have $x \in C$, and since Y is normal in P , it follows that P stabilizes C . This implies that $C = Q$. Hence, P/Y acts on Q . Notice that $\mathbf{C}_{P/Y}(x) = \mathbf{C}_P(x)/Y \leq P'/Y$ for every $x \in Q \setminus \{1\}$. Also, since Y is central and P has nilpotence class 3, it follows that P/Y is not abelian, and hence, P/Y is a Camina p -group. Applying the inductive hypothesis, we have that P/Y is the quaternions. Since p is odd, this is a contradiction.

We now have $\mathbf{C}_P(x) \cap Z = 1$ for all $x \in Q \setminus \{1\}$. This implies that Z acts Frobeniusly on Q . Hence, Z is cyclic. Therefore, we may view Q as an irreducible module for $F[P]$ where F is the field of q elements. Let K be the ring of $F[P]$ -endomorphisms of Q , and by Schur's lemma, K is a division ring. It is finite, so K is a field, and we can view Q as a $K[P]$ -module. (Notice that we are not changing Q as a set, we just view it differently.) Now, Q will be absolutely irreducible as a $K[P]$ -module. By [6, Theorem 9.14], Q gives rise to an irreducible representation of

algebraically closed field that contains K . Hence, Q will determine an irreducible q -Brauer character for G (see [6, page 264]). Since q does not divide $|P|$, the irreducible q -Brauer characters of P are just the ordinary irreducible characters of P [6, Theorem 15.13]. Thus, Q corresponds to a complex irreducible character χ of P . Notice that $\ker(\chi) = \mathbf{C}_P(Q)$ is contained in the centralizer of all elements of Q , so $\ker(\chi) \cap Z = 1$, and this implies that $\ker(\chi) = 1$. Since G is nonabelian, this implies that $\chi(1) \geq p$.

Now, P is an M -group, so there is a subgroup R and a linear character $\lambda \in \text{Irr } R$ so that $\lambda^P = \chi$. Also, there is a $K[R]$ -module W corresponding to λ so that $W^P = Q$. Since λ is linear, it follows that $R/\ker(\lambda) = R/\mathbf{C}_R(W)$ is cyclic. We can write $Q = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ where $r = |P : R| = \chi(1)$ and $W_1 \cong W$. In [8], we saw that irreducible characters of Camina p -groups with nilpotence class 3 whose kernels do not contain the center are fully-ramified with respect to the center. It follows that χ is fully-ramified with respect to P/Z .

We claim that it suffices to find an element $g \in P \setminus RP'$ so that $g^p = 1$. Suppose such an element g exists. We can relabel the W_i so that for $i = \{1, \dots, p\}$, we have $W_i = W_1 g^{i-1}$. Fix $w_1 \in W_1 \setminus \{0\}$, and set $w_i = w_1 g^{i-1}$. It follows that $w_i g = w_{i+1}$ for $1 \leq i \leq p-1$ and $w_p g = w_1$. Now, take $x \in Q$ so that $x = (w_1, \dots, w_p, 0, \dots, 0)$. Notice that $x \neq 0$ and g just permutes the w_i 's, so g will centralize x . Since g is not in P' , this will violate $\mathbf{C}_P(x) \leq P'$.

Before we work to find the element x , we gather some information regarding R . Notice that $\mathbf{C}_R(W)$ is contained in the centralizers of elements of Q whose only nonzero component lies in W . It follows that $\mathbf{C}_R(W) \leq P'$ and $\mathbf{C}_R(W) \cap Z = 1$. Since χ is induced from R , we have $Z \leq R$.

We know from [9] that $|P : P'| = |P' : Z|^2$ and so, $|P : Z| = |P' : Z|^3$. Since χ is fully-ramified with respect to P/Z , it follows that $|P : R| = \chi(1) = |P : Z|^{1/2} = |P' : Z|^{3/2}$. We also have $|P' : Z|^2 = |P : P'|$. This implies $|P : R| = |P : P'|^{3/4}$. By [9], P/P' is elementary abelian. Since RP'/P' is cyclic, we conclude that $|RP' : P'| = p$. We now have that $|P : R| \geq |P : RP'| = |P : P'|/p$ and hence, $|P : P'| \leq p^4$. From [9], we know that $|P' : Z| \geq p^2$, so $|P : P'| \geq p^4$. We conclude that $|P : P'| = p^4$. This implies that $|P : R| = p^3 = |P : RP'|$, and hence, $P' \leq R$ and $|R : P'| = p$. By [11], we know that P' is elementary

abelian. Since $P'/\mathbf{C}_R(W)$ is a subgroup of $R/\mathbf{C}_R(W)$, it is cyclic and, hence, has order p . It follows that $R/\mathbf{C}_R(W)$ is cyclic of order at least p^2 .

Since R/P' is cyclic and P'/Z is central in P/Z , so R/Z is cyclic-by-central. This implies that R/Z is abelian. Hence, $R' \leq Z$. We also know that $R' \leq \mathbf{C}_R(W)$. We conclude that $R' = 1$, and R is abelian. Since $P' \leq R$, this implies that $RP' = R$ and R centralizes P' . Recall that P' is elementary abelian, and we saw in the previous paragraph that R does not have exponent p . Since R is abelian, the set $\Omega_1(R)$ of elements of order p in R is a subgroup of R , and $P' \leq \Omega_1(R) < R$. We saw that $|R : P'| = p$, so $\Omega_1(R) = P'$. It follows that the elements in $R \setminus P'$ all have order at least p^2 .

We now work to obtain the element $x \in P \setminus R$ so that $x^p = 1$. (Notice that $RP' = R$, so this is the previous statement.) We take an element $a \in P \setminus \mathbf{C}_P(P')$. Notice that $R \leq \mathbf{C}_P(P')$, so if $a^p = 1$, then we take $x = a$. Hence, we may suppose that $a^p \neq 1$. By [11], we know that P/Z has exponent p , so $a^p = z \in Z$. Since P is a Camina group and $a \in P \setminus P'$, it follows that the conjugacy class of a is aP' . This implies that $|P : \mathbf{C}_P(a)| = |P'|$, and so, $|\mathbf{C}_P(a)| = |P : P'| = p^4$. We know that P' is not centralized by a , so $\mathbf{C}_{P'}a < P'$. Now, since Z is cyclic and P' is elementary abelian, we deduce that $|Z| = p$. Also, since $|P : P'| = p^4$ and $|P : P'| = |P' : Z|^2$, we deduce that $|P' : Z| = p^2$, and hence, $|P'| = p^3$. In other words, $|\mathbf{C}_{P'}(a)| \leq p^2$, and $|\langle a \rangle \mathbf{C}_{P'}(a)| \leq p^3$. Therefore, there exists an element $b \in \mathbf{C}_P(a) \setminus \langle a \rangle \mathbf{C}_{P'}(a)$.

First, suppose that $b^p = 1$. If $b \in R$, then $b \in \Omega_1(R) = P'$, and so, $b \in \mathbf{C}_P(a) \cap P' = \mathbf{C}_{P'}(a)$, which is a contradiction. Thus, $b \in P \setminus R$, and we take $x = b$. Hence, we may assume that $b^p \neq 1$. We know that $b^p \in Z$, so $b^p = z^i$ for some integer $1 \leq i \leq p - 1$. There is an integer j so that $ij \equiv -1 \pmod{p}$. We take $x = ab^j$. Notice that $x \in \mathbf{C}_P(a)$ and $x^p = (ab^j)^p = a^p b^{jp} = z z^{ij} = z z^{-1} = 1$. If $x \in R$, then $x \in \Omega_1(R) = P'$, and hence, $x \in \mathbf{C}_{P'}(a)$. This implies that $ab^j \in \mathbf{C}_{P'}(a)$, and thus, $b^j \in a^{-1} \mathbf{C}_{P'}(a)$. Since j is coprime to p , we end up with $b \in \langle a \rangle \mathbf{C}_{P'}(a)$, which is a contradiction. Therefore, $x \in P \setminus R$, and we have found the desired element. \square

In [7, Theorem 2.1], Isaacs proved that if (G, K) is a Camina pair where G/K is nilpotent, then either G is a Frobenius group with

Frobenius kernel K or G/K is a p -group for some prime p , and if G/K is a p -group, then G has a normal p -complement M and $\mathbf{C}_G(m) \leq K$ for all $m \in M \setminus \{1\}$. We do not define Camina pairs in this paper; however, for our purposes, it is enough to know that G is a Camina group if and only if (G, G') is a Camina pair. In [13], Ren has noted that Theorem 2.1 of [7] together with Lemma 3.1 of [7] imply that, if G is a Camina group, then one of the following occurs:

1. G is Frobenius group with G' its Frobenius kernel.
2. G is a Camina p -group.
3. $G = RP$ where P is a Sylow p -subgroup of G for some prime p and R is a normal p -complement for G with $R < G'$. In addition, if P has nilpotence class 2, then P is the quaternions and G is a Frobenius group whose Frobenius kernel is R .

Essentially, we will prove using Theorem 2 that if (3) occurs, then P must have nilpotence class 2. This observation underlies our argument.

Proof of Theorem 1. It is easy to see that each of the groups mentioned are Camina groups. Thus, we will assume that G is a Camina group and prove that it is one of the groups listed. As we have seen, [7, Theorem 2.1] can be restated in terms of Camina groups as saying that, if G is a Camina group, then either G is a Frobenius group with G' as its Frobenius kernel or G/G' is a p -group for some prime p and G has a normal p -complement Q where $\mathbf{C}_G(x) \leq G'$ for all $x \in Q \setminus \{1\}$.

If G is a Frobenius group with G' as its Frobenius kernel, then the Frobenius complements for G must be abelian, and hence, cyclic. Thus, we may assume that G/G' is a p -group for some prime p . If G is a p -group, then we are done since we know that Camina p -groups have nilpotence class 2 or 3 by [2]. Thus, we may assume that G is not a p -group, and so, G has a nontrivial normal p -complement Q . Let P be a Sylow p -subgroup of G . Notice that $G' = P'Q$, and so, $P \cap G' = P'$. We have $\mathbf{C}_P(x) \leq P \cap G' = P'$ for all $x \in Q \setminus \{1\}$. Notice that $P \cong G/Q$ is either abelian or a Camina group. If P is abelian, we have $\mathbf{C}_P(x) = 1$ for all $x \in Q \setminus \{1\}$ and $Q = G'$. In particular, G is a Frobenius group with G' as its Frobenius group, and we are done as before. Finally, we have the case where P is a Camina group where $\mathbf{C}_P(x) \leq P'$ for all $x \in Q \setminus \{1\}$. We now apply Theorem 2 to see

that P acts Frobeniusly on Q and is the quaternions. Therefore, G is a Frobenius group with Frobenius kernel Q and Frobenius complement P where P is isomorphic to the quaternions. \square

Acknowledgments. We would like to thank David Chillag, Marcel Herzog, Marty Isaacs and the referee for their useful comments on this paper.

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