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EXISTENCE OF NONOSCILLATORY SOLUTIONS TO SECOND-ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

JIN GAO AND QIRU WANG

ABSTRACT. By employing Kranoselskii's fixed point theorem, we establish the existence of nonoscillatory solutions to the second-order nonlinear neutral dynamic equation $[r(t)(x(t)+p(t)x(g(t)))^{\Delta}]^{\Delta} + f(t, x(h(t))) = 0$ on a time scale. In particular, one interesting example is included to illustrate the versatility of our results.

1. Introduction. Consider second-order nonlinear neutral dynamic equations of the form

$$[r(t)(x(t) + p(t)x(g(t)))^{\Delta}]^{\Delta} + f(t, x(h(t))) = 0 \tag{1}$$

on a time scale $\mathbf{T}$. The motivation originates from [**6, 8**], where some open problems were presented in [**6**] and some conditions for the existence of nonoscillatory solutions of first-order nonlinear neutral dynamic equation $[x(t)+p(t)x(g(t))]^{\Delta}+f(t, x(h(t))) = 0$ were presented in [**8**]. In this paper, by employing Kranoselskii's fixed point theorem, we try to find some conditions for the existence of nonoscillatory solutions of (1). We remark that there has been a number of researchers studying oscillatory behaviors for dynamic equations on time scales, see, e.g., [**1–3, 5–7**] and the references therein. However, there are few papers discussing the existence of nonoscillatory solutions for neutral functional dynamic equations on time scales. For a neutral functional dynamic equation, the highest derivative of the unknown function appears with the argument t (the present state of the system) as well as one or more deviating arguments (the past or future state of the system).

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For convenience, we recall some concepts related to time scales. More details can be found in [**1, 2**].

Definition 1. A time scale is an arbitrary nonempty closed subset of the set $\mathbf{R}$ of real numbers with the topology and ordering inherited from $\mathbf{R}$. Let $\mathbf{T}$ be a time scale, for $t \in \mathbf{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbf{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbf{T} : s < t\}$ and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf \varnothing := \sup \mathbf{T}$ and $\sup \varnothing := \inf \mathbf{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The set $\mathbf{T}^\kappa$ is defined as follows: if $\mathbf{T}$ has a left-scattered maximum m, then $\mathbf{T}^\kappa = \mathbf{T} - \{m\}$; otherwise, $\mathbf{T}^\kappa = \mathbf{T}$.

Definition 2. For a function $f : \mathbf{T} \to \mathbf{R}$ and $t \in \mathbf{T}^\kappa$, we define the delta-derivative $f^\Delta(t)$ of $f(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbf{T}$ for some δ) such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We say that f is delta-differentiable (or in short: differentiable) on $\mathbf{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbf{T}^\kappa$.

It is easily seen that if f is continuous at $t \in \mathbf{T}$ and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Moreover, if t is right-dense, then f is differential at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In addition, if $f^\Delta \geq 0$, then f is nondecreasing.

Definition 3. Let $f : \mathbf{T} \to \mathbf{R}$ be a function, f is called *right-dense continuous* (rd-continuous) if it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbf{T}$. A function $F : \mathbf{T} \to \mathbf{R}$ is called an *antiderivative* of f provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbf{T}^k$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\Delta s = F(b) - F(a)$, and $\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s$.

Let $C_{rd}(\mathbf{T}, \mathbf{R})$ denote the set of all rd-continuous functions mapping $\mathbf{T}$ to $\mathbf{R}$. It is shown in [**2**] that every rd-continuous function has an antiderivative. Since we are interested in the nonoscillatory behavior of (1), we assume throughout that the time scale $\mathbf{T}$ under consideration satisfies $\inf \mathbf{T} = t_0$ and $\sup \mathbf{T} = \infty$.

As usual, by a solution of (1) we mean a function $x(t)$ which is defined on $\mathbf{T}$ and satisfies (1) for $t \geq t_0$. A solution x of (1) is said to be eventually positive (or eventually negative) if there exists a $c \in \mathbf{T}$ such that $x(t) > 0$ (or $x(t) < 0$) for all $t \geq c$ in $\mathbf{T}$. A solution x of (1) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

2. Main results. For $T_0, T_1 \in \mathbf{T}$, let $[T_0, \infty)_\mathbf{T} := \{t \in \mathbf{T} : t \geq T_0\}$ and $[T_0, T_1]_\mathbf{T} := \{t \in \mathbf{T} : T_0 \leq t \leq T_1\}$. Further, let $C([T_0, \infty)_\mathbf{T}, \mathbf{R})$ denote all continuous functions mapping $[T_0, \infty)_\mathbf{T}$ into $\mathbf{R}$, and

(2)
$$BC[T_0, \infty)_\mathbf{T} := \Big\{x : x \in C([T_0, \infty)_\mathbf{T}, \mathbf{R}) \text{ and } \sup_{t\in[T_0,\infty)_\mathbf{T}} |x(t)| < \infty\Big\}.$$

Endowed on $BC[T_0, \infty)_\mathbf{T}$ with the norm $||x|| = \sup_{t\in[T_0,\infty)_\mathbf{T}} |x(t)|$, $(BC[T_0, \infty)_\mathbf{T}, ||\cdot||)$ is a Banach space. Letting $X \subseteq BC[T_0, \infty)_\mathbf{T}$, we say that X is uniformly Cauchy if, for any given $\varepsilon > 0$, there exists a $T_1 \in [T_0, \infty)_\mathbf{T}$ such that, for any $x \in X$,

$$|x(t_1) - x(t_2)| < \varepsilon \quad \text{for all } t_1, t_2 \in [T_1, \infty)_\mathbf{T}.$$

X is said to be equicontinuous on $[a, b]_\mathbf{T}$ if, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ and $t_1, t_2 \in [a, b]_\mathbf{T}$ with

$|t_1 - t_2| < \delta$,

$$|x(t_1) - x(t_2)| < \varepsilon.$$

The following is an analogue of the Arzela-Ascoli theorem on time scales.

Lemma 1 [**8**, Lemma 4]. *Suppose that $X \subseteq BC[T_0, \infty)_{\mathbf{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equicontinuous on $[T_0, T_1]_{\mathbf{T}}$ for any $T_1 \in [T_0, \infty)_{\mathbf{T}}$. Then X is relatively compact.*

In this section, we will employ Kranoselskii's fixed point theorem (see [**4**]) to establish the existence of nonoscillatory solutions for (1). For the sake of convenience, we state this theorem as follows.

Lemma 2 (Kranoselskii's fixed point theorem). *Suppose that X is a Banach space and Ω is a bounded, convex and closed subset of X. Suppose further that two operators $U, S : \Omega \to X$ exist such that*

(i) *$Ux + Sy \in \Omega$ for all $x, y \in \Omega$;*

(ii) *U is a contraction mapping;*

(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in Ω.

Throughout this section, we will assume in (1) that

(A1) $r \in C_{rd}(\mathbf{T}, (0, \infty))$ and $\int_{t_0}^{\infty} 1/r(s)\Delta s < \infty$;

(A2) $g, h \in C_{rd}(\mathbf{T}, \mathbf{T}), g(t) \leq t, \lim_{t\to\infty} g(t) = \infty, \lim_{t\to\infty} h(t) = \infty$, and there exists a $\{c_k\}_{k\geq 0}$ such that $\lim_{k\to\infty} c_k = \infty$ and $g(c_{k+1}) = c_k$;

(A3) $p \in C_{rd}(\mathbf{T}, \mathbf{R})$ and there exists a constant p_0 with $|p_0| < 1$ such that $\lim_{t\to\infty} p(t) = p_0$;

(A4) $f \in C(\mathbf{T} \times \mathbf{R}, \mathbf{R}), f(t, x)$ is nondecreasing in x and $xf(t, x) > 0$ for $t \in \mathbf{T}$ and $x \neq 0$.

We note by the assumptions above that, if $x(t)$ is an eventually negative solution of (1), then $y(t) = -x(t)$ satisfies

$$[r(t)(y(t) + p(t)y(g(t)))^{\Delta}]^{\Delta} - f(t, -y(h(t))) = 0.$$

We further note that $\overline{f}(t,u) := -f(t,-u)$ is nondecreasing in the second variable and $u\overline{f}(t,u) > 0$ for $t \in \mathbf{T}$ and $u \neq 0$. Hence, in the following, we will restrict our attention to eventually positive solutions of (1).

In the sequel, we use the notation

$$z(t) = x(t) + p(t)x(g(t)). \tag{3}$$

Now, we present our first theorem for a classification scheme of the eventually positive solutions to equation (1).

Theorem 1. *If $x(t)$ is an eventually positive solution of* (1)*, then either* $\lim_{t\to\infty} x(t) = a > 0$ *or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). In view of conditions (A2) and (A3), $T_1 \in \mathbf{T}$ and $|p_0| < p_1 < 1$ exist such that $x(h(t)) > 0$, $x(g(t)) > 0$ and $|p(t)| \leq p_1$ for all $t \in [T_1, \infty)_{\mathbf{T}}$. Then, from (1) and (3), we have $[r(t)z^{\Delta}(t)]^{\Delta} < 0$ on $[T_1,\infty)_{\mathbf{T}}$, which means that $r(t)z^{\Delta}(t)$ is decreasing on $[T_1,\infty)_{\mathbf{T}}$. Then

$$r(t)z^{\Delta}(t) \leq r(T_1)z^{\Delta}(T_1)$$

or

$$z^{\Delta}(t) \leq \frac{r(T_1)z^{\Delta}(T_1)}{r(t)} \tag{4}$$

on $[T_1,\infty)_{\mathbf{T}}$. If there exists a $t_1 \in \mathbf{T}$ with $t_1 \geq T_1$ such that $z^{\Delta}(t_1) \leq 0$, then $r(t)z^{\Delta}(t) \leq 0$ for $t \geq t_1$ and so $z^{\Delta}(t)$ is eventually negative. Otherwise, if, for every $t \geq T_1$, $z^{\Delta}(t) > 0$, then $z^{\Delta}(t)$ is eventually positive. Hence, $z(t)$ is always monotonic eventually.

Integrating (4) from T_1 to $t(\geq T_1)$, by (A1), we have

$$\begin{aligned} z(t) - z(T_1) &\leq r(T_1)z^{\Delta}(T_1)\int_{T_1}^{t} \frac{1}{r(s)}\Delta s \\ &< r(T_1)\left|z^{\Delta}(T_1)\right|\int_{T_1}^{\infty} \frac{1}{r(s)}\Delta s, \end{aligned}$$

which implies that $z(t)$ is upper bounded.

Now, we claim that $\lim_{t\to\infty} z(t)$ exists (finite) and is nonnegative. Otherwise, $\lim_{t\to\infty} z(t) < 0$ or $\lim_{t\to\infty} z(t) = -\infty$, which implies that there exists a $T_2 \geq T_1$ such that

$$z(t) < 0 \text{ or } x(t) < -p(t)x(g(t)) < p_1 x(g(t)) \text{ for } t \in [T_2, \infty)_{\mathbf{T}}.$$

By (A2), we can choose some positive integer k_0 such that $c_k \geq T_2$ for all $k \geq k_0$. Then, for any $k \geq k_0 + 1$, we have

$$\begin{aligned} x(c_k) &< p_1 x(g(c_k)) = p_1 x(c_{k-1}) < p_1^2 x(g(c_{k-1})) = p_1^2 x(c_{k-2}) \\ &< \cdots < p_1^{k-k_0} x(g(c_{k_0+1})) = p_1^{k-k_0} x(c_{k_0}). \end{aligned}$$

The inequality above implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows from (3) that $\lim_{k\to\infty} z(c_k) = 0$ which contradicts $\lim_{t\to\infty} z(t) < 0$ or $\lim_{t\to\infty} z(t) = -\infty$.

Next, we assert that $x(t)$ is bounded. If this is not true, $\{t_k\}$ exists with $t_k \to \infty$ as $k \to \infty$ such that

$$x(t_k) = \max_{t_0 \leq s \leq t_k} x(s) \quad \text{and} \quad \lim_{k\to\infty} x(t_k) = \infty.$$

Since $g(t) \leq t$ and

$$z(t_k) = x(t_k) + p(t_k)x(g(t_k)) \geq (1 - |p(t_k)|)x(t_k),$$

it follows from (A3) that $\lim_{k\to\infty} z(t_k) = \infty$, which contradicts the conclusion above that $\lim_{t\to\infty} z(t) = b \geq 0$ and b is finite. Hence, $x(t)$ is bounded.

Finally, we assume that

$$\limsup_{t\to\infty} x(t) = \overline{x}, \qquad \liminf_{t\to\infty} x(t) = \underline{x}.$$

If $0 \leq p_0 < 1$, we have

$$b \geq \overline{x} + p_0 \underline{x} \quad \text{and} \quad b \leq \underline{x} + p_0 \overline{x},$$

which implies that $\overline{x} \leq \underline{x}$. Thus, $\overline{x} = \underline{x}$ when $0 \leq p_0 < 1$.

If $-1 < p_0 < 0$, we have

$$b \geq \overline{x} + p_0\overline{x} \quad \text{and} \quad b \leq \underline{x} + p_0\underline{x},$$

which implies that $\overline{x} \leq \underline{x}$. Thus, $\overline{x} = \underline{x}$ when $-1 < p_0 < 0$.

To sum up, we see that $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = b/(1+p_0)$. The proof is complete. □

Next, we will give the existence criteria for each type of solution.

Theorem 2. *Equation* (1) *has an eventually positive solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) = a > 0$ *if and only if there exists some constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{f(t,K)}{r(s)} \Delta t \Delta s < \infty. \tag{5}$$

Proof. Suppose that $x(t)$ is an eventually positive solution of (1) satisfying $\lim_{t\to\infty} x(t) = a > 0$. Then $\lim_{t\to\infty} z(t) = (1+p_0)a$, and $T_1 \in \mathbf{T}$ exists such that $x(h(t)) \geq a/2$ for $t \in [T_1, \infty)_{\mathbf{T}}$. Integrating (1) from T_1 to $s(\geq T_1)$, we have

$$r(s)z^{\Delta}(s) - r(T_1)z^{\Delta}(T_1) = -\int_{T_1}^{s} f(u, x(h(u)))\Delta u,$$

or

$$z^{\Delta}(s) = \frac{r(T_1)z^{\Delta}(T_1)}{r(s)} - \frac{\int_{T_1}^{s} f(u, x(h(u)))\Delta u}{r(s)}. \tag{6}$$

Integrating (6) from T_1 to $t(\geq T_1)$, we obtain

$$z(t) - z(T_1) = r(T_1)z^{\Delta}(T_1) \int_{T_1}^{t} \frac{1}{r(s)} \Delta s - \int_{T_1}^{t} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s. \tag{7}$$

Letting $t \to \infty$ in (7), we have

$$\int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s < \infty.$$

In view of (A4), we have

$$f\left(u, \frac{a}{2}\right) \leq f(u, x(h(u))), \quad \text{for } u \in [T_1, \infty)_{\mathbf{T}}$$

and

$$\int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, (a/2))}{r(s)} \Delta u \Delta s \leq \int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s < \infty,$$

which means that (5) holds. The necessary condition is proved.

Conversely, suppose that there exists some constant $K > 0$ such that (5) holds. There are two cases to be considered: $0 \leq p_0 < 1$ and $-1 < p_0 < 0$.

In the case $0 \leq p_0 < 1$, take p_1 so that $p_0 < p_1 < (1 + 4p_0)/5 < 1$; then $p_0 > (5p_1 - 1)/4$.

Since $\lim_{t \to \infty} p(t) = p_0$ and (5) holds, we can choose $T_0 \in \mathbf{T}$ large enough such that

$$\frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \quad t \in [T_0, \infty)_{\mathbf{T}} \tag{8}$$

and

$$\int_{T_0}^{\infty} \int_{t_0}^{s} \frac{f(t, K)}{r(s)} \Delta t \Delta s \leq \frac{(1 - p_1)K}{8}. \tag{9}$$

Furthermore, from (A2), we see that $T_1 \in \mathbf{T}$ exists with $T_1 > T_0$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbf{T}}$.

Define the Banach space $BC[t_0, \infty)_{\mathbf{T}}$ as in (2), and let

$$\Omega = \left\{ x = x(t) \in BC[t_0, \infty)_{\mathbf{T}} : \frac{K}{2} \leq x(t) \leq K \right\}. \tag{10}$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC[t_0, \infty)_{\mathbf{T}}$. By (A4), we have that, for any $x \in \Omega$,

$$f(t, x(h(t))) \leq f(t, K), \quad t \in [T_1, \infty)_{\mathbf{T}}.$$

Now we define two operators S_1 and $S_2 : \Omega \to BC[t_0, \infty)_{\mathbf{T}}$ as follows:

$$(S_1x)(t) = \begin{cases} (3Kp_1)/4 - p(t)x(g(t)) & t \in [T_1, \infty)_{\mathbf{T}}, \\ (S_1x)(T_1) & t \in [t_0, T_1]_{\mathbf{T}}, \end{cases}$$

and

(11)
$$(S_2x)(t) = \begin{cases} (3K/4) + \int_t^\infty \int_{t_0}^s [(f(u, x(h(u))))/r(s)] \Delta u \Delta s & t \in [T_1, \infty)_{\mathbf{T}}, \\ (S_2x)(T_1) & t \in [t_0, T_1]_{\mathbf{T}}. \end{cases}$$

Next, we will show that S_1 and S_2 satisfy the conditions in Lemma 2.

(i) We first prove that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. Note that, for any $x, y \in \Omega$, $K/2 \le x \le K$ and $K/2 \le y \le K$. For any $x, y \in \Omega$ and $t \in [T_1, \infty)_{\mathbf{T}}$, by (8) and (9), we have:

$$\begin{aligned} &(S_1x)(t) + (S_2y)(t) \\ &\quad = \frac{3(1+p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, y(h(u)))}{r(s)} \Delta u \Delta s \\ &\quad \ge \frac{3(1+p_1)K}{4} - p_1K = \frac{(3-p_1)K}{4} > \frac{K}{2}, \end{aligned}$$

and

$$\begin{aligned} &(S_1x)(t) + (S_2y)(t) \\ &\quad \le \frac{3(1+p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, K)}{r(s)} \Delta u \Delta s \\ &\quad \le \frac{3(1+p_1)K}{4} - \frac{5p_1 - 1}{4} \times \frac{K}{2} + \frac{(1-p_1)K}{8} \\ &\quad = K. \end{aligned}$$

Similarly, we can prove that $K/2 \le (S_1x)(t) + (S_2y)(t) \le K$ for any $x, y \in \Omega$ and $t \in [t_0, T_1]_{\mathbf{T}}$. Hence, $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

(ii) We prove that S_1 is a contraction mapping. Indeed, for $x, y \in \Omega$, we have

$$\begin{aligned} |(S_1x)(t) - (S_1y)(t)| &= |p(t)[x(g(T_1)) - y(g(T_1))]| \\ &\le p_1 \sup_{t \in [t_0, \infty)_{\mathbf{T}}} |x(t) - y(t)| \end{aligned}$$

for $t \in [t_0, T_1]_{\mathbf{T}}$ and

$$|(S_1x)(t) - (S_1y)(t)| = |p(t)[(x(g(t)) - y(g(t))]| \\ \leq p_1 \sup_{t\in[t_0,\infty)_{\mathbf{T}}} |x(t) - y(t)|$$

for $t \in [T_1, \infty)_{\mathbf{T}}$. Therefore, we have

$$||S_1x - S_1y|| \leq p_1||x - y||$$

for any $x, y \in \Omega$. Hence, S_1 is a contraction mapping.

(iii) We will prove that S_2 is a completely continuous mapping.

First, for $t \in [t_0, \infty)_{\mathbf{T}}$, we see that $(S_2x)(t) > 3K/4$ and $(S_2x)(t) \leq 3K/4+ (1 - p_1)K/8 = (7 - p_1)K/8 < K$. That is, S_2 maps Ω into Ω.

Second, we consider the continuity of S_2. Let $x_n \in \Omega$ and $||x_n - x|| \to 0$ as $n \to \infty$. Then $x \in \Omega$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \infty)_{\mathbf{T}}$. Consequently, for $t \in [T_1, \infty)_{\mathbf{T}}$,

$$|f(t, x_n(h(t))) - f(t, x(h(t)))| \longrightarrow 0 \quad \text{as } n \to \infty,$$

and, for $t \in \mathbf{T}$,

$$|(S_2x_k)(t) - (S_2x)(t)| \\ \leq \int_{T_1}^{\infty} \int_{t_0}^{s} \frac{|f(u, x_k(h(u))) - f(u, x(h(u)))|}{r(s)} \Delta u \Delta s.$$

By applying the Lebesgue dominated convergence theorem, we conclude that

$$||(S_2x_k)(t) - (S_2x)(t)|| \longrightarrow 0 \quad (k \to \infty).$$

This means that S_2 is continuous.

Third, we show $S_2\Omega$ is relatively compact. According to Lemma 1, it suffices to show that $S_2\Omega$ is bounded, uniformly Cauchy and equicontinuous. The boundedness is obvious. Since $\int_{t_0}^{\infty} \int_{t_0}^{s} [f(t, K)/r(s)] \Delta t \Delta s < \infty$, for any $\varepsilon > 0$, there exists a $T_2 \in [T_1, \infty)_{\mathbf{T}}$ large enough so that $\int_{T_2}^{\infty} \int_{t_0}^{s} [f(u, K)/r(s)] \Delta u \Delta s < (\varepsilon/2)$. Then, for any $x \in \Omega$ and

$t_1, t_2 \in [T_2, \infty)_{\mathbf{T}}$, we have

$$\begin{aligned}|(S_2x)(t_1) - (S_2x)(t_2)| &= \bigg| \int_{t_1}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \\ &\quad - \int_{t_2}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \bigg| \\ &\le \int_{t_1}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \\ &\quad + \int_{t_2}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \\ &\le 2 \int_{T_2}^{\infty} \int_{t_0}^{s} \frac{f(u, K)}{r(s)} \Delta u \Delta s < \varepsilon.\end{aligned}$$

Thus, $S_2\Omega$ is uniformly Cauchy.

Also, for $x \in \Omega, t_1, t_2 \in [\min\{T_1 - 1, t_0\}, T_2 + 1]_{\mathbf{T}}$, we have

$$\begin{aligned}|(S_2x)(t_1) - (S_2x)(t_2)| &\le \bigg| \int_{t_1}^{t_2} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \bigg| \\ &\le \bigg| \int_{t_1}^{t_2} \int_{t_0}^{s} \frac{f(u, K)}{r(s)} \Delta u \Delta s \bigg|.\end{aligned}$$

Then $0 < \delta < 1$ exists such that $|(S_2x)(t_1) - (S_2x)(t_2)| < \varepsilon$ if $|t_2 - t_1| < \delta$.

For any $x \in \Omega, t_1, t_2 \in [t_0, T_1]_{\mathbf{T}}$, it is easy to see that $|(S_2x)(t_1) - (S_2x)(t_2)| = 0 < \varepsilon$. This means that $S_2\Omega$ is equicontinuous.

It follows from Lemma 1 that $S_2\Omega$ is relatively compact, and then S_2 is completely continuous.

By Lemma 2, $x \in \Omega$ exists such that $(S_1 + S_2)x = x$, which means that $x(t)$ is a solution of (1). In particular, we have

$$\begin{aligned}x(t) &= \frac{3(1 + p_1)K}{4} - p(t)x(g(t)) \\ &\quad + \int_{t}^{\infty} \int_{t_0}^{s} \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s, \quad t \in [T_1, \infty)_{\mathbf{T}}.\end{aligned}$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} z(t) = 3(1 + p_1)K/4$, and then $\lim_{t\to\infty} x(t) = 3(1 + p_1)K/(4 + 4p_0) > 0$. The sufficiency holds when $0 \le p_0 < 1$.

In the case $-1 < p_0 < 0$, take p_1 so that $-p_0 < p_1 < (1-4p_0)/5 < 1$; then $p_0 < (1-5p_1)/4$. Since $\lim_{t\to\infty} p(t) = p_0$ and (5) holds, we can choose $T_0 \in \mathbf{T}$ large enough such that (9) holds and

$$\frac{5p_1 - 1}{4} \le -p(t) \le p_1 < 1, \quad t \in [T_0, \infty)_{\mathbf{T}}.$$

Take $T_1 \in \mathbf{T}$ with $T_1 > T_0$ so that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbf{T}}$. Similarly, we introduce the Banach space $BC[t_0, \infty)_{\mathbf{T}}$ and its subset Ω as in (10). Define operator S_2 as in (11) and operator S_1 on Ω as follows

$$(S_1x)(t) = \begin{cases} -(3Kp_1)/4 - p(t)x(g(t)) & t \in [T_1, \infty)_{\mathbf{T}}, \\ (S_1x)(T_1) & t \in [t_0, T_1]_{\mathbf{T}}. \end{cases}$$

Next, we prove that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. Indeed, for any $x, y \in \Omega$ and $t \in [T_1, \infty)_{\mathbf{T}}$, we have

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= \frac{3(1-p_1)K}{4} - p(t)x(g(t)) \\ &\quad + \int_t^\infty \int_{t_0}^s \frac{f(u, y(h(u)))}{r(s)} \Delta u \Delta s \\ &\ge \frac{3(1-p_1)K}{4} - p(t)x(g(t)) \\ &\ge \frac{3(1-p_1)K}{4} + \frac{5p_1 - 1}{4} \times \frac{K}{2} \\ &= \frac{(5-p_1)K}{8} > \frac{K}{2} \end{aligned}$$

and

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &\le \frac{3(1-p_1)K}{4} - p(t)x(g(t)) \\ &\quad + \int_t^\infty \int_{t_0}^s \frac{f(u, K)}{r(s)} \Delta u \Delta s \\ &\le \frac{3(1-p_1)K}{4} + p_1 K + \frac{(1-p_1)K}{8} \\ &= \frac{(7+p_1)K}{8} < K. \end{aligned}$$

That is, $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

The following proof is similar to that of the case $0 \le p_0 < 1$ and is omitted. By Lemma 2, $x \in \Omega$ exists such that $(S_1 + S_2)x = x$, which means that $x(t)$ is a solution of (1). In particular, we have

$$x(t) = \frac{3(1-p_1)K}{4} - p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s, \quad t \in [T_1, \infty)_{\mathbf{T}}.$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} z(t) = 3(1-p_1)K/4$, and then $\lim_{t\to\infty} x(t) = 3(1-p_1)K/(4+4p_0) > 0$. The sufficiency also holds when $-1 < p_0 < 0$.

The proof is complete. □

Theorem 3. If $T_0 \in \mathbf{T}$ exists with $T_0 > 0$ such that

$$p(t)e^{-g(t)} \le -e^{-t}, \ t \in [T_0, \infty)_{\mathbf{T}} \tag{12}$$

and

$$\int_t^\infty \int_{t_0}^s \frac{f(u, \frac{1}{h(u)})}{r(s)} \Delta u \Delta s \le \frac{1}{t} + \frac{p(t)}{g(t)}, \quad t \in [T_0, \infty)_{\mathbf{T}}, \tag{13}$$

then equation (1) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$.

Proof. Take $T_1 \in \mathbf{T}$ with $T_1 > T_0$ so that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbf{T}}$. Define the Banach space $BC[t_0, \infty)_{\mathbf{T}}$ as in (2). Let

$$\Omega = \Big\{ x \in BC[t_0, \infty)_{\mathbf{T}} : e^{-t} \le x(t) \le \frac{1}{t} \quad \text{for } t \in [T_1, \infty)_{\mathbf{T}} \text{ and } e^{-T_1} \le x(t) \le \frac{1}{t} \quad \text{for } t \in [t_0, T_1]_{\mathbf{T}} \Big\};$$

then Ω is a bounded, convex and closed subset of $BC[t_0, \infty)_{\mathbf{T}}$. Define an operator S on Ω as follows:

$$(Sx)(t) = \begin{cases} -p(T_1)x(g(T_1)) \\ \quad + \int_{T_1}^\infty \int_{t_0}^s [f(u, x(h(u)))/r(s)] \Delta u \Delta s & t \in [t_0, T_1]_{\mathbf{T}}, \\ -p(t)x(g(t)) \\ \quad + \int_t^\infty \int_{t_0}^s [f(u, x(h(u)))/r(s)] \Delta u \Delta s & t \in [T_1, \infty)_{\mathbf{T}}. \end{cases}$$

First, we show that $Sx \in \Omega$ for all $x \in \Omega$. Indeed, from (12) and (13), we have for $t \in [T_1, \infty)_{\mathbf{T}}$,

$$\begin{aligned}(Sx)(t) &= -p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, x(h(u)))}{r(s)} \Delta u \Delta s \\ &\le -p(t)x(g(t)) + \int_t^\infty \int_{t_0}^s \frac{f(u, [1/h(u)])}{r(s)} \Delta u \Delta s \\ &\le \frac{-p(t)}{g(t)} + \frac{1}{t} + \frac{p(t)}{g(t)} = \frac{1}{t}\end{aligned}$$

and

$$(Sx)(t) \ge -p(t)x(g(t)) \ge -p(t)e^{-g(t)} \ge e^{-t}.$$

Also, $e^{-T_1} \le (Sx)(t) \le 1/t$ for $t \in [t_0, T_1]_{\mathbf{T}}$. Thus, we have proved that $Sx \in \Omega$ for all $x \in \Omega$. The rest of the proof is similar to Theorem 2 and hence is omitted.

By Lemma 2 with the operator $U = 0$, there exists an $x \in \Omega$ such that $Sx = x$, which means that $x(t)$ is a solution of (1). Note from the definition of Ω, we see that $x(t)$ is eventually positive and $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Example. Let $q > 1$ and $\mathbf{T} = \{q^n : n \in \mathbf{N}_0\}$, where $\mathbf{N}_0$ is the set of nonnegative integers. Consider the following equation

$$\left[t^2\left[x(t) + \frac{t+1}{2t}x(\rho(t))\right]^\Delta\right]^\Delta + \frac{x(\sigma(t))}{t^2\sigma(t)} = 0, \quad t \in \mathbf{T}. \tag{14}$$

Then $p(t) = (t+1)/2t$, $g(t) = \rho(t)$, $h(t) = \sigma(t)$, $f(t,x) = x/(t^2\sigma(t))$ and $r(t) = t^2$. It is easy to see that all the conditions (A1)–(A4) are satisfied. Also, $\int_1^\infty \int_1^s [f(t,K)/r(s)] \Delta t \Delta s < \infty$ for any given $K > 0$. By Theorem 2, equation (14) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = a > 0$.

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REFERENCES

1. R. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: A survey*, J. Comput. Appl. Math. **141** (2002), 1–26.

2. M. Bohner and A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, Boston, 2001.

3. ———, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

4. Y.S. Chen, *Existence of nonoscillatory solutions of nth order neutral delay differential equations*, Funk. Ekvac. **35** (1992,) 557–570.

5. L.H. Erbe, Q.K. Kong and B.G. Zhang, *Oscillation theory for functional-differential equations*, Mono. Text. Pure Appl. Math. **190**, Marcel Dekker, Inc., New York, 1995.

6. R.M. Mathsen, Q.R. Wang and H.W. Wu, *Oscillation for neutral dynamic functional equations on time scales*, J. Differ. Equat. Appl. **10** (2004), 651–659.

7. Z.Q. Zhu and Q.R. Wang, *Frequency measures on time scales with applications*, J. Math. Anal. Appl. **319** (2006), 398–409.

8. ———, *Existence of nonoscillatory solutions to neutral dynamic equations on time scales*, J. Math. Anal. Appl. **335** (2007), 751–762.

School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, P.R. China
Email address: 306996140@qq.com

School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, P.R. China
Email address: mcswqr@mail.sysu.edu.cn