

A UNIVERSAL SURVIVAL RING OF CONTINUOUS FUNCTIONS WHICH IS NOT A UNIVERSAL LYING-OVER RING

DAVID E. DOBBS, RONALD LEVY AND JAY SHAPIRO

ABSTRACT. The ring R of continuous real-valued functions on the one-point compactification of the discrete space of cardinality \aleph_1 is a universal survival ring, yet is not a ULO-ring. Chains of prime ideals of R of cardinality \aleph_1 exist. Moreover, R/P is a divided domain for each $P \in \text{Spec}(R)$. If the Continuum Hypothesis holds, then there exists a minimal prime ideal P of R such that R/P is an infinite-dimensional valuation domain; however, it is consistent with ZFC that no such minimal primes exist.

1. Introduction. All rings considered below are commutative, with $1 \neq 0$; all ring homomorphisms and ring extensions are unital. If A is a ring, then $Z(A)$ denotes the set of zero-divisors of A ; $\text{tq}(A) := A_{A \setminus Z(A)}$, the total quotient ring of A ; and $\text{Spec}(A)$ denotes the set of all prime ideals of A . As usual, “dim(ension)” refers to the Krull dimension. Following [11, page 28], we use LO to denote the lying-over property of ring extensions. Recall from [5, page 419] that a ring extension $A \subseteq B$ is said to satisfy QLO if, whenever $P \in \text{Spec}(A)$ is such that $PB \neq B$; then there exists a $Q \in \text{Spec}(B)$ such that $Q \cap A = P$. It is clear that $\text{LO} \Rightarrow \text{QLO}$, while any nontrivial ring of fractions (for instance, $\mathbf{Z} \subset \mathbf{Q}$) shows that $\text{QLO} \not\Rightarrow \text{LO}$. Slightly modifying terminology from [11, page 35], we say that a ring extension $A \subseteq B$ is a *survival extension* if $PB \neq B$ whenever $P \in \text{Spec}(A)$. It is clear that each ring extension that satisfies LO must be a survival extension; once again, examples such as $\mathbf{Z} \subset \mathbf{Q}$ show that the converse is false. Note that a survival extension satisfies LO if (and only if) it satisfies QLO.

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In [7] (respectively, [8]), the first and third authors developed the theory of rings A such that $A \subseteq B$ satisfies LO (respectively, QLO) for each ring extension B of A . Any such A was called a ULO-ring (respectively, a UQLO-ring). Analogously, we will say that a ring A is a *universal survival ring* (in short, a US-ring) if each ring extension of the form $A \subseteq B$ is a survival extension. In view of the above comments it is clear that any ULO-ring is both a QLO-ring and a US-ring and that a US-ring is a ULO-ring if (and only if) it is a UQLO-ring. Moreover, it was shown in [6, Proposition 2.13] that if $1 \leq n \leq \infty$, then there exists an n -dimensional ULO-ring. While there are conditions that can force a UQLO-ring to be a ULO-ring (cf. [7, Proposition 2.9]), there exist n -dimensional UQLO-rings that are not ULO-rings, for each $n, 1 \leq n \leq \infty$ [7, Proposition 2.10]. (The case $n = 0$ is avoided because any zero-dimensional ring must be a ULO-ring [6, Proposition 2.1].) It is natural to ask if there exist any US-rings that are not ULO-rings.

Corollary 2.4 answers this question in the affirmative, by giving n -dimensional examples, for each integer $n \geq 2$, of US-rings which are not even UQLO-rings. These examples, as is the case with most of the interesting examples from [6, 7], are built using the $A+B$ construction from [10]. (We review the $A+B$ construction and some of its properties in Section 2.) In fact, Corollary 2.4 results by combining a sufficient condition for US-rings that is given in Proposition 2.2 with a family of $A+B$ constructions from [7, Corollary 2.15]. Thus, the question naturally arises whether there is a basically new way to construct infinite-dimensional US-rings which are not UQLO-rings. We answer this in Section 2 by developing such a ring R , which is the ring of continuous real-valued functions on a certain compact Hausdorff space. Not only is R infinite-dimensional and a US-ring which is not a UQLO-ring, but we also show that R cannot be obtained from any $A+B$ construction. The most arduous verification involves showing that $R \subset T$ does not satisfy LO for a certain ring of (continuous real-valued) functions $T \supset R$.

While Section 2 contains enough analysis of ultrafilters and prime ideals to obtain the above information about the ring of functions R , Sections 3 and 4 develop additional properties of this very interesting ring. For instance, Corollary 3.2 shows that, if $P \in \text{Spec}(R)$, then R/P is a divided domain (in the sense of [4]). However, Corollary 3.9 shows that not every such factor domain of R can be a valuation domain,

while Corollary 3.10 shows that if the continuum hypothesis holds, then there does exist a minimal prime ideal P of R such that R/P is an infinite-dimensional valuation domain. Also, while it is in general quite difficult to explicitly describe the prime spectrum of a ring of functions, Section 4 explains how to obtain each prime ideal of R as a union or an intersection of the special types of prime ideals of R that are considered in the earlier sections.

Besides the notation and conventions mentioned above, we adopt the following. If A is a ring, then $\text{Max}(A)$ (respectively, $\text{Min}(A)$) denotes the set of maximal (respectively, minimal prime) ideals of A . As usual, \mathfrak{c} denotes the cardinality of \mathbf{R} ; \mathbf{N} denotes the set of natural numbers; $C(W)$ denotes the ring of continuous real-valued functions defined on a topological space W ; “Ann” denotes an annihilator; and \subset and \supset denote proper containments. Any unexplained material is standard, as in [11].

2. Construction of an infinite-dimensional US-ring $C(Y)$. We begin by recording two facts from the introduction.

Proposition 2.1. (a) *Each ULO-ring is both a UQLO-ring and a US-ring.*

(b) *If A is a US-ring, then A is a ULO-ring if (and only if) A is a UQLO-ring.*

In [10], Huckaba introduced a ring-theoretic property called Property A as a generalization of the Noetherian property. Specifically, a ring A is said to have (or satisfy) Property A if, whenever I is a finitely generated ideal of A such that $I \subseteq Z(A)$, we have that $\text{Ann}(I) \neq 0$. Proposition 2.1 (b) can be viewed as an analogue of the result [7, Proposition 2.9] that, if A is a ring that has Property A and $A = \text{tq}(A)$, then A is a QLO-ring if (and only if) A is a UQLO-ring. The hypotheses of the latter result are also the hypotheses of the useful sufficient condition for US-rings in Proposition 2.2 (a).

Proposition 2.2. (a) *If A is a ring satisfying Property A such that $A = \text{tq}(A)$, then A is a US-ring.*

(b) *If A is a reduced US-ring, then A satisfies Property A.*

Proof. For (a), rework the first two paragraphs of the proof of [6, Theorem 2.6]. For (b), rework the proof of [6, Proposition 2.5]. \square

If one chases the proof given below for Corollary 2.4 back to the proof of [6, Corollary 2.15], we see that the rings in Corollary 2.4 were built via the $A+B$ construction. It is convenient next to recall the definition and some of the basic properties of this construction from [10].

Let D be a reduced ring, and let \mathcal{P} be a nonempty subset of $\text{Spec}(D)$. If \mathcal{A} is an indexing set for \mathcal{P} , let $I := \mathcal{A} \times \mathbf{N}$. For each $i = (\alpha, n) \in I$, let $P_i := P_\alpha$ and $D_i := D/P_i$. Let $\prod D_i$ be the product of $\{D_i \mid i \in I\}$, and let $B := \sum_{i \in I} D_i$. Define $\varphi : D \rightarrow \prod D_i$ by $\varphi(d) := (d + P_i)_{i \in I}$. If A is the image of φ , define $S = A + B$. Note that S is a reduced ring, B is an ideal of S and $S/B \cong A$.

Lemma 2.3 [10, Theorems 26.1, 26.2, 26.4 and 27.1]. *Let S be an $A+B$ ring, as defined above. Then:*

(a) *The minimal prime ideals of S that do not contain B are the ideals of the form $M_i = \{(r_j) \in S \mid r_i = 0\}$, as i varies over I . Moreover, $S/M_i \cong D_i$ for each i .*

(b) *If $\mathcal{P} = \text{Max}(D)$ and J is the Jacobson radical of D , then the prime ideals Q of S that contain B are in one to one order-preserving correspondence with the prime ideals P of D that contain J (via $Q = \varphi(P) + B$), and $S/B \cong D/J$.*

(c) *If $\mathcal{P} = \text{Max}(D)$, then $S = A + B$ is its own total quotient ring and has Property A.*

Corollary 2.4. *Let $2 \leq n \leq \infty$. Then there exists an n -dimensional US-ring A which is not a ULO-ring (and hence not a UQLO-ring). It may be arranged that A is reduced, $A = \text{tq}(A)$, and A satisfies Property A.*

Proof. The parenthetical assertion follows from Proposition 2.1 (b). Hence, if $n \neq \infty$, it is enough to combine Proposition 2.2 (a) with [6, Corollary 2.15]. In fact, the same method can be shown to work for the case $n = \infty$, the point being that [14] can be used to build an example of an infinite-dimensional h -local domain D with infinitely

many maximal ideals. To avoid going off on a tangent, