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COMPARISON OF SPECTRA OF ABSOLUTELY REGULAR DISTRIBUTIONS AND APPLICATIONS

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ABSTRACT. We study the reduced Beurling spectra $sp_{\mathcal{A},V}(F)$ of functions $F \in L^1_{\text{loc}}(\mathbf{J}, X)$ relative to certain function spaces $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ and $V \subset L^1(\mathbf{R})$ and compare them with other spectra including the weak Laplace spectrum. Here $\mathbf{J}$ is $\mathbf{R}_+$ or $\mathbf{R}$ and X is a Banach space. If F belongs to the space $\mathcal{S}'_{ar}(\mathbf{R}, X)$ of absolutely regular distributions and has uniformly continuous indefinite integral with $0 \notin sp_{\mathcal{A},\mathcal{S}(\mathbf{R})}(F)$ (for example, if F is slowly oscillating and $\mathcal{A}$ is $\{0\}$ or $C_0(\mathbf{J}, X)$), then F is ergodic. If $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ and $M_h F(\cdot) = (1/h)\int_0^h F(\cdot + s)\, ds$ is bounded for all $h > 0$ (for example, if F is ergodic) and if $sp_{C_0(\mathbf{R},X),\mathcal{S}}(F) = \varnothing$, then $F * \psi \in C_0(\mathbf{R}, X)$ for all $\psi \in \mathcal{S}(\mathbf{R})$. We show that Tauberian theorems for Laplace transforms follow from results about reduced spectra. Our results are more general than previous ones, and we demonstrate this through examples.

1. Introduction. The goal of this paper is to study the asymptotic behavior of certain locally integrable functions $F : \mathbf{J} \to X$ where $\mathbf{J}$ denotes $\mathbf{R}$ or $\mathbf{R}_+$ and X is a complex Banach space. Such a study has a long history. It is motivated by Loomis's theorem (see [**24**, Theorem 4, pages 92, 97] and [**25**]) which gives spectral conditions under which a function $F : \mathbf{R} \to \mathbf{C}$ is almost periodic and by the Tauberian theorem of Ingham (see [**5**, Theorem 4.9.5, page 326] and [**22**]) which gives conditions under which $\lim_{t\to\infty} F(t) = 0$. Many notions of the spectrum of a function have since been introduced in order to obtain (vector valued) extensions of these results and we will review and compare some of these in this paper. In particular, we develop the reduced spectrum $sp_{\mathcal{A}}(F)$ of F relative to various closed subspaces $\mathcal{A}$ of $BUC(\mathbf{J}, X)$, a spectrum that was introduced before in this context (see [**2**, page 371], [**6, 7, 8, 17**], [**24**, Chapter 6.4, page

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91]). Typically, the reduced spectrum gives stronger results than other spectra, as we shall see. The style of definition and its properties are similar to those of the Beurling spectrum (see [**6**, Theorem 4.1.4]) and it is widely applicable. For $F \in BUC(\mathbf{J}, X)$ and $\mathcal{A} \subset BUC(\mathbf{J}, X)$, there is also an operator theoretical approach using C_0-semigroups and groups (see [**19**], for example). Attempts by Minh [**26**] to define a reduced spectrum of bounded, not necessarily uniformly continuous, functions $F \in BC(J, X)$ using the operator theoretical approach failed even for the case $\mathbf{J} = \mathbf{R}$ (see [**27**]).

The reduced spectrum was replaced by the local Arveson spectrum in [**2**] and [**5**, page 296] and by the Laplace spectrum $sp^{\mathcal{L}}(F)$ in [**3, 14, 15**]. To extend the theory, a smaller spectrum, the half-line spectrum $sp_+(F)$, was introduced in [**4**, page 474]. Then Chill introduced a still smaller spectrum, the weak Laplace spectrum $sp^{w\mathcal{L}}(F)$ in [**17**, page 25] and [**18**, Definition 1.1]. Typically, $sp^{w\mathcal{L}}(F) \subset sp_+(F) \subset sp^{\mathcal{L}}(F)$.

The important Theorem 5 of Chill and Fasangova [**19**] (see also [**10**, Theorem 3.10]), establishing that the reduced spectrum coincides with the local Arveson spectrum for the group induced by the shifts on a quotient space, shows that some results of Arendt and Batty in [**2, 3**] using the local Arveson spectrum follow from earlier results in [**6, 7**]. This point is noted in [**19**, Theorem 10]. Similarly, noting that $sp_{\mathcal{A}}(F) \subset sp_{C_0}(F) \subset sp^{w\mathcal{L}}(F) \subset sp^{\mathcal{L}}(F)$ (Proposition 4.2), the general Tauberian theorems [**3**, Theorem 2.3] and [**15**, Theorem 4.1] using the Laplace spectrum $sp^{\mathcal{L}}(F)$ are consequences of Theorems 4.2.5 and 4.2.6 of [**6**] which use the reduced spectrum. The latter theorems are stronger than the main Tauberian theorem of [**15**], and their proofs are simpler. A difficulty with $sp^{\mathcal{L}}(F)$ and $sp^{w\mathcal{L}}(F)$ is that it is unclear whether they satisfy the useful property $sp^*(F*g) \subset sp^*(F) \cap \operatorname{supp} \widehat{g}$, for appropriate g, even when F is bounded; but see Proposition 3.4 (i) for reduced spectra.

In (3.4) we define a more general spectrum $sp_{\mathcal{A},V}(F)$, the reduced spectrum of $F \in L^1_{\mathrm{loc}}(\mathbf{J}, X)$ relative to $(\mathcal{A}, V)$, where $V \subset L^1(\mathbf{R})$, a spectrum closely related to the one given in (3.4*) which was first studied in [**10**]. We are able to strengthen further the improvements made by Chill ([**17**, Lemma 1.16] and [**18**, Proposition 1.3, Theorem 1.5, Corollary 1.7]). In particular, we replace $sp^{w\mathcal{L}}(F)$ by $sp_{\mathcal{A},V}(F)$ for some $\mathcal{A} \supset C_0(\mathbf{R}_+, X)$. We are able to consider functions whose Fourier transforms are not regular distributions (see Example 3.12) and avoid some geometrical restrictions on X that were imposed in [**17**, Theorem 1.23] and [**18**, Proposition 2.1] for example. Moreover, our methods lead to new results for the weak Laplace spectrum (see Theorem 4.3). Finally, spectral criteria for solutions of evolution equations on $\mathbf{R}_+$ or $\mathbf{R}$ are readily related to reduced spectra (see Theorem 4.6 and [**7**, (1.7)]).

In Section 2 we describe our notation and prove some preliminary results. We are particularly interested in functions $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, the space of absolutely regular distributions (see (2.1)).

In Section 3 we study $sp_{\mathcal{A},V}(F)$, where V is one of the spaces $\mathcal{D} = \mathcal{D}(\mathbf{R})$, $\mathcal{S} = \mathcal{S}(\mathbf{R})$ or $L^1 = L^1(\mathbf{R})$. If $F \in L^\infty(\mathbf{R}, X)$ and $\mathcal{A} = 0 := \{0\}$, then by [**10**, (3.3)] (see also [**5**, Proposition 4.8.4, (4.26)], [**28**, 0.5, page 19], [**29**, page 183]),

$$sp_{0,\mathcal{S}}(F) = sp_{0,L^1}(F) = sp^{\mathcal{C}}(F) = sp^B(F) = sp_0(F), \tag{1.1}$$

where $sp^{\mathcal{C}}(F)$ is the Carleman spectrum and $sp^B(F)$ is the Beurling spectrum. In Proposition 3.1 (i) we prove that our definition (3.4) coincides with (3.4*) for the spaces $V \in \{\mathcal{D}, \mathcal{S}, L^1\}$. In (3.3) and Proposition 3.2, we study the conditions imposed on $\mathcal{A}$ and relate them to others in the literature, in particular to the translation-biinvariance used in [**17**, Definition 1.2, page 17] and [**3**, Section 2]. In Remark 3.3, we show that the converse of Proposition 3.2 (i) is false and that $F \in \mathcal{A}$ does not imply in general that $sp_{\mathcal{A}}(F) = \varnothing$. We develop some basic properties of the reduced spectrum in Proposition 3.4. Our main results are stated in Theorems 3.5, 3.6, 3.7 and 3.8.

In Theorems 3.5 and 3.6 we prove ergodicity results for functions $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$. If $\mathbf{J} = \mathbf{R}$, $0 \notin sp_{0,\mathcal{S}}(F)$ and the indefinite integral $PF \in UC(\mathbf{R}, X)$, then $PF \in BUC(\mathbf{R}, X)$ and F is ergodic. For a variety of classes $\mathcal{A}$ including $C_0(\mathbf{J}, X)$, $EAP_0(\mathbf{J}, X)$, $EAP(\mathbf{J}, X)$, $AAP(\mathbf{J}, X)$ and $AP(\mathbf{R}, X)$, if $0 \notin sp_{\mathcal{A},\mathcal{S}}(F)$, then F is ergodic.

Theorem 3.7 deals with functions $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$ with $sp_{\mathcal{A},\mathcal{S}}(F)$ countable. It is a generalized Tauberian theorem providing spectral conditions under which F has various types of asymptotic behavior. For example, (Theorem 3.7 (v)), if $M_h F$ is bounded for each $h > 0$, $sp_{\mathcal{A},\mathcal{S}}(F)$ is countable and $\gamma_{-\omega} F$ is ergodic for each $\omega \in sp_{\mathcal{A},\mathcal{S}}(F)$, then $(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}$ for all $\psi \in \mathcal{S}(\mathbf{R})$. (For the definition of $\overline{F}$, see (2.5)). Versions of Theorem 3.6, Theorem 3.7 (i)–iv) and Corollary 3.9

are already known when $\mathbf{J} = \mathbf{R}_+$ and $sp_{\mathcal{A},\mathcal{S}}(F)$ is replaced by the larger spectrum $sp^{w\mathcal{L}}(F)$ (see Remark 4.4 (i)). Corollary 3.10 states that if $F \in L^1_{\text{loc}}(\mathbf{J}, X)$ with $sp_{C_0(\mathbf{J},X),\mathcal{D}}(F) = \varnothing$ and, if the convolution $(\overline{F} * \psi) \mid \mathbf{J}$ is uniformly continuous for some $\psi \in \mathcal{D}(\mathbf{R})$, then $\overline{F} * \psi \in C_0(\mathbf{R}, X)$. For the case $\mathbf{J} = \mathbf{R}$, Chill [**18**, Proposition 3.1] obtained this same conclusion under the assumptions that $F \in L^1_{\text{loc}}(\mathbf{R}, X)$ and $\widehat{F} \in \mathcal{S}'_{ar}(\mathbf{R}, X)$. In particular, if $F \in L^p(\mathbf{R}, X)$ where $1 \leq p < \infty$, then F satisfies the assumptions of Corollary 3.10. However, as is well known, when $p > 2$, there are functions $F \in L^p(\mathbf{R}, X)$ for which $\widehat{F}$ is not a regular distribution and so the result of [**18**] does not apply. Even when $1 < p \leq 2$ special geometry on X is required in order that every $F \in L^p(\mathbf{R}, X)$ has a Fourier transform which is regular.

In Proposition 4.1, we establish some properties shared by the weak Laplace, Laplace and Carleman spectra. In Proposition 4.2, we prove the inclusion $sp_{\mathcal{A},V}(F) \subset sp^{w\mathcal{L}}(F)$ for $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$ and $\mathcal{A} \supset C_0(\mathbf{R}_+, X)$. This enables us to prove a new Tauberian theorem (Theorem 4.3) and also to deduce the main Tauberian results of Chill mentioned above (see Remark 4.4). In Example 4.5, we demonstrate how to use Proposition 4.1 to calculate Laplace spectra. Finally, in Theorem 4.6, we obtain a spectral condition satisfied by bounded mild solutions of the evolution equation $du(t)/dt = Au(t) + \phi(t)$, $u(0) \in X$, $t \in \mathbf{J}$, where A is a closed linear operator on X and $\phi \in L^\infty(\mathbf{J}, X)$. This generalizes earlier results where it is assumed that $u, \phi \in BUC(\mathbf{J}, X)$ (see [**5**, Proposition 5.6.7] and [**7**, Theorem 3.3, Corollary 3.4]).

2. Notation, definitions and preliminaries. Throughout the paper, $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R}_- = (-\infty, 0]$, $\mathbf{J} \in \{\mathbf{R}_+, \mathbf{R}\}$, $\mathbf{N} = \{1, 2, \dots\}$, $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ and $\mathbf{C}_- = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda < 0\}$. By X we denote a complex Banach space. If Y and Z are locally convex topological spaces, $L(Y, Z)$ denotes the space of all bounded linear operators from Y to Z. The Schwartz spaces of test functions and rapidly decreasing functions are denoted by $\mathcal{D}(\mathbf{R})$ and $\mathcal{S}(\mathbf{R})$, respectively. Then $\mathcal{D}'(\mathbf{R}, X) = L(\mathcal{D}(\mathbf{R}), X)$ is the space of X-valued distributions and $\mathcal{S}'(\mathbf{R}, X) = L(\mathcal{S}(\mathbf{R}), X)$ is the space of X-valued tempered distributions (see [**5**, page 480], [**33**, page 149] for $X = \mathbf{C}$). The space of regular distributions $\mathcal{S}'_r(\mathbf{R}, X) = \mathcal{S}'(\mathbf{R}, X) \cap L^1_{\text{loc}}(\mathbf{R}, X)$.

We make much use of the spaces

(2.1)
$$\mathcal{S}'_{ar}(\mathbf{J},X) = \{H \in L^1_{\rm loc}(\mathbf{J},X) : H\varphi \mid \mathbf{J} \in L^1(\mathbf{J},X) \text{ for all } \varphi \in \mathcal{S}(\mathbf{R})\}.$$

The space $\mathcal{S}'_{ar}(\mathbf{R},X)$ is the space of absolutely regular tempered distributions defined in [**23**, page 137] when $X = \mathbf{C}$. By a simple application of the closed graph theorem, $\mathcal{S}'_{ar}(\mathbf{R},X) \subset \mathcal{S}'_r(\mathbf{R},X)$ (see [**23**, Lemma]).

The action of an element $S \in \mathcal{D}'(\mathbf{R},X)$ or $\mathcal{S}'(\mathbf{R},X)$ on $\varphi \in \mathcal{D}(\mathbf{R})$ or $\mathcal{S}(\mathbf{R})$ is denoted by $\langle S,\varphi\rangle$. If F is an X-valued function defined on $\mathbf{J}$ and $s \in \mathbf{J}$, then F_s, $\Delta_s F$, $|F|$ stand for the functions defined on $\mathbf{J}$ by $F_s(t) = F(t+s)$, $\Delta_s F(t) = F_s(t) - F(t)$ and $|F|(t) = \|F(t)\|$. Also $\|F\|_\infty = \sup_{t\in\mathbf{J}} \|F(t)\|$. If $F \in L^1_{\rm loc}(\mathbf{J},X)$ and $h > 0$, then PF, M_hF and $\check{F}$ (when $\mathbf{J} = \mathbf{R}$) denote the *indefinite integral, mollifier* and *reflection* of F defined respectively by $PF(t) = \int_0^t F(s)\,ds$, $M_hF(t) = (1/h)\int_0^h F(t+s)\,ds$ for $t \in \mathbf{J}$ and $\check{F}(t) = F(-t)$ for $t \in \mathbf{R}$. For $g \in L^1(\mathbf{R})$ and $F \in L^\infty(\mathbf{R},X)$ or $g \in L^1(\mathbf{R},X)$ and $F \in L^\infty(\mathbf{R})$ the *Fourier transform* $\widehat{g}$ and *convolution* $F * g$ are defined respectively by $\widehat{g}(\omega) = \int_{-\infty}^\infty \gamma_{-\omega}(t)\,g(t)\,dt$ and $F*g(t) = \int_{-\infty}^\infty F(t-s)\,g(s)\,ds$, where $\gamma_\omega(t) = e^{i\,\omega t}$ and $\omega \in \mathbf{R}$. The *Fourier transform* of $H \in \mathcal{S}'(\mathbf{R},X)$ is the tempered distribution $\widehat{H}$ defined by

$$\langle \widehat{H},\varphi\rangle = \langle H,\widehat{\varphi}\rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}). \tag{2.2}$$

Set $\widehat{\mathcal{D}}(\mathbf{R}) = \{\widehat{\varphi} : \varphi \in \mathcal{D}(\mathbf{R})\} \subset \mathcal{S}(\mathbf{R})$. The *Fourier transform* of $F \in L^1_{\rm loc}(\mathbf{R},X)$ is the distribution $\widehat{F} \in L(\widehat{\mathcal{D}}(\mathbf{R}),X)$ defined by

$$\langle \widehat{F},\psi\rangle = \langle F,\widehat{\psi}\rangle \quad \text{for all } \psi \in \widehat{\mathcal{D}}(\mathbf{R}). \tag{2.3}$$

Throughout the paper all integrals are Lebesgue-Bochner integrals (see [**5**, page 6], [**20**, page 318], [**21**, page 76]). All convolutions are understood as convolutions of functions defined on $\mathbf{R}$. Given $F \in W(\mathbf{J},X)$ where

$$W(\mathbf{J},X) \in \{L^1_{\rm loc}(\mathbf{J},X), \mathcal{S}'_{ar}(\mathbf{J},X), L^\infty(\mathbf{J},X)\}, \tag{2.4}$$

we denote by $\overline{F} : \mathbf{R} \to X$ the function given by

$$\overline{F}|\mathbf{J} = F \quad \text{and, if } \mathbf{J} = \mathbf{R}_+,\ \overline{F}|(-\infty,0) = 0. \tag{2.5}$$

Then $\overline{F} \in W(\mathbf{R}, X)$. In addition, if $g \in L_c^\infty(\mathbf{R}) = \{f \in L^\infty(\mathbf{R}) : f$ has compact support$\}$, then for some constant t_g,

$$\overline{F} * g \in W(\mathbf{R}, X) \cap C(\mathbf{R}, X) \quad \text{and, if } \mathbf{J} = \mathbf{R}_+, \quad \overline{F} * g(t) = 0 \quad \text{for all } t \le t_g. \tag{2.6}$$

It follows that if $h > 0$ and $s_h = (1/h)\chi_{(-h,0)}$, where $\chi_{(-h,0)}$ is the characteristic function of $(-h, 0)$, then

$$\overline{F} * s_h \in W(\mathbf{R}, X) \cap C(\mathbf{R}, X), \quad M_h F = (\overline{F} * s_h) \mid \mathbf{J} \text{ and, if } \mathbf{J} = \mathbf{R}_+, \ \overline{F} * s_h(t) = 0 \text{ for all } t \le -h. \tag{2.7}$$

We use convolutions of functions $F \in W = W(\mathbf{J}, X)$ and $g \in V = V(\mathbf{R}) \in \{\mathcal{D}(\mathbf{R}), \mathcal{S}(\mathbf{R}), L^1(\mathbf{R})\}$, with

$$V = \mathcal{D}(\mathbf{R}) \quad \text{if } W = L^1_{\text{loc}}(\mathbf{J}, X), \ V = \mathcal{S}(\mathbf{R}) \text{ if } W = \mathcal{S}'_{ar}(\mathbf{J}, X) \tag{2.8}$$

and

$$V = L^1(\mathbf{R}) \quad \text{if } W = L^\infty(\mathbf{J}, X).$$

The following properties of the convolution are repeatedly used (see [**30**, 7.19 Theorem (a), (b), pages 179–180] and [**33**, page 156, (4)] when $X = \mathbf{C}$):

If $F \in W(\mathbf{J}, X)$ and $\varphi \in V(\mathbf{R})$ where (2.4) and (2.8) are satisfied, then

$$\overline{F} * \varphi \in W(\mathbf{R}, X) \cap C(\mathbf{R}, X). \tag{2.9}$$

Indeed, the cases $W = L^1_{\text{loc}}(\mathbf{J}, X)$ and $W = L^\infty(\mathbf{J}, X)$ are obvious. If $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$, then $|\overline{F}| \in \mathcal{S}'_{ar}(\mathbf{R}, \mathbf{C})$. By [**23**, Theorem (b)] there is an integer $k \in \mathbf{N}$ such that

$$\frac{|\overline{F}|}{w_k} = f \in L^1(\mathbf{R}), \quad \text{where } w_k(t) = (1 + t^2)^k. \tag{2.10}$$

Using (2.10), we easily conclude (2.9).

Moreover, if $\psi \in V(\mathbf{R})$ or $\psi \in L^\infty_c(\mathbf{R})$, then

$$(\overline{F} * \varphi) * \psi = \overline{F} * (\varphi * \psi) = (\overline{F} * \psi) * \varphi. \tag{2.11}$$

Now let $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ and $\varphi \in \mathcal{S}(\mathbf{R})$ be such that $\widehat{\varphi} \in \mathcal{D}(\mathbf{R})$ and $\widehat{\varphi} = 1$ on $[-\delta, \delta]$ for some $\delta > 0$. Then

$$0 \notin sp_{0,\mathcal{S}}(F - F * \varphi). \tag{2.12}$$

Indeed, if $\psi \in \mathcal{S}(\mathbf{R})$, $\operatorname{supp} \widehat{\psi} \subset [-\delta, \delta]$ and $\widehat{\psi}(0) = 1$, then $\varphi * \psi = \psi$. So, by (2.11), $(F - F * \varphi) * \psi = 0$.

For the benefit of the reader we include the proofs of the following elementary but necessary results.

Lemma 2.1. *Let $f \in L^1(\mathbf{R})$ with $\widehat{f} \neq 0$ on a compact set K. Then there exists $g \in L^1(\mathbf{R})$ such that $\widehat{g} \cdot \widehat{f} = 1$ on K. Moreover, one can choose g such that $\widehat{g}$ has compact support and, if $f \in \mathcal{S}(\mathbf{R})$, with $g \in \mathcal{S}(\mathbf{R})$.*

Proof. Choose a bounded open set U such that $K \subset U$ and $\widehat{f} \neq 0$ on $\overline{U}$ the closure of U. By [**16**, Proposition 1.1.5 (b), page 22], there is $k \in L^1(\mathbf{R})$ such that $\widehat{k} \cdot \widehat{f} = 1$ on $\overline{U}$. Now, choose $\varphi \in \mathcal{D}(\mathbf{R})$ such that $\varphi = 1$ on K and $\operatorname{supp} \varphi \subset \overline{U}$. Also choose $\psi \in \mathcal{S}(\mathbf{R})$ such that $\widehat{\psi} = \varphi$ and take $g = k * \psi$. Then $\widehat{g}$ has compact support and, if $f \in \mathcal{S}(\mathbf{R})$, then $\widehat{g} \in \mathcal{D}(\mathbf{R})$ and so $g \in \mathcal{S}(\mathbf{R})$. ◻

In the following lemma ψ will denote an element of $\mathcal{S}(\mathbf{R})$ with the properties:

$$\widehat{\psi} \quad \text{has compact support, } \widehat{\psi}(0) = 1 \text{ and } \psi \text{ is non-negative.}$$

An example of such ψ is given by $\psi = \widehat{\varphi}^2$, where $\varphi(t) = ae^{1/(t^2-1)}$ for $|t| \leq 1$ and $\varphi = 0$ elsewhere on $\mathbf{R}$ for some suitable constant a.

Lemma 2.2. (i) *The sequence $\psi_n(t) = n\,\psi(n\,t)$ is an approximate identity for the space of uniformly continuous functions $UC(\mathbf{R}, X)$, that is* $\lim_{n\to\infty} \|u * \psi_n - u\|_\infty = 0$ *for all $u \in UC(\mathbf{R}, X)$.*

(ii) $\lim_{h\searrow 0} \|M_h u - u\|_\infty = 0$ *for all* $u \in UC(\mathbf{J}, X)$. *In particular if* $M_h u \in BUC(\mathbf{J}, X)$ *for all* $h > 0$, *then* $u \in BUC(\mathbf{J}, X)$.

Proof. (i) Given $u \in UC(\mathbf{R}, X)$ and $\varepsilon > 0$ there exists $k > 0$ such that $\|u(t+s) - u(t)\| \le k|s| + \varepsilon$ for all $t, s \in \mathbf{R}$. In particular $u \in \mathcal{S}'_{ar}(\mathbf{R}, X)$. Also, $u * \psi_n(t) - u(t) = \int_{-\infty}^{\infty} [u(t - (s/n)) - u(t)]\psi(s)\, ds$ which gives $\|u * \psi_n - u\|_\infty \le (k/n) \int_{-\infty}^{\infty} |s|\psi(s)\, ds + \varepsilon \int_{-\infty}^{\infty} \psi(s)\, ds$ and (i) follows.

(ii) Since $\|M_h u - u\|_\infty \le \sup_{t\in\mathbf{J}, 0\le s\le h} \|u(t+s) - u(t)\|$, part (ii) follows. $\square$

Lemma 2.3. *Let* $F \in W(\mathbf{R}, X)$ *and* $g \in V(\mathbf{R})$ *with* (2.4) *and* (2.8) *satisfied.*

(i) *If* $F|\mathbf{R}_- = 0$, *then* $(F * g)|\mathbf{R}_- \in C_0(\mathbf{R}_-, X)$.

(ii) *If* $F|\mathbf{R}_+ = 0$, *then* $(F * g)|\mathbf{R}_+ \in C_0(\mathbf{R}_+, X)$. $\square$

Proof. (i) The cases $W = L^1_{\mathrm{loc}}(\mathbf{R}, X)$ and $W = L^\infty(\mathbf{R}, X)$ can be shown by simple calculations. If $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, then from (2.10) $|F|/w_k = f \in L^1(\mathbf{R})$ for some $w_k(t) = (1 + t^2)^k$. Since $\varphi \in \mathcal{S}(\mathbf{R})$, $\|w_k\varphi\|_\infty = c_k < \infty$. It follows that $\|F * \varphi(t)\| = \| \int_0^\infty \varphi(t - s)F(s)\, ds\| \le \int_0^\infty |\varphi|(t-s)|F|(s)\, ds \le c_k \int_0^\infty w_k(s)/w_k(t-s) f(s)\, ds$. Since $w_k(s)/w_k(t-s) \le 1$ for each $t \le 0$, $s \ge 0$ and $\lim_{t\to-\infty} w_k(s)/w_k(t-s) = 0$ for each $s \ge 0$, it follows that $\lim_{t\to-\infty} \|F * \varphi(t)\| = 0$ by the Lebesgue convergence theorem. By (2.9), the result follows.

(ii) This follows by applying part (i) to $\check{F}$. $\square$

3. Reduced spectra for regular distributions. In this section we introduce the reduced spectrum $sp_{\mathcal{A},V}(F)$ of a function $F \in L^1_{\mathrm{loc}}(\mathbf{J}, X)$ relative to $\mathcal{A}, V$, where $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ and $V \subset L^1(\mathbf{R})$. We usually impose the following conditions on $\mathcal{A}$.

(3.1) $\mathcal{A}$ is a closed subspace of $L^\infty(\mathbf{J}, X)$ and is *BUC-invariant*; that is, if $F \in BUC(\mathbf{R}, X)$ and $F \mid \mathbf{J} \in \mathcal{A}$, then $F_t \mid \mathbf{J} \in \mathcal{A}$ for each $t \in \mathbf{R}$.

The property of being *BUC-invariant* was first introduced in [**6**, $(P.\Lambda)$, Definition 1.3.1] and called the Loomis property for classes $\mathcal{A} \subset BUC(\mathbf{J}, X)$. The notion was extended to classes $\mathcal{A} \subset L^1_{\mathrm{loc}}(\mathbf{J}, X)$ in [**8**, $(1.\mathrm{III}_{ub})$]. In [**10**], this property was called C_{ub}-invariance. In the proof of Theorem 2.2.4 in [**6**, page 13], it is shown that, if $\mathbf{J} = \mathbf{R}_+$ and $\mathcal{A}$ satisfies (3.1), then

$$C_0(\mathbf{R}_+, X) \subset \mathcal{A}_{ub}, \quad \text{where } \mathcal{A}_{ub} = \mathcal{A} \cap BUC(\mathbf{R}_+, X). \tag{3.2}$$

We note that if $\mathbf{J} = \mathbf{R}$, then $\mathcal{A}$ is BUC-invariant if and only if $\mathcal{A} \cap BUC(\mathbf{R}, X)$ is a translation invariant subspace of $BUC(\mathbf{R}, X)$. If $\mathbf{J} = \mathbf{R}_+$, then $\mathcal{A}$ is BUC-invariant if and only if $\mathcal{A} \cap BUC(\mathbf{R}_+, X)$ is a positive invariant subspace of $BUC(\mathbf{R}_+, X)$ (that is, $F_t \in \mathcal{A}$ whenever $F \in \mathcal{A}$, $t \geq 0$) with the additional property that $F \in \mathcal{A}$ whenever $F \in BUC(\mathbf{R}_+, X)$ and $F_t \in \mathcal{A}$ for some $t \geq 0$. Such subspaces of $BUC(\mathbf{R}_+, X)$ were called *translation-biinvariant* ([**3**], [**17**, (1.1), page 17]). So, for a closed subspace $\mathcal{A}$ of $BUC(\mathbf{R}_+, X)$,

(3.3) $\mathcal{A}$ is BUC-invariant if and only if $\mathcal{A}$ is translation-biinvariant.

For $\mathcal{A}$ satisfying (3.1), $V \subset L^1(\mathbf{R})$ and $F \in L^1_{\text{loc}}(\mathbf{J}, X)$, a point $\omega \in \mathbf{R}$ is called $(\mathcal{A}, V)$-*regular* for F or $\overline{F}$, if there is $\varphi \in V$ such that $\widehat{\varphi}(\omega) \neq 0$ and $(\overline{F} * \varphi) \mid \mathbf{J} \in \mathcal{A}$. The *reduced Beurling spectrum* of F or $\overline{F}$ relative to $(\mathcal{A}, V)$ is defined by

$$\begin{aligned} sp_{\mathcal{A},V}(F) &= \{\omega \in \mathbf{R} : \omega \text{ is not an } (\mathcal{A}, V)\text{-regular point for } F\} \\ &= \{\omega \in \mathbf{R} : \varphi \in V, (\overline{F} * \varphi) \mid \mathbf{J} \in \mathcal{A} \text{ implies } \widehat{\varphi}(\omega) = 0\} \\ &= sp_{\mathcal{A},V}(\overline{F}), \end{aligned} \tag{3.4}$$

provided the convolution $\overline{F} * \varphi$ and the restriction $\overline{F} * \varphi) \mid \mathbf{J}$ are defined for all $\varphi \in V$. Clearly, $sp_{\mathcal{A},V}(F)$ is a closed subset of $\mathbf{R}$. Further, if $F \in L^1_{\text{loc}}(\mathbf{R}, X)$ and $\mathcal{A} \subset L^\infty(\mathbf{R}_+, X)$ we also explore the following spectrum first introduced in [**10**, Definition 3.1]

(3.4*)

$$sp_{\mathcal{A},V}(F) = \{\omega \in \mathbf{R} : \varphi \in V, (F * \varphi) \mid \mathbf{R}_+ \in \mathcal{A} \quad \text{implies } \widehat{\varphi}(\omega) = 0\}.$$

We give conditions in Proposition 3.1 under which $sp_{\mathcal{A},V}(F) = sp_{\mathcal{A},V}(F|\mathbf{R}_+)$.

If $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$, then $sp_{\mathcal{A},\mathcal{S}}(F) \subset sp_{\mathcal{A},\mathcal{D}}(F)$ since $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$. Moreover, the first inclusion might be proper. For example, take $F \in L^\infty(\mathbf{R}, X)$ with $sp^B(F) = sp_{0,\mathcal{S}}(F)$ (see (1.1)) uncountable but not $\mathbf{R}$. If $0 \neq \varphi \in \mathcal{D}(\mathbf{R})$, then $F * \varphi \neq 0$, otherwise $sp^B(F)$ is countable. It follows that $sp_{0,\mathcal{D}}(F) = \mathbf{R}$.

For $F \in L^\infty(\mathbf{J}, X)$ and $V = L^1(\mathbf{R}) = L^1$, we write $sp_{\mathcal{A}}(F) = sp_{\mathcal{A},L^1}(F)$.

If $F \in W(\mathbf{J}, X)$ and $V = V(\mathbf{R})$ satisfies (2.8), then the convolution $\overline{F} * g$ and the restriction $(\overline{F} * g) \mid \mathbf{J}$ are defined for all $g \in V(\mathbf{R})$. So, $sp_{\mathcal{A},V}(F)$ is well defined. This is an extension of the definitions in [**6**, (4.1.1)], [**7**, (2.9)] and [**17**, Definition 1.14, page 24]. In those references the conditions on $\mathcal{A}$ are more restrictive and $F \in L^\infty(\mathbf{R}, X)$.

If $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, then $sp_{0,\mathcal{S}}(F) = \operatorname{supp} \widehat{F} = sp^{\mathcal{C}}(F)$, where $sp^{\mathcal{C}}(F)$ is the Carleman spectrum, see [**28**, Proposition 0.5, page 22].

If $F \in L^\infty(\mathbf{J}, X)$, then $\overline{F} * f \in BUC(\mathbf{R}, X)$ for all $f \in L^1(\mathbf{R})$. It follows that

$$sp_{\mathcal{A}}(F) = sp_{\mathcal{A}_{ub}}(F), \quad \text{where } \mathcal{A}_{ub} = \mathcal{A} \cap BUC(\mathbf{J}, X). \tag{3.5}$$

A sufficient condition to have the property $sp_{\mathcal{A}}(F) = \varnothing$ for each $F \in \mathcal{A} \subset L^\infty(\mathbf{J}, X)$ is the following:

$$(\overline{F} * f) \mid \mathbf{J} \in \mathcal{A}_{ub} \quad \text{for each } F \in \mathcal{A} \text{ and } f \in L^1(\mathbf{R}). \tag{3.6}$$

Examples of spaces $\mathcal{A}$ satisfying (3.6) include (using $\mathcal{A}(\mathbf{J}, X) = \mathcal{A}(\mathbf{R}, X) \mid \mathbf{J}$)

$$\begin{aligned}
&\{0\}, \quad C_0(\mathbf{J}, X), \quad AP = AP(\mathbf{R}, X), \quad LAP_b(\mathbf{R}, X),\\
&AA = AA(\mathbf{R}, X), \quad EAP(\mathbf{J}, X) = EAP_0(\mathbf{J}, X) \oplus AP(\mathbf{J}, X),\\
&AAP(\mathbf{J}, X) = C_0(\mathbf{J}, X) \oplus AP(\mathbf{J}, X),\\
&ALAP_b(\mathbf{J}, X) = C_0(\mathbf{J}, X) \oplus LAP_b(\mathbf{J}, X) \quad \text{and}\\
&AAA(\mathbf{J}, X) = C_0(\mathbf{J}, X) \oplus AA(\mathbf{J}, X).
\end{aligned}$$

These are the spaces consisting respectively of the zero function (when $\mathbf{J} = \mathbf{R}$), continuous functions vanishing at infinity, almost periodic ([**1, 6, 24**]), Levitan bounded almost periodic [**24**], almost automorphic functions [**8**], Eberlein (weakly) almost periodic ([**6**, Definition 2.3.1]), asymptotically almost periodic functions (when $\mathbf{J} = \mathbf{R}_+$)([**6**, Definitions 2.2.1, 2.3.1, (2.3.2)]), asymptotically Levitan bounded almost periodic functions and asymptotically almost automorphic functions.

For $\lambda \in \mathbf{C}_+$, set

$$f_\lambda(t) = \begin{cases} e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \text{and} \quad f_{-\lambda} = -\check{f}_\lambda.$$

Then $f_\lambda, \check{f}_\lambda \in L^1(\mathbf{R})$ for all $\lambda \in \mathbf{C} \setminus i\mathbf{R}$. If $H \in L^\infty(\mathbf{R}, X)$, then $H * \check{f}_\lambda \in BUC(\mathbf{R}, X)$ for all $\lambda \in \mathbf{C} \setminus i\mathbf{R}$. We will consider the property

$$(\overline{F} * \check{f}_\lambda) \mid \mathbf{J} \in \mathcal{A}_{ub} \quad \text{for each } F \in \mathcal{A} \text{ and } \lambda \in \mathbf{C} \setminus i\mathbf{R}. \tag{3.7}$$

Proposition 3.1. *Let $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ be a closed subspace satisfying* (3.1) *and* (3.6). *Assume that $H \in W(\mathbf{R}, X)$ and $F = H \mid \mathbf{J}$, where $W(\mathbf{R}, X)$ and $V(\mathbf{R})$ satisfy* (2.4) *and* (2.8).

(i) $sp_{\mathcal{A},V}(H) = sp_{\mathcal{A},V}(F) = sp_{\mathcal{A},V}(\overline{F})$.

(ii) *If $F \in \mathcal{A}$, then $(H * f) \mid \mathbf{J} \in \mathcal{A}_{ub}$ for each $f \in V(\mathbf{R})$.*

(iii) *If $H \in L^\infty(\mathbf{R}, X)$, then $sp_{\mathcal{A},\mathcal{S}}(H) = sp_{\mathcal{A},\mathcal{S}}(F) = sp_{\mathcal{A}}(F) = sp_{\mathcal{A}}(H)$.*

(iv) *If $H \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ and 0 is an $(\mathcal{A}, \mathcal{S})$-regular point for H, then there is $\delta > 0$ and $\psi \in \mathcal{S}(\mathbf{R})$ such that $\widehat{\psi} \in \mathcal{D}(\mathbf{R})$, $\widehat{\psi} = 1$ on $[-\delta, \delta]$ and $(H * \psi) \mid \mathbf{J} \in \mathcal{A}_{ub}$.*

Proof. (i) If $\mathbf{J} = \mathbf{R}$, there is nothing to prove so take $\mathbf{J} = \mathbf{R}_+$. For $\varphi \in V(\mathbf{R})$, we have $H * \varphi = \overline{F} * \varphi + (H - \overline{F}) * \varphi$. By Lemma 2.3 (ii), $((H - \overline{F}) * \varphi) \mid \mathbf{R}_+ \in C_0(\mathbf{R}_+, X)$, so by (3.2) it follows that $(H * \varphi) \mid \mathbf{R}_+ \in \mathcal{A}$ if and only if $(\overline{F} * \varphi) \mid \mathbf{R}_+ \in \mathcal{A}$.

(ii) Again, we need only consider the case $\mathbf{J} = \mathbf{R}_+$. For $f \in V(\mathbf{R})$, we have $(H * f) \mid \mathbf{R}_+ = (\overline{F} * f) \mid \mathbf{R}_+ + \xi$ where $\xi = ((H - \overline{F}) * f) \mid \mathbf{R}_+ \in C_0(\mathbf{R}_+, X)$ as in part (i). By (3.6), it follows that $(\overline{F} * f) \mid \mathbf{R}_+ \in \mathcal{A}_{ub}$. Hence, $(H * f) \mid \mathbf{R}_+ \in \mathcal{A}_{ub}$ by (3.2).

(iii) By part (i) we have $sp_{\mathcal{A},\mathcal{S}}(H) = sp_{\mathcal{A},\mathcal{S}}(F)$ and $sp_{\mathcal{A},L^1}(H) = sp_{\mathcal{A},L^1}(F)$. Moreover, $sp_{\mathcal{A},L^1}(F) \subset sp_{\mathcal{A},\mathcal{S}}(F)$. For the reverse inclusion, let $\omega_0 \in \mathbf{R}$ be $(\mathcal{A}, L^1)$-regular for F. So there is $h_0 \in L^1(\mathbf{R})$ such that $\widehat{h_0}(\omega_0) \neq 0$ and $(\overline{F} * h_0) \mid \mathbf{J} \in \mathcal{A}$. Choose $\delta > 0$ such that $\widehat{h_0} \neq 0$ on $[\omega_0 - \delta, \omega_0 + \delta]$ and, by Lemma 2.1, $k_0 \in L^1(\mathbf{R})$ such that $\widehat{k_0} \cdot \widehat{h_0} = 1$ on $[\omega_0 - \delta, \omega_0 + \delta]$. Let $\varphi \in \mathcal{S}(\mathbf{R})$, $\widehat{\varphi}(\omega_0) \neq 0$ and $\operatorname{supp} \widehat{\varphi} \subset [\omega_0 - \delta, \omega_0 + \delta]$. By (2.11), we have $\overline{F} * \varphi = \overline{F} * (h_0 * k_0 * \varphi) = (\overline{F} * h_0) * (k_0 * \varphi)$. So, $(\overline{F} * \varphi) \mid \mathbf{J} \in \mathcal{A}$ by part (ii), and therefore ω_0 is $(\mathcal{A}, \mathcal{S})$-regular for F.

(iv) By (i), 0 is an $(\mathcal{A}, \mathcal{S})$-regular point for F, so there is $\delta > 0$ and $\varphi \in \mathcal{S}(\mathbf{R})$ such that $\widehat{\varphi} \neq 0$ on $[-\delta, \delta]$ and $(\overline{F} * \varphi) \mid \mathbf{J} \in \mathcal{A}$. If $\mathbf{J} = \mathbf{R}$, then $H = F = \overline{F}$, and so $H * \varphi \in \mathcal{A}$. If $\mathbf{J} = \mathbf{R}_+$, then $(H - \overline{F}) * \varphi \mid \mathbf{R}_+ \in C_0(\mathbf{R}_+, X)$ by Lemma 2.3 (ii). So, $(H * \varphi) \mid \mathbf{R}_+ \in \mathcal{A}$

by (3.2). By Lemma 2.1, there is $g \in \mathcal{S}(\mathbf{R})$ such that $\widehat{g} \in \mathcal{D}(\mathbf{R})$ and $\widehat{\varphi} \cdot \widehat{g} = 1$ on $[-\delta, \delta]$. Obviously, $\psi = \varphi * g \in \mathcal{S}(\mathbf{R})$ and $\widehat{\psi} \in \mathcal{D}(\mathbf{R})$. Since $(H * \varphi) \mid \mathbf{J} \in \mathcal{A} \subset L^\infty(\mathbf{J}, X)$, $(H * \psi) \mid \mathbf{J} = ((H * \varphi) * g) \mid \mathbf{J} \in \mathcal{A}_{ub}$ by (2.11) and part (ii). $\square$

Proposition 3.2. *Let $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ be a closed subspace.*

(i) *If $\mathcal{A}$ satisfies* (3.6)*, then $\mathcal{A}$ is BUC-invariant.*

(ii) *$\mathcal{A}$ satisfies* (3.7) *if and only if $\mathcal{A}$ satisfies* (3.6)*.*

Proof. (i) Take $F \in BUC(\mathbf{R}, X)$ with $F \mid \mathbf{J} \in \mathcal{A}$, and take $t \in \mathbf{R}$. By Proposition 3.1 (ii), for each $f \in L^1(\mathbf{R})$, we have $(F_t * f) \mid \mathbf{J} = (F * f_t) \mid \mathbf{J} \in \mathcal{A}$. Using the approximate identity of Lemma 2.2 (i), we conclude that $F_t \mid \mathbf{J} \in \mathcal{A}$.

(ii) Obviously, (3.6) implies (3.7). For the converse, we begin by showing that $E = \operatorname{span}\{f_\lambda : \operatorname{Re}\lambda \neq 0\}$ is a dense subspace of $L^1(\mathbf{R})$. Indeed, if E is not dense in $L^1(\mathbf{R})$, then by the Hahn-Banach theorem, there is $0 \neq \phi \in L^\infty(\mathbf{R}) = (L^1(\mathbf{R}))^*$ such that $\mathcal{C}\phi(\lambda) = \int_0^\infty e^{-\lambda t}\phi(t)\, dt = 0$ if $\operatorname{Re}\lambda > 0$ and $\mathcal{C}\phi(\lambda) = -\int_0^\infty e^{\lambda t}\phi(-t)\, dt = 0$ if $\operatorname{Re}\lambda < 0$. This means that the Carleman transform $\mathcal{C}\phi$ is zero on $\mathbf{C} \setminus i\mathbf{R}$ (see (4.5) below) and implies $sp^C(\phi) = \varnothing$ and so $\phi = 0$ by [**28**, Proposition 0.5 (ii)]. This is a contradiction showing that E is dense in $L^1(\mathbf{R})$. Given (3.7), it follows that $(\overline{F} * f) \mid \mathbf{J} \in \mathcal{A}$ for each $F \in \mathcal{A}$ and $f \in E$. Since E is a dense subspace of $L^1(\mathbf{R})$ and $\mathcal{A}$ is closed, (3.6) follows. $\square$

Remark 3.3. (a) If $\mathcal{A} \subset BUC(\mathbf{J}, X)$ satisfies (3.1), then using the properties of Bochner integration (see [**6**, Lemma 1.2.1]) we find that $\mathcal{A}$ satisfies (3.6).

(ii) The converse of Proposition 3.2 (i) is false in general. The Banach space

$$\mathcal{A}_g = g \cdot AP(\mathbf{R}, X) \quad \text{with} \quad g(t) = e^{it^2} \text{ for } t \in \mathbf{R} \tag{3.8}$$

satisfies (3.1) but does not satisfy (3.6).

(iii) As $\mathcal{A}_g \cap BUC(\mathbf{R}, X) = \{0\}$, we conclude that if $0 \neq F \in BC(\mathbf{R}, X)$, then by (3.5) and (1.1),

$$sp_{\mathcal{A}_g}(F) = sp_0(F) = sp^B(F) \neq \varnothing. \tag{3.9}$$

In particular, $sp_{\mathcal{A}_g}(F) \neq \varnothing$ for each $0 \neq F \in \mathcal{A}_g$.

Proposition 3.4. *Let $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ be a closed subspace satisfying* (3.6)*. Let W, V satisfy* (2.4)*,* (2.8) *and $F, H \in W(\mathbf{J}, X)$.*

(i) *If $g \in V(\mathbf{R})$ or $g \in L_c^\infty(\mathbf{R})$, then $sp_{\mathcal{A},V}(\overline{F} * g) \subset sp_{\mathcal{A},V}(F) \cap \operatorname{supp} \widehat{g}$.*

(ii) *$sp_{\mathcal{A},V}(F) = \cup_{h>0} sp_{\mathcal{A},V}(M_h F)$.*

(iii) *If $t \in \mathbf{R}$ and $0 \neq c \in \mathbf{C}$, then $sp_{\mathcal{A},V}(c(\overline{F})_t) = sp_{\mathcal{A},V}(F)$.*

(iv) *$sp_{\mathcal{A},V}(F + H) \subset sp_{\mathcal{A},V}(F) \cup sp_{\mathcal{A},V}(H)$.*

(v) *If $\gamma_\lambda \mathcal{A} \subset \mathcal{A}$ for all $\lambda \in \mathbf{R}$, then $sp_{\mathcal{A},V}(\gamma_\omega F) = \omega + sp_{\mathcal{A},V}(F)$ for all $\omega \in \mathbf{R}$.*

Proof. (i) Assume $\omega \notin sp_{\mathcal{A},V}(F)$. Then there is $\varphi \in V(\mathbf{R})$ with $\widehat{\varphi}(\omega) \neq 0$ and $(\overline{F} * \varphi) \mid \mathbf{J} \in \mathcal{A}$. By (2.11), we have $(\overline{F} * g) * \varphi = (\overline{F} * \varphi) * g = \overline{F} * (\varphi * g)$. So, by Proposition 3.1 (ii), we get $((\overline{F} * \varphi) * g) \mid \mathbf{J} \in \mathcal{A}$ proving $\omega \notin sp_{\mathcal{A},V}(\overline{F} * g)$. On the other hand, if $\omega \notin \operatorname{supp} \widehat{g}$, then there is $\varphi \in V(\mathbf{R})$ with $\widehat{\varphi}(\omega) \neq 0$ and $\varphi * g = 0$. So, $\omega \notin sp_{\mathcal{A},V}(\overline{F} * g)$. For the case $\mathcal{A} = \{0\}$, see also [**28**, Proposition 0.6 (i)].

(ii) We note from (2.7) that $M_h F = (\overline{F} * s_h) \mid \mathbf{J}$, where $s_h \in L_c^\infty(\mathbf{R})$ for each $h > 0$. Hence, $sp_{\mathcal{A},V}(\overline{F} * s_h) \subset sp_{\mathcal{A},V}(\overline{F})$. By Proposition 3.1 (i), we have $sp_{\mathcal{A},V}(M_h F) = sp_{\mathcal{A},V}(\overline{F} * s_h)$ and so $\cup_{h>0} sp_{\mathcal{A},V}(M_h F) \subset sp_{\mathcal{A},V}(F)$. Now, let $\omega \in sp_{\mathcal{A},V}(F)$. There is $h > 0$ such that $\widehat{s_h}(\omega) \neq 0$. Assume that $\omega \notin sp_{\mathcal{A},V}(M_h F) = sp_{\mathcal{A},V}(\overline{F} * s_h)$. There is $\psi \in V(\mathbf{R})$ such that $\widehat{\psi}(\omega) \neq 0$ and $((\overline{F} * s_h) * \psi) \mid \mathbf{J} \in \mathcal{A}$. By (2.11), $(\overline{F} * s_h) * \psi = \overline{F} * (s_h * \psi)$ so $(\overline{F} * (s_h * \psi)) \mid \mathbf{J} \in \mathcal{A}$. Since $s_h * \psi \in V(\mathbf{R})$ and $\widehat{s_h * \psi}(\omega) \neq 0$, we conclude that $\omega \notin sp_{\mathcal{A},V}(F)$, a contradiction which shows $\omega \in sp_{\mathcal{A},V}(M_h F)$. This proves $sp_{\mathcal{A},V}(F) \subset \cup_{h>0} sp_{\mathcal{A},V}(M_h F)$.

The proofs of (iii), (iv) and (v) are similar to the case $\mathcal{A} = \{0\}$ [**28**, Proposition 0.4]. $\square$

We recall (see [**8**, page 118], [**9**, page 1007], [**13, 32**]) that a function $F \in L^1_{\mathrm{loc}}(\mathbf{J}, X)$ is called *ergodic* if there is a constant $m(F) \in X$ such that

$$\sup_{t \in \mathbf{J}} \left\| \frac{1}{T} \int_0^T F(t+s)\, ds - m(F) \right\| \longrightarrow 0 \quad \text{as } T \to \infty.$$

The limit $m(F)$ is called the *mean* of F. The set of all such ergodic functions will be denoted by $\mathcal{E}(\mathbf{J},X)$. We set $\mathcal{E}_0(\mathbf{J},X) = \{F \in \mathcal{E}(\mathbf{J},X) : m(F) = 0\}$, $\mathcal{E}_b(\mathbf{J},X) = \mathcal{E}(\mathbf{J},X) \cap L^\infty(\mathbf{J},X)$, $\mathcal{E}_{b,0}(\mathbf{J},X) = \{F \in \mathcal{E}_b(\mathbf{J},X) : m(F) = 0\}$, $\mathcal{E}_{ub}(\mathbf{J},X) = \mathcal{E}(\mathbf{J},X) \cap BUC(\mathbf{J},X)$ and $\mathcal{E}_{u,0}(\mathbf{J},X) = \mathcal{E}_{ub}(\mathbf{J},X) \cap \mathcal{E}_{b,0}(\mathbf{J},X)$.

If $F \in L^1_{\mathrm{loc}}(\mathbf{J},X)$ and $\gamma_\omega F \in \mathcal{E}(\mathbf{J},X)$ for some $\omega \in \mathbf{R}$, then

$$\gamma_\omega M_h\, F \in \mathcal{E}(\mathbf{J},X) \quad \text{and} \quad M_h\gamma_\omega F \in \mathcal{E}_b(\mathbf{J},X) \text{ for all } h > 0. \tag{3.10}$$

Moreover, if $F \in L^\infty(\mathbf{J},X)$ and $\gamma_\omega F \in \mathcal{E}_b(\mathbf{J},X)$ for some $\omega \in \mathbf{R}$, then

$$\gamma_\omega(\overline{F} * g) \mid \mathbf{J} \in \mathcal{E}_{ub}(\mathbf{J},X) \quad \text{for all } g \in L^1(\mathbf{R}). \tag{3.11}$$

To prove (3.10), note that

$$M_T\gamma_\omega M_h F = \gamma_\omega M_h\gamma_{-\omega}M_T\gamma_\omega F \quad \text{and} \quad M_T M_h\gamma_\omega F = M_h M_T\gamma_\omega F.$$

It follows that $\gamma_\omega M_h F$, $M_h\gamma_\omega F \in \mathcal{E}(\mathbf{J},X)$ for all $h > 0$. By [**9**, (2.4)], $M_h\gamma_\omega F \in C_b(\mathbf{J},X)$ and so $M_h\gamma_\omega F \in \mathcal{E}_b(\mathbf{J},X)$. For (3.11), note that, if $F \in L^\infty(\mathbf{J},X)$, then $M_h F = (\overline{F} * s_h) \mid \mathbf{J}$ (see (2.7)) is bounded and uniformly continuous. So, $\gamma_\omega(\overline{F} * s_h) \in \mathcal{E}_{ub}(\mathbf{J},X)$ by (3.10). A similar calculation gives $\gamma_\omega(\overline{F} * \check{s_h}) \in \mathcal{E}_{ub}(\mathbf{J},X)$. It follows that $\gamma_\omega(\overline{F} * g) \mid \mathbf{J} \in \mathcal{E}_{ub}(\mathbf{J},X)$ for any step function g. Since step functions are dense in $L^1(\mathbf{R})$, (3.11) follows.

Also, we note that

$$\mathcal{E}_u(\mathbf{J},X) := UC(\mathbf{J},X) \cap \mathcal{E}(\mathbf{J},X) = \mathcal{E}_{ub}(\mathbf{J},X). \tag{3.12}$$

This follows by Lemma 2.2 (ii) using (3.10) (see also [**9**, Proposition 2.9]).

Next, we recall the definition of the class of slowly oscillating functions

$$SO(\mathbf{J},X) = UC(\mathbf{J},X) + L^1_{\mathrm{loc},0}(\mathbf{J},X),$$

where (see [**5**, Proposition 4.2.2] and [**18**, Lemma 1.6] for the case $\mathbf{J} = \mathbf{R}_+$)

$$L^1_{\mathrm{loc},0}(\mathbf{J},X) = \{F \in L^1_{\mathrm{loc}}(\mathbf{J},X) : \lim_{|t|\to\infty, t\in\mathbf{J}} F(t) = 0\}.$$

It follows that, if $F \in L^1_{\mathrm{loc},0}(\mathbf{J}, X)$ and $\psi \in \mathcal{S}(\mathbf{R})$, then

$$F \in \mathcal{E}_0(\mathbf{J}, X) \quad \text{and} \quad \overline{F} \in L^1_{\mathrm{loc},0}(\mathbf{R}, X); \tag{3.13}$$

$$M_h F \in C_0(\mathbf{J}, X) \quad \text{for all } h > 0 \text{ and } \overline{F} * \psi \in C_0(\mathbf{R}, X). \tag{3.14}$$

Also, it is readily verified that, for $F \in SO(\mathbf{J}, X)$,

$$\overline{F} * \psi, \qquad M_h \overline{F} \in UC(\mathbf{R}, X) \quad \text{for each } \psi \in \mathcal{S}(\mathbf{R}), h > 0. \tag{3.15}$$

Theorem 3.5. *Let $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ and $0 \notin sp_{0,\mathcal{S}}(F)$.*

(i) *If $PF \in UC(\mathbf{R}, X)$, then $PF \in BUC(\mathbf{R}, X)$ and $F \in \mathcal{E}_0(\mathbf{R}, X)$.*

(ii) *If $F \in L^\infty(\mathbf{R}, X)$ or $F \in SO(\mathbf{R}, X)$, then $PF \in BUC(\mathbf{R}, X)$.*

Proof. (i) Choose $\varphi \in \mathcal{S}(\mathbf{R})$ such that $\operatorname{supp} \widehat{\varphi} \subset [-\delta, \delta]$ and $\widehat{\varphi} = 1$ on $[-\delta/2, \delta/2]$, where $\delta > 0$ and $[-\delta, \delta] \cap sp_{0,\mathcal{S}}(F) = \varnothing$. Then $sp_{0,\mathcal{S}}(F * \varphi) = \varnothing$ by Proposition 3.4 (i). So $F * \varphi = 0$ by [**28**, Proposition 0.5 (ii)]. Since $(PF * \varphi)' = F * \varphi = 0$, we get $PF * \varphi = \text{constant}$. By (2.12), we have $0 \notin sp_{0,\mathcal{S}}(PF - PF * \varphi)$. Hence, $(PF - PF * \varphi) \in BUC(\mathbf{R}, X)$ by [**12**, Theorem 4.2] implying $PF \in BUC(\mathbf{R}, X)$. It is readily verified that $F \in \mathcal{E}_0(\mathbf{R}, X)$.

(ii) If $F \in L^\infty(\mathbf{R}, X)$, then clearly $PF \in UC(\mathbf{R}, X)$. So suppose that $F \in SO(\mathbf{R}, X)$ and $h > 0$. By (3.15) and Proposition 3.4 (ii) we have $M_h F \in UC(\mathbf{R}, X)$ and $0 \in sp_{0,\mathcal{S}}(M_h F)$. Again, $M_h F \in BUC(\mathbf{R}, X)$ by [**12**, Theorem 4.2]. As $\Delta_h PF = h M_h F$, one gets that $PF \in UC(\mathbf{R}, X)$ by [**9**, Proposition 1.4]. It follows that $PF \in BUC(\mathbf{R}, X)$ by part (i). □

A slight modification of the proof of Theorem 3.5 (i) gives the following sharper result. If $u \in UC(\mathbf{R}, X)$ and if $0 \notin \operatorname{supp} \widehat{u'}$, in the distributional sense, then $u \in BUC(\mathbf{R}, X)$.

We are now ready to state and prove our main results.

Theorem 3.6. *Let $\mathcal{A} \subset L^\infty(\mathbf{J}, X)$ be a closed subspace satisfying (3.6) and $\gamma_\lambda \mathcal{A} \subset \widetilde{\mathcal{E}} \in \{\mathcal{E}(\mathbf{J}, X), \mathcal{E}_0(\mathbf{J}, X)\}$ for all $\lambda \in \mathbf{R}$. Let $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$ and $0 \notin sp_{\mathcal{A},\mathcal{S}}(F)$.*

(i) $\overline{F} = H + G$, *where* $H \in BUC(\mathbf{R}, X)$, $H \mid \mathbf{J} \in \mathcal{A}_{ub}$ *and* $0 \notin sp_{0,\mathcal{S}}(G)$.

(ii) *If* $PF \in UC(\mathbf{J}, X)$ *or* $F \in L^\infty(\mathbf{J}, X)$ *or* $F \in SO(\mathbf{J}, X)$, *then* $F \in \widetilde{\mathcal{E}}$. *If also* $F \in UC(\mathbf{J}, X)$, *then* $F \in \widetilde{\mathcal{E}} \cap BUC(\mathbf{J}, X)$.

Proof. (i) By Proposition 3.1 (iv), there is $\delta > 0$ and $\psi \in \mathcal{S}(\mathbf{R})$ such that $\operatorname{supp} \widehat{\psi}$ is compact, $\widehat{\psi} = 1$ in a neighborhood of 0 and $(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}_{ub} \subset \widetilde{\mathcal{E}}$. Set $H = \overline{F} * \psi$ and $G = \overline{F} - H$. By (2.9), H is continuous and by Lemma 2.3, we conclude that $H \in BUC(\mathbf{R}, X)$. By (2.12), $0 \notin sp_{0,\mathcal{S}}(G)$.

(ii) Since $PH \in UC(\mathbf{R}, X)$ it follows that $PG \in UC(\mathbf{R}, X)$ and, by Theorem 3.5, $G \in \mathcal{E}_0(\mathbf{R}, X)$. This and part (i) give $F = (H + G) \mid \mathbf{J} \in \widetilde{\mathcal{E}}$. The last assertion follows by (3.12). $\square$

Consider the conditions

$$\begin{aligned} &\mathcal{A}_0(\mathbf{J}, X) \in \{C_0(\mathbf{J}, X), EAP_0(\mathbf{J}, X)\}, \\ &\mathcal{A}(\mathbf{J}, X) = \mathcal{A}_0(\mathbf{J}, X) \oplus AP(\mathbf{J}, X) \end{aligned} \tag{3.16}$$

and

$$\mathcal{A}_*(\mathbf{J}, X) \in \{\mathcal{A}_0(\mathbf{J}, X), \mathcal{A}(\mathbf{J}, X)\}. \tag{3.17}$$

In the notation of [**6**], $\mathcal{A}(\mathbf{J}, X)$ is a Λ-class and by [**31**, Remark, page 18], $F \in \mathcal{A}(\mathbf{J}, X)$ implies $F \in \mathcal{A}_0(\mathbf{J}, X)$ if and only if $\gamma_\lambda F \in \mathcal{E}_{u,0}(\mathbf{J}, X)$ for all $\lambda \in \mathbf{R}$.

Theorem 3.7. *Let* $\mathcal{A}_0, \mathcal{A}, \mathcal{A}_*$ *satisfy* (3.16) *and* (3.17). *Assume that* $F \in \mathcal{S}'_{ar}(\mathbf{J}, X)$, $sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$ *is countable and* $\gamma_{-\omega} F \in \mathcal{E}(\mathbf{J}, X)$ *for all* $\omega \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$.

(i) *If* $F \in UC(\mathbf{J}, X)$, *then* $F \in \mathcal{A}(\mathbf{J}, X)$. *If also* $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$, *then* $F \in \mathcal{A}_0(\mathbf{J}, X)$.

(ii) *If* $F \in SO(\mathbf{J}, X)$, *then* $F \in \mathcal{A}(\mathbf{J}, X) + L^1_{\mathrm{loc},0}(\mathbf{J}, X)$. *If also* $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$, *then* $F \in \mathcal{A}_0(\mathbf{J}, X) + L^1_{\mathrm{loc},0}(\mathbf{J}, X)$.

(iii) *If* $F = H \mid \mathbf{J}$ *where* $H \in L^\infty(\mathbf{R}, X)$ *and, if* $f \in L^1(\mathbf{R})$, *then* $(H * f) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$.

(iv) *If* $sp_{\mathcal{A}_*(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$ *and if* $\psi \in \mathcal{S}(\mathbf{R})$ *with* $\widehat{\psi} \in \mathcal{D}(\mathbf{R})$*, then* $(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}_*(\mathbf{J}, X)$ *and when* $\mathbf{J} = \mathbf{R}_+$*,* $(\overline{F} * \psi)|\mathbf{R}_- \in C_0(\mathbf{R}_-, X)$*.*

(v) *If* $M_h F \in BC(\mathbf{J}, X)$ *for all* $h > 0$ *(for example, if* F *is ergodic) and if* $\psi \in \mathcal{S}(\mathbf{R})$*, then* $(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$ *and, when* $\mathbf{J} = \mathbf{R}_+$*,* $(\overline{F} * \psi) \mid \mathbf{R}_- \in C_0(\mathbf{R}_-, X)$*. If also* $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$*, then* $(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}_0(\mathbf{J}, X)$*.*

Proof. Assume that F satisfies the assumptions of one of the parts (i)–(iii). We note that $\mathcal{A}(\mathbf{J}, X)$ satisfies the assumptions of Theorem 3.6 with $\widetilde{\mathcal{E}} = \mathcal{E}(\mathbf{J}, X)$; so, if $0 \notin sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$, then F is ergodic by Theorem 3.6. If $0 \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$, then F is ergodic by assumption.

(i) By (3.12), we get $F \in \mathcal{E}_{ub}(\mathbf{J}, X)$. Let $\widetilde{F} \in BUC(\mathbf{R}, X)$ be an extension of F. By Proposition 3.1 (iii), $sp_{\mathcal{A}(\mathbf{J},X)}(\widetilde{F}) = sp_{\mathcal{A}(\mathbf{J},X)}(F)$ which is countable. By [**6**, Theorem 4.2.6], $F = \widetilde{F} \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$. If $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$, then since $\mathcal{A}_0(\mathbf{J}, X) \subset \mathcal{E}_{u,0}(\mathbf{J}, X)$ and, by Theorem 3.6 (ii), $\gamma_\lambda F \in \mathcal{E}_{u,0}(\mathbf{J}, X)$ for all $\lambda \in \mathbf{R}$. This implies $F \in \mathcal{A}_0(\mathbf{J}, X)$.

(ii) Let $F = u + \xi$, where $u \in UC(\mathbf{J}, X)$, $\xi \in L^1_{\mathrm{loc},0}(\mathbf{J}, X)$. We note that $sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(\xi) \subset sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(\xi) = \varnothing$ by (3.14) and $\gamma_\lambda \xi \in \mathcal{E}_0(\mathbf{J}, X)$ for all $\lambda \in \mathbf{R}$, by (3.13). Also, we have $sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(M_h F)$ is countable by Proposition 3.4 (ii). By (3.10), we get $\gamma_{-\omega} M_h F \in \mathcal{E}(\mathbf{J}, X)$ for all $\omega \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$. By Proposition 3.4 (iv), $sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(M_h u)$ is countable and $\gamma_{-\omega} M_h u \in \mathcal{E}(\mathbf{J}, X)$ for all $\omega \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$. So, by part (i), we conclude that $M_h u \in \mathcal{A}(\mathbf{J}, X)$ for all $h > 0$. By Lemma 2.2 (ii), $u = \lim_{h \to 0} M_h u \in \mathcal{A}(\mathbf{J}, X)$. It follows that $F \in \mathcal{A}(\mathbf{J}, X) + L^1_{\mathrm{loc},0}(\mathbf{J}, X)$. If $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$, then again $\gamma_\lambda F \in \mathcal{E}_0(\mathbf{J}, X)$ for all $\lambda \in \mathbf{R}$, by Theorem 3.6 (ii). This implies that $F \in \mathcal{A}_0(\mathbf{J}, X) + L^1_{\mathrm{loc},0}(\mathbf{J}, X)$.

(iii) Let $f \in L^1(\mathbf{R})$. Then $\overline{F} * f \in BUC(\mathbf{R}, X)$. By Proposition 3.4 (i), we deduce that $sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(\overline{F} * f)$ is countable. By (3.11), we find that $\gamma_{-\omega}(\overline{F} * f) \mid \mathbf{J} \in \mathcal{E}_{ub}(\mathbf{J}, X)$ for all $\omega \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(\overline{F} * f)$. It follows that $(\overline{F} * f) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X))$, by part (i). By Lemma 2.3 (ii) and (3.2), we have $((H - \overline{F}) * f) \mid \mathbf{J} \in C_0(\mathbf{J}, X) \subset \mathcal{A}_0(\mathbf{J}, X)$. Hence, $(H * f) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$.

(iv) Let $\omega \in K = \operatorname{supp} \widehat{\psi}$. Since $\mathcal{A}_*(\mathbf{J}, X)$ satisfies (3.1) and (3.6), by Proposition 3.1 (iv), there is $f^\omega \in \mathcal{S}(\mathbf{R})$ such that $\widehat{f^\omega}$ has compact support, $\widehat{f^\omega} = 1$ on an open neighborhood V^ω of ω and $(\overline{F} * f^\omega) \mid \mathbf{J} \in \mathcal{A}_*(\mathbf{J}, X)$. Take $k^\omega = f^\omega * g^\omega$, where $g^\omega(t) = \overline{f^\omega(-t)}$. By (2.11) and Proposition 3.1 (ii), we conclude that $(\overline{F} * k^\omega) \mid \mathbf{J} \in \mathcal{A}_*(\mathbf{J}, X)$. Consider the open covering $\{V^\omega : \omega \in K\}$. By compactness, there is a finite sub-covering $\{V^{\omega_1}, \cdots, V^{\omega_n}\}$ of K. One has $k = \sum_{i=1}^n k^{\omega_i} \in \mathcal{S}(\mathbf{R})$, $\operatorname{supp} \widehat{k}$ is compact, $\widehat{k} \geq 1$ on K and $(\overline{F} * k) \mid \mathbf{J} \in \mathcal{A}_*(\mathbf{J}, X)$. By Lemma 2.1, there is $h \in \mathcal{S}(\mathbf{R})$ such that $\widehat{h} \cdot \widehat{k} = 1$ on K. Again, by (3.6) and Proposition 3.1 (ii), it follows that $(\overline{F} * \psi) \mid \mathbf{J} = ((\overline{F} * k) * h * \psi) \mid \mathbf{J} \in \mathcal{A}_*(\mathbf{J}, X)$. By Lemma 2.3 (i), if $\mathbf{J} = \mathbf{R}_+$, then $(\overline{F} * \psi) \mid \mathbf{R}_- \in C_0((\mathbf{R}_-, X)$.

(v) Let $h > 0$. By (3.10) and Proposition 3.4 (ii), $\gamma_{-\omega} M_h F \in \mathcal{E}_b(\mathbf{J}, X)$ for all $h > 0$ and $\omega \in sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(M_h F) \subset sp_{\mathcal{A}(\mathbf{J},X),\mathcal{S}}(F)$. Therefore, by part (iii), $((\overline{F} * s_h) * g) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$ for all $g \in L^1(\mathbf{R})$. Take $\psi \in \mathcal{S}(\mathbf{R})$. It follows that $M_h(\overline{F} * \psi) \mid \mathbf{J} = ((\overline{F} * s_h) * \psi) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$ and also $(\Delta_h(\overline{F} * \psi)) \mid \mathbf{J} = (\overline{F} * \Delta_h \psi) \mid \mathbf{J} = (\overline{F} * h M_h \psi') \mid \mathbf{J} = (\overline{F} * (s_h * \psi')) \mid \mathbf{J} = ((\overline{F} * s_h) * \psi')) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X) \subset BUC(\mathbf{J}, X)$. By [**9**, Proposition 1.4], one gets $(\overline{F} * \psi) \mid \mathbf{J}$ is uniformly continuous. This implies $(\overline{F} * \psi) \mid \mathbf{J} = \lim_{h \searrow 0} M_h(\overline{F} * \psi) \mid \mathbf{J} \in \mathcal{A}(\mathbf{J}, X)$, by Lemma 2.2 (ii). If $\mathbf{J} = \mathbf{R}_+$ we proceed as in (iv). If $sp_{\mathcal{A}_0(\mathbf{J},X),\mathcal{S}}(F) = \varnothing$, the result follows by (i). $\square$

In the following, we demonstrate again how one can extend the results proved for the case $F \in BUC(\mathbf{R}, X)$ to the case $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$.

Theorem 3.8. *Assume that $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, $sp_{AP,\mathcal{S}}(F)$ is countable and $\gamma_{-\omega} F \in \mathcal{E}(\mathbf{R}, X)$ for all $\omega \in sp_{AP,\mathcal{S}}(F)$.*

(i) *If $F \in UC(\mathbf{R}, X)$, then $F \in AP(\mathbf{R}, X)$.*

(ii) *If $F \in SO(\mathbf{R}, X)$, then $F = u$ almost everywhere for some $u \in AP(\mathbf{R}, X)$.*

(iii) *If $F \in L^\infty(\mathbf{R}, X)$ and if $\psi \in L^1(\mathbf{R})$, then $F * \psi \in AP(\mathbf{R}, X)$.*

(iv) *If $sp_{AP,\mathcal{S}}(F) = \varnothing$ and if $\psi \in \mathcal{S}(\mathbf{R})$ with $\widehat{\psi} \in \mathcal{D}(\mathbf{R})$, then $F * \psi \in AP(\mathbf{R}, X)$.*

(v) *If $M_h F \in BC(\mathbf{R}, X)$ for all $h > 0$, then $F * \psi \in AP(\mathbf{R}, X)$ for each $\psi \in \mathcal{S}(\mathbf{R})$.*

Proof. Since $AP(\mathbf{R},X) \subset AAP(\mathbf{R},X)$, we conclude that $sp_{AAP,\mathcal{S}}(F) \subset sp_{AP,\mathcal{S}}(F)$.

(i) This follows by Theorem 3.7 (i) and [**6**, Theorem 4.2.6].

(ii) Let $h > 0$. By (3.15), $M_h F$ is uniformly continuous. By (i), $M_h F \in AP(\mathbf{R},X)$. So $sp_{AP,\mathcal{S}}(F) = \varnothing$ by (3.6) and Proposition 3.4 (ii). By Theorem 3.7 (ii) with $\mathcal{A}_0 = C_0(\mathbf{R},X)$, one has $F = u + \xi$ where $u \in AP(\mathbf{R},X)$ and $\xi \in L^1_{\mathrm{loc},0}(\mathbf{R},X)$. By Proposition 3.4 (iv), we get $sp_{AP,\mathcal{S}}(\xi) = sp_{AP,\mathcal{S}}(F-u) \subset sp_{AP,\mathcal{S}}(F) \cup sp_{AP,\mathcal{S}}(-u) = \varnothing$. By (3.14), if $\varphi \in \mathcal{S}(\mathbf{R})$, then $\xi * \varphi \in AP(\mathbf{R},X)$ if and only if $\xi * \varphi = 0$. It follows that $sp_{0,\mathcal{S}}(\xi) = sp_{AP,\mathcal{S}}(\xi) = \varnothing$. Hence, $\xi = 0$ (almost everywhere) by [**28**, Proposition 0.5 (ii)].

(iii)–(v) By Theorem 3.7 (iii)–(v), $F * \psi \in AAP(\mathbf{R},X)$. Hence, $F * \psi \in AP(\mathbf{R},X)$ by Proposition 3.4 (i) and part (i) above. □

Corollary 3.9. *Assume that $F \in \mathcal{S}'_{ar}(\mathbf{J},X)$, $\psi \in \mathcal{S}(\mathbf{R})$ with $\widehat{\psi} \in \mathcal{D}(\mathbf{R})$ and $sp_{C_0(\mathbf{J},X),\mathcal{S}}(F) \cap \operatorname{supp}\widehat{\psi} = \varnothing$. Then $\overline{F} * \psi \in C_0(\mathbf{R},X)$.*

Proof. Choose $\rho \in \mathcal{S}(\mathbf{R})$ such that $\widehat{\rho} \in \mathcal{D}(\mathbf{R})$ and $\widehat{\rho} = 1$ on an open neighborhood of $\operatorname{supp}\widehat{\psi}$. Then $(\overline{F} * \psi) * \rho = \overline{F} * (\psi * \rho) = \overline{F} * \psi$. By Proposition 3.4 (i), $sp_{C_0(\mathbf{J},X),\mathcal{S}}(\overline{F} * \psi) = \varnothing$, so $(\overline{F} * \psi) * \rho \in C_0(\mathbf{R},X)$ by Theorem 3.7 (iv). □

Corollary 3.10. *Assume $F \in L^1_{\mathrm{loc}}(\mathbf{J},X)$ and $sp_{C_0(\mathbf{J},X),\mathcal{D}}(F) = \varnothing$. If $(\overline{F} * \psi) \mid \mathbf{J}$ is uniformly continuous for some $\psi \in \mathcal{D}(\mathbf{R})$, then $\overline{F} * \psi \in C_0(\mathbf{R},X)$.*

Proof. By $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$ and Proposition 3.4 (i),

$$sp_{C_0(\mathbf{R},X),\mathcal{S}}(\overline{F} * \psi) \subset sp_{C_0(\mathbf{R},X),\mathcal{D}}(\overline{F} * \psi) \subset sp_{C_0(\mathbf{R},X),\mathcal{D}}(F) = \varnothing.$$

So, $sp_{C_0(\mathbf{R},X),\mathcal{S}}(\overline{F} * \psi) = \varnothing$. The result follows from Theorem 3.7 (i). □

The following example shows that the assumption of uniform continuity is essential in Corollary 3.10.

Example 3.11. If $F(t) = e^t$ for $t \in \mathbf{R}$, then $sp_{C_0(\mathbf{J},X),\mathcal{D}}(F) = \varnothing$ but $(F * \psi) \mid \mathbf{J}$ is unbounded for each $\psi \in \mathcal{D}(\mathbf{R})$ with $\int_{-\infty}^{\infty} e^{-s}\psi(s)\,ds \neq 0$.

Proof. For any $\omega \in \mathbf{R}$, choose $a > 0$ such that $\cos \omega t$ does not change sign on $[0, a]$. Take $\varphi \in \mathcal{D}(\mathbf{R})$ such that $\varphi > 0$ on $(0, a)$ and $\operatorname{supp} \varphi = [0, a]$. Let $f(t) = \varphi(t)$ for $t \geq 0$, $f(t) = -e^{2t}\varphi(-t)$ for $t < 0$. It follows that $f \in \mathcal{D}(\mathbf{R})$, $F * f = 0$ and $\widehat{f}(\omega) \neq 0$. This means $sp_{C_0(\mathbf{J},X),\mathcal{D}}(F) = \varnothing$. Moreover, for $\psi \in \mathcal{D}(\mathbf{R})$, we have $F * \psi(t) = ce^t$, where $c = \int_{-\infty}^{\infty} e^{-s}\psi(s)\, ds$. So, $(F * \psi) \mid \mathbf{J}$ is unbounded if $c \neq 0$. ◻

In the following example we calculate reduced spectra of some functions whose Fourier transforms may not be regular distributions.

Example 3.12. (i) If $F \in L^p(\mathbf{J}, X)$ for some $1 \leq p < \infty$, then $M_h F \in C_0(\mathbf{J}, X)$ for all $h > 0$ and $sp_{C_0(\mathbf{R},X),V}(F) = \varnothing$ for any $V \in \{\mathcal{D}(\mathbf{R}), \mathcal{S}(\mathbf{R})\}$.

(ii) Let $F \in \mathcal{E}_{ub}(\mathbf{J}, X)$ and either $F' \in L^p(\mathbf{J}, X)$ for some $1 \leq p < \infty$ or more generally $F' \in L^1_{\text{loc}}(\mathbf{J}, X)$ with $M_h F' \in C_0(\mathbf{J}, X)$ for all $h > 0$. Then $F \in X \oplus C_0(\mathbf{J}, X)$ and $sp_{C_0(\mathbf{J},X)}(F) \subset \{0\}$.

Proof. (i) By Hölder's inequality, $\|M_h F(t)\| = (1/h)\| \int_0^h F(t + s)\, ds\| \leq h^{-1/p}(\int_0^h \|F(t+s)\|^p\, ds)^{1/p})$, so $M_h F \in C_0(\mathbf{J}, X)$ for all $h > 0$. By (2.7) and Lemma 2.3 (i), we get $\overline{F} * s_h \in C_0(\mathbf{R}, X)$ for all $h > 0$. So, $sp_{C_0(\mathbf{R},X),V}(\overline{F} * s_h) = sp_{C_0(\mathbf{J},X),V}(M_h F) = \varnothing$ for all $h > 0$. Hence, $sp_{C_0(\mathbf{R},X),V}(F) = \varnothing$ by Proposition 3.4 (ii).

(ii) By part (i), we have $hM_h F'(\cdot) = F(\cdot + h) - F(\cdot) \in C_0(\mathbf{J}, X)$ for all $h > 0$. Let $\widetilde{F} \in BUC(\mathbf{R}, X)$ be given by $\widetilde{F} = F$ on $\mathbf{J}$ and $\widetilde{F}(t) = F(0)$ on $\mathbf{R} \backslash \mathbf{J}$. It follows that $\Delta_s \widetilde{F} \in C_0(\mathbf{R}, X)$ for all $s \in \mathbf{R}$. By [**6**, Theorem 4.2.2, Corollary 4.2.3], we conclude that $F = \widetilde{F} \mid \mathbf{J} \in X \oplus C_0(\mathbf{J}, X)$. This implies $sp_{C_0(\mathbf{J},X)}(F) \subset \{0\}$. ◻

The following result shows that the ergodicity condition in Theorem 3.7 parts (i), (ii), (v) is necessary.

Example 3.13. Let $F \in C_0(\mathbf{J}, X)$ with PF unbounded. Then $sp_{C_0(\mathbf{J},X),\mathcal{S}}(PF) = \{0\}$ and $P\overline{F} * \psi \mid \mathbf{J} \notin C_0(\mathbf{J}, X)$ for each $\psi \in \mathcal{S}(\mathbf{R})$ with $\widehat{\psi}(0) \neq 0$.

Proof. Note that $P\overline{F} = \overline{PF} \in UC(\mathbf{R}, X)$. Set $\varphi = \psi'$ where $\psi \in \mathcal{S}(\mathbf{R})$. Then $P\overline{F} * \varphi \mid \mathbf{J} = \overline{F} * \psi \mid \mathbf{J} \in C_0(\mathbf{J}, X)$. This shows that $sp_{C_0(\mathbf{J},X),\mathcal{S}}(PF) \subset \{0\}$. If $sp_{C_0(\mathbf{J},X),\mathcal{S}}(PF) = \varnothing$, we conclude that $PF \in C_0(\mathbf{J}, X)$ by Theorem 3.7 (i). But PF is assumed to be unbounded, so $sp_{C_0(\mathbf{J},X),\mathcal{S}}(PF) = \{0\}$. Now, let $\psi \in \mathcal{S}(\mathbf{R})$ with $\widehat{\psi}(0) \neq 0$. If $P\overline{F} * \psi \mid \mathbf{J} \in C_0(\mathbf{R}, X)$, then $0 \notin sp_{C_0(\mathbf{J},X),\mathcal{S}}(PF)$, a contradiction which proves $P\overline{F} * \psi \mid \mathbf{J} \notin C_0(\mathbf{R}, X)$. $\square$

4. Properties of the weak Laplace spectra. In this section we establish some new properties of the Laplace and weak Laplace spectrum for regular tempered distributions and show that they are similar to those of the Carleman spectrum (see [**28**, Proposition 0.6]). We use the functions e_a for $a \geq 0$ defined on $\mathbf{R}$ or $\mathbf{R}_+$ by $e_a(t) = e^{-at}$.

If $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$, then $e_a F \in L^1(\mathbf{R}_+, X)$ for all $a > 0$ and so the *Laplace transform* $\mathcal{L}F$ may be defined by

$$\mathcal{L}F(\lambda) = \int_0^\infty e^{-\lambda t} F(t)\, dt \quad \text{for } \lambda \in \mathbf{C}_+. \tag{4.1}$$

For a function $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ the *Carleman transform* $\mathcal{C}F$ is defined by

$$\mathcal{C}F(\lambda) = \begin{cases} \mathcal{L}^+F(\lambda) = \int_0^\infty e^{-\lambda t} F(t)\, dt & \text{for } \lambda \in \mathbf{C}_+ \\ \mathcal{L}^-F(\lambda) = -\int_0^\infty e^{\lambda t} F(-t)\, dt & \text{for } \lambda \in \mathbf{C}_-. \end{cases} \tag{4.2}$$

If $F \in L^1(\mathbf{R}_+, X)$, then $\mathcal{L}F$ has a continuous extension to $\mathbf{C}_+ \cup i\mathbf{R}$ given also by the integral in (4.1). By the Riemann-Lebesgue lemma $\widehat{\overline{F}} = \mathcal{L}F(i\cdot) \in C_0(\mathbf{R}, X)$.

If $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$, then $\widehat{\overline{F}} \in \mathcal{S}'(\mathbf{R}, X)$ and $\mathcal{L}F(a + i\cdot) = \widehat{e_a\overline{F}} \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ for all $a > 0$. Moreover, for $\varphi \in \mathcal{S}(\mathbf{R})$,

$$\langle \mathcal{L}F(a + i\cdot), \varphi \rangle = \langle \widehat{e_a\overline{F}}, \varphi \rangle = \langle e_a\overline{F}, \widehat{\varphi} \rangle \longrightarrow \langle \overline{F}, \widehat{\varphi} \rangle = \langle \widehat{\overline{F}}, \varphi \rangle, \tag{4.3}$$

where the limit exists as $a \searrow 0$ by the Lebesgue convergence theorem. This means that $\lim_{a\searrow 0} \mathcal{L}F(a + i\cdot) = \widehat{\overline{F}}$ with respect to the weak dual topology on $\mathcal{S}'(\mathbf{R}, X)$.

For a holomorphic function $\zeta : \Sigma \to X$, where $\Sigma = \mathbf{C}_+$ or $\Sigma = \mathbf{C}\backslash i\,\mathbf{R}$, the point $i\omega \in i\,\mathbf{R}$ is called a *regular point* for ζ or ζ is called *holomorphic* at $i\omega$, if ζ has an extension $\widetilde{\zeta}$ which is holomorphic in a neighbourhood $V \subset \mathbf{C}$ of $i\omega$.

Points $i\omega$ which are *not regular points* are called *singular points*.

The *Laplace spectrum* of a function $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$ is defined by

$$sp^{\mathcal{L}}(F) = \{\omega \in \mathbf{R} : i\omega \text{ is a singular point for } \mathcal{L}F\}. \tag{4.4}$$

The *Carleman spectrum* of a function $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$ is defined (see [**5**, (4.26)]) by

$$sp^{\mathcal{C}}(F) = \{\omega \in \mathbf{R} : i\omega \text{ is a singular point for } \mathcal{C}F\}. \tag{4.5}$$

The Laplace spectrum is also called the half-line spectrum [**5**, page 275].

Note that, if $\widetilde{\mathcal{L}}\gamma_{-\omega}F$ and $\widetilde{\mathcal{C}}\gamma_{-\omega}F$ are holomorphic extensions of $\mathcal{L}\gamma_{-\omega}F$ and $\mathcal{C}\gamma_{-\omega}F$ respectively, which are holomorphic in a neighborhood of 0, then

$$\begin{aligned} &\lim_{\lambda\to 0} \mathcal{L}\gamma_{-\omega}F(\lambda) = \widetilde{\mathcal{L}}\gamma_{-\omega}F(0) \text{ if } \omega \notin sp^{\mathcal{L}}(F), \text{ and} \\ &\lim_{\lambda\to 0} \mathcal{C}\gamma_{-\omega}F(\lambda) = \widetilde{\mathcal{C}}\gamma_{-\omega}F(0) \text{ if } \omega \notin sp^{\mathcal{C}}(F). \end{aligned} \tag{4.6}$$

If $F \in L^\infty(\mathbf{R}_+, X)$ and $sp^{\mathcal{L}}(F) = \varnothing$, then by Zagier's result [**34**, Analytic theorem] we conclude that $\widehat{F}(\omega) = \int_0^\infty e^{-i\omega\, t} F(t)\, dt$ exists as an improper integral (and by (4.6) equals $\widetilde{\mathcal{L}}\gamma_{-\omega}F(0)$) for each $\omega \in \mathbf{R}$. Zagier's analytic theorem does not hold for unbounded functions. Indeed, the Laplace spectrum of $F(t) = te^{it^2}$ is empty (see Example 4.5 below), and it can be verified that $\int_0^\infty e^{-i\omega\, t} F(t)\, dt$ does not exist as an improper Riemann integral for any $\omega \in \mathbf{R}$.

For a holomorphic function $\zeta : \mathbf{C}_+ \to X$, the point $i\omega \in i\mathbf{R}$ is called a *weakly regular point* for ζ if there exist $\varepsilon > 0$ and $h \in L^1((\omega - \varepsilon, \omega + \varepsilon), X)$ such that

$$\lim_{a\searrow 0} \int_{-\infty}^{\infty} \zeta(a + i\, s)\varphi(s)\, ds = \int_{\omega-\varepsilon}^{\omega+\varepsilon} h(s)\varphi(s)\, ds \tag{4.7}$$
$$\text{for all } \varphi \in \mathcal{D}(\mathbf{R}) \text{ with } \operatorname{supp}\varphi \subset (\omega - \varepsilon, \omega + \varepsilon).$$

See [**4**, page 474] for the particular case $h \in C(\omega-\varepsilon, \omega+\varepsilon)$. The points $i\omega$ which are not weakly regular points are called *weakly singular points.*

The *weak Laplace spectrum* of $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$ is defined (see [**5**, page 324]) by

$$sp^{w\mathcal{L}}(F) = \{\omega \in \mathbf{R} : i\omega \text{ is not a weakly regular point for } \mathcal{L}F\}. \tag{4.8}$$

For $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, we define $sp^{w\mathcal{L}}(F) := sp^{w\mathcal{L}}(F|\mathbf{R}_+)$. It follows readily that, if $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, then

$$\begin{aligned} &sp^{w\mathcal{L}}(F) \subset sp^{\mathcal{L}}(F) \subset sp^{\mathcal{C}}(F); \text{ and,} \\ &\text{if } F \in L^1(\mathbf{R}_+, X), sp^{w\mathcal{L}}(F) = \varnothing. \end{aligned} \tag{4.9}$$

In the following sp^* denotes $sp^{\mathcal{L}}$ or $sp^{w\mathcal{L}}$ or $sp^{\mathcal{C}}$. Note that $sp^*(F)$ is closed for any $F \in \mathcal{S}'_{ar}(\mathbf{R}, X)$.

Proposition 4.1. *If $F, G \in \mathcal{S}'_{ar}(\mathbf{R}, X)$, then*

(i) $sp^*(F) = sp^*(F_s) = sp^*(cF)$ *for each* $s \in \mathbf{R}$, $0 \neq c \in \mathbf{C}$.

(ii) $sp^*(F) = \cup_{h>0} sp^*(M_h F)$.

(iii) $sp^*(\gamma_\omega F) = \omega + sp^*(F)$.

(iv) $sp^*(F+G) \subset sp^*(F) \cup sp^*(G)$.

Proof. (i) A simple calculation shows that, for $\lambda \in \mathbf{C}^{\pm}$,

$$\mathcal{L}^{\pm} F_s(\lambda) = e^{\lambda s} \mathcal{L}^{\pm} F(\lambda) - e^{\lambda s} \int_0^s e^{-\lambda t} F(t)\, dt. \tag{4.10}$$

Note that the second term on the right of (4.10) is entire in λ for each $s \in \mathbf{R}$. It follows that $\mathcal{L}^+ F$ (respectively $\mathcal{C}F$) is holomorphic at $i\omega$ if and only if $\mathcal{L}^+ F_s$ (respectively, $\mathcal{C}F_s$) is holomorphic at $i\omega$. This proves (i) for $sp^{\mathcal{L}}$ and $sp^{\mathcal{C}}$. Now, assume $i\omega$ is a weakly regular point for $\mathcal{L}F$. So there exists $\varepsilon > 0$ and $h \in L^1(\omega-\varepsilon, \omega+\varepsilon)$ satisfying $\lim_{a\searrow 0} \int_{-\infty}^{\infty} \mathcal{L}^+ F(a+i\,\eta)\varphi(\eta)\, d\eta = \int_{\omega-\varepsilon}^{\omega+\varepsilon} h(\eta)\varphi(\eta)\, d\eta$ for all $\varphi \in \mathcal{D}(\mathbf{R})$ with $\operatorname{supp}\varphi \subset (\omega-\varepsilon, \omega+\varepsilon)$. Then, by [**30**, Theorem 6.18, page 146] (valid also for X-valued distributions), $\lim_{a\searrow 0} \int_{-\infty}^{\infty} \mathcal{L}F(a+ i\,\eta)e^{(a+i\,\eta)s}\varphi(\eta)\, d\eta = \int_{\omega-\varepsilon}^{\omega+\varepsilon} h(\eta)e^{i\eta\, s}\varphi(\eta)\, d\eta$ for all $\varphi \in \mathcal{D}(\mathbf{R})$ with

$\operatorname{supp}\varphi \subset (\omega-\varepsilon, \omega+\varepsilon)$. It follows that $i\omega$ is a weakly regular point for $\mathcal{L}F_s$.

(ii) Another calculation shows that, for $\lambda \in \mathbf{C}^{\pm}$,
(4.11)

$$\mathcal{L}^{\pm} M_h F(\lambda) = g(\lambda h)\mathcal{L}^{\pm} F(\lambda) - (1/h)\int_0^h \left(e^{\lambda v}\int_0^v e^{-\lambda t}F(t)\,dt\right)dv,$$

where g is the entire function given by $g(\lambda) = (e^{\lambda}-1)/\lambda$ for $\lambda \neq 0$. Let $i\omega \in i\mathbf{R}$ be a regular point for $\mathcal{L}^+F$, and let $\widetilde{\mathcal{L}}^+F : V \to X$ be a holomorphic extension of $\mathcal{L}^+F$ to a neighborhood $V \subset \mathbf{C}$ of $i\omega$. Then $\widetilde{\mathcal{L}}^+ M_h F(\lambda) = g(\lambda h)\widetilde{\mathcal{L}}^+F(\lambda) - (1/h)\int_0^h (e^{\lambda v}\int_0^v e^{-\lambda t}F(t)\,dt)\,dv$, $\lambda \in V$, is a holomorphic extension of $\mathcal{L}^+M_hF$. So $i\omega$ is a regular point for $\mathcal{L}^+M_hF$. Conversely, suppose $i\omega \in i\mathbf{R}$ is a regular point of $\mathcal{L}M_hF$ for each $h > 0$. Choose $h_0 > 0$ such that $g(i\omega h_0) \neq 0$. Then $i\omega$ is a regular point for $\mathcal{L}^+F$. This proves (ii) for $sp^{\mathcal{L}}$. The case $sp^{\mathcal{C}}$ follows similarly noting that (4.11) implies $\mathcal{C}M_hF(\lambda) = g(\lambda h)\mathcal{C}F(\lambda) - (1/h)\int_0^h (e^{\lambda v}\int_0^v e^{-\lambda t}F(t)\,dt)\,dv$. The proof for $sp^{w\mathcal{L}}$ is similar to the one in part (i).

(iii) This follows easily from the definitions noting that $\mathcal{L}^+(\gamma_\omega F)(\lambda) = \mathcal{L}^+F(\lambda - i\omega)$ and $\mathcal{C}(\gamma_\omega F)(\lambda) = \mathcal{C}F(\lambda - i\omega)$.

(iv) This follows directly from the definition. ◻

The following result was obtained in [**10**, (3.12)] in the case $F \in L^\infty(\mathbf{R}_+, X)$ since then $sp_{C_0(\mathbf{R}_+,X),\mathcal{S}}(F) = sp_{C_0(\mathbf{R}_+,X)}(F)$ (see also [**17**, Lemma 1.16] for $\mathcal{A} = C_0(\mathbf{R}_+, X)$).

Proposition 4.2. *If $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$ and $\mathcal{A} \subset L^\infty(\mathbf{R}_+, X)$ satisfies* (3.1)*, then $sp_{\mathcal{A},\mathcal{S}}(F) \subset sp_{C_0(\mathbf{R}_+,X),\mathcal{S}}(F) \subset sp^{w\mathcal{L}}(F)$.*

Proof. By (3.2), $C_0(\mathbf{R}_+, X) \subset \mathcal{A}$ and so $sp_{\mathcal{A},\mathcal{S}}(F) \subset sp_{C_0(\mathbf{R}_+,X),\mathcal{S}}(F)$. Let $\omega \notin sp^{w\mathcal{L}}(F)$. Choose $\varepsilon > 0$ and $\varphi \in \mathcal{S}(\mathbf{R})$ such that $sp^{w\mathcal{L}}(F) \cap [\omega-\varepsilon, \omega+\varepsilon] = \varnothing$, $\widehat{\varphi}(\omega) = 1$ and $\operatorname{supp}\widehat{\varphi} \subset [\omega-\varepsilon, \omega+\varepsilon]$. By [**18**, Proposition 1.3], $\overline{F} * \varphi \in C_0(\mathbf{R}, X)$, and so $\omega \notin sp_{C_0(\mathbf{R}_+,X),\mathcal{S}}(F)$. ◻

Theorem 4.3. *Let $F \in \mathcal{S}'_{ar}(\mathbf{R}_+, X)$.*

(i) *If $0 \notin sp^{\mathcal{L}}(F)$ and $PF \in UC(\mathbf{R}_+, X)$, then $PF \in BUC(\mathbf{R}_+, X)$*

and $F \in \mathcal{E}_0(\mathbf{R}_+, X)$. *If also* $F \in L^\infty(\mathbf{R}_+, X)$ *or* $F \in SO(\mathbf{R}_+, X)$, *then* $PF \in BUC(\mathbf{R}_+, X)$.

(ii) *If* $M_h F \in BC(\mathbf{J}, X)$ *for all* $h > 0$, *if* $sp^{w\mathcal{L}}(F)$ *is countable and* $\gamma_{-\omega} F \in \mathcal{E}(\mathbf{J}, X)$ *for all* $\omega \in sp^{w\mathcal{L}}(F)$, *and, if* $\psi \in \mathcal{S}(\mathbf{R})$, *then* $(\overline{F} * \psi) \mid \mathbf{R}_+ \in AAP(\mathbf{R}_+, X)$ *for each* $\psi \in \mathcal{S}(\mathbf{R})$. *If also* $sp^{w\mathcal{L}}(F) = \varnothing$, *then* $\overline{F} * \psi \in C_0(\mathbf{R}, X)$.

Proof. (i) Let $h > 0$. We have $M_h F = (1/h)\Delta_h PF \in BUC(\mathbf{R}_+, X)$ and $0 \notin sp^{\mathcal{L}}(M_h F)$ by Proposition 4.1 (ii). By [**5**, Corollary 4.4.4], $PM_h F \in BUC(\mathbf{R}_+, X)$. As $M_h PF = PM_h F$, by Lemma 2.2 (ii) we conclude that $PF \in BUC(\mathbf{R}_+, X)$, and hence $F \in \mathcal{E}_0(\mathbf{R}_+, X)$.

If $F \in L^\infty(\mathbf{R}_+, X)$, then clearly $PF \in UC(\mathbf{R}_+, X)$. So, assume that $F \in SO(\mathbf{R}_+, X)$. Then $M_h F \in UC(\mathbf{R}_+, X)$ by (3.15) and $0 \notin sp^{\mathcal{L}}(M_h F)$ by Proposition 4.1 (ii). It follows that $M_h F \in BUC(\mathbf{R}_+, X)$ by [**18**, Proposition 1.3, Remark 1.4]. This implies $PF \in UC(\mathbf{R}_+, X)$ by [**9**, Proposition 1.4].

(ii) By Proposition 4.2, we have $sp_{AAP,\mathcal{S}}(F) \subset sp_{C_0,\mathcal{S}}(F) \subset sp^{w\mathcal{L}}(F)$, and the result follows by Theorem 3.7 (v) with $\mathcal{A} = AAP(\mathbf{R}_+, X)$ and $\mathcal{A}_0 = C_0(\mathbf{R}_+, X)$. $\square$

Remark 4.4. (i) Ingham's Tauberian theorem [**22**] (see [**5**, Theorem 4.4.1]) asserts that if $F \in BUC(\mathbf{R}_+, X)$ and $sp^{\mathcal{L}}(F) = \varnothing$, then $F \in C_0(\mathbf{R}_+, X)$. Chill proved this theorem replacing $sp^{\mathcal{L}}(F)$ by $sp^{w\mathcal{L}}(F)$ (see [17, Theorem 1.22]). Noting that by Proposition 4.2, we have (*) $sp_{C_0(\mathbf{R}_+,X),\mathcal{S}}(F) \subset sp^{w\mathcal{L}}(F) \subset sp^{\mathcal{L}}(F)$, Ingham's theorem and its generalization by Chill, follow by Theorem 3.7 (i). Similarly, Lemma 1.16 in [**17**, page 25] which states that if $F \in L^\infty(\mathbf{R}_+, X)$ and $0 \notin sp^{w\mathcal{L}}(F)$, then $0 \notin sp_{C_0(\mathbf{R}_+,X)}$ and $F \in \mathcal{E}_0(\mathbf{R}_+, X)$ follows from $(*)$ and Theorem 3.6 (b). Also, [**18**, Theorem 1.5] is a consequence of Theorem 3.7 (ii).

By the same argument, several Tauberian results in [**2, 3**], [**5**, Theorem 4.7.7, Corollary 4.7.10, Theorems 4.9.5, 4.9.7, Lemma 4.10.2], [**14, 15, 18**] are consequences of Theorems 3.6 and 3.7. Our proofs are simpler and different. Replacing Laplace and weak Laplace spectra by reduced spectra we are able to strengthen and unify these previous results. Theorems 3.7 (v) and 4.3 seem to be new for any spectrum.

(ii) If F in Theorem 3.6 is not bounded or slowly oscillating, then F is not necessarily ergodic. For example, if $g(t) = e^{it^2}$ and $F = g^{(n)}$ for some $n \in \mathbf{N}$, then by Example 4.5 below and (4.9), we find $sp^{w\mathcal{L}}(F) = \varnothing$. By Proposition 4.2, we get $sp_{C_0(\mathbf{R}_+,\mathbf{C}),\mathcal{S}}(F) = \varnothing$ but $F \mid \mathbf{R}_+$ is neither bounded nor ergodic when $n \geq 2$. If $n = 1$, F is ergodic but not bounded.

(iii) If $F \in C_0(\mathbf{J}, X)$ satisfies $F(t) = 1/|t|$ for $t \in \mathbf{J}$, $|t| > 1$, then $PF \in UC(\mathbf{J}, X)$ but PF is not bounded. This means that Theorem 3.5 (i) and Theorem 4.3 (i) are not valid if we replace $sp_{0,\mathcal{S}}(F)$ or $sp^{\mathcal{L}}(F)$ by $sp_{C_0(\mathbf{J},X),\mathcal{S}}(F)$.

In the following we use our results to calculate some Laplace spectra.

Example 4.5. Take $g(t) = e^{it^2}$ for $t \in \mathbf{R}$. Then $sp^{\mathcal{C}}(g) = \mathbf{R}$ and $sp^{\mathcal{L}}(g) = sp^{\mathcal{L}}(g^{(n)}) = \varnothing$ for any $n \in \mathbf{N}$. Moreover, $M_h g \in C_0(\mathbf{R}, \mathbf{C})$ and $sp^{\mathcal{L}}(M_h g) = \varnothing$ for all $h > 0$.

Proof. By Proposition 4.1 (i) and (iii), it is readily verified that $sp^{\mathcal{L}}(g) = sp^{\mathcal{L}}(g_a) = 2a + sp^{\mathcal{L}}(g)$ for each $a \in \mathbf{R}$. This implies that either $sp^{\mathcal{L}}(g) = \varnothing$ or $sp^{\mathcal{L}}(g) = \mathbf{R}$. Similarly either $sp^{\mathcal{C}}(g) = \varnothing$ or $sp^{\mathcal{C}}(g) = \mathbf{R}$. But $g \neq 0$ and so by [**28**, Proposition 0.5 (ii)], $sp^{\mathcal{C}}(g) = \mathbf{R}$.

Next note that $y(\lambda) = \mathcal{L}^+ g(\lambda)$ is a solution of the differential equation $y'(\lambda) + (\lambda/2i)y(\lambda) = 1/2i$ for $\lambda \in \mathbf{C}_+$. Solving the equation we find $y(\lambda) = e^{-\lambda^2/4i}(c + (1/2i)\int_0^\lambda e^{z^2/4i}\, dz$ for some choice of $c \in \mathbf{C}$. As this last function is entire, we conclude that $sp^{\mathcal{L}}(g) = \varnothing$. Since $\int_0^\infty e^{i\,t^2}\, dt$ converges as an improper Riemann integral and $M_h g(t) = (1/h)[Pg(t+h) - Pg(t)]$, it follows that $M_h g \in C_0(\mathbf{R}, \mathbf{C})$ for each $h > 0$. Moreover, by Proposition 4.1, $sp^{\mathcal{L}}(M_h g) \subset sp^{\mathcal{L}}(g) = \varnothing$ and $sp^{\mathcal{L}}(g') = \cup_{h>0} sp^{\mathcal{L}}(M_h g') = \cup_{h>0} sp^{\mathcal{L}}((1/h)(g_h - g)) = \varnothing$. It follows that $sp^{\mathcal{L}}(g^{(n)}) = \varnothing$ for any $n \in \mathbf{N}$. $\square$

Finally, we demonstrate that our results can be used to deduce spectral criteria for bounded solutions of evolution equations of the form

$$\frac{du(t)}{dt} = Au(t) + \phi(t), \quad u(0) \in X,\ t \in \mathbf{J}, \tag{4.12}$$

where A is a closed linear operator on X and $\phi \in L^\infty(\mathbf{J}, X)$.

Theorem 4.6. *Let $\phi \in L^\infty(\mathbf{J}, X)$ and u be a bounded mild solution of* (4.12)*. Let $\mathcal{A}$ satisfy* (3.1)*,* (3.6)*, $\gamma_\lambda \mathcal{A} \subset \mathcal{A}$ for all $\lambda \in \mathbf{R}$ and contain all constants.*

(i) *If $\mathbf{J} = \mathbf{R}_+$, then $i\, sp^{\mathcal{L}}(u) \subset (\sigma(A) \cap i\mathbf{R}) \cup i\, sp^{\mathcal{L}}(\phi)$.*

(ii) *If $sp_{\mathcal{A}}(\phi) = \varnothing$, then $i\, sp_{\mathcal{A}}(u) \subset \sigma(A) \cap i\mathbf{R}$.*

Proof. As $u, \phi \in L^\infty(\mathbf{J}, X)$, we get $M_h u, M_h \phi \in BUC(\mathbf{J}, X)$ and $v = M_h u$ is a classical solution of $v'(t) = Av(t) + M_h\phi(t)$, $v(t) \in D(A)$, $t \in \mathbf{J}$ for each $h > 0$.

(i) By [**5**, Proposition 5.6.7, page 380], we have

$$i\, sp^{\mathcal{L}}(M_h u) \subset (\sigma(A) \cap i\mathbf{R}) \cup i\, sp^{\mathcal{L}}(M_h \phi) \quad \text{for all } h > 0.$$

Taking the union of both sides, we get

$$\cup_{h>0}\, i\, sp^{\mathcal{L}}(M_h u) \subset (\sigma(A) \cap i\mathbf{R}) \cup (\cup_{h>0}\, i\, sp^{\mathcal{L}}(M_h(\phi))).$$

Applying Proposition 4.1 (ii) to both sides, we conclude that

$$i\, sp^{\mathcal{L}}(u) \subset (\sigma(A) \cap i\mathbf{R}) \cup i\, sp^{\mathcal{L}}(\phi).$$

(ii) Take $h > 0$. Since $sp_{\mathcal{A}}(\phi) = \varnothing$, it follows that $sp_{\mathcal{A}}(M_h\phi) = \varnothing$, by Proposition 3.4 (ii). Hence, $M_h\phi \in \mathcal{A}$ by [**6**, Theorem 4.2.1]. Using [**7**, Corollary 3.4 (i)], we conclude that $i\, sp_{\mathcal{A}}(M_h u) \subset \sigma(A) \cap i\mathbf{R}$. By Proposition 3.4 (ii), we conclude that $i\, sp_{\mathcal{A}}(u) \subset \sigma(A) \cap i\mathbf{R}$. □

Remark 4.7. (i) There are some inclusions of this general type in [**4**, (4.4), (4.5)].

(ii) In [**11**], the inclusion $i\, sp_{\mathcal{A}}(u) \subset (\sigma(A) \cap i\mathbf{R}) \cup i\, sp_{\mathcal{A}}(\phi)$ was proved. □

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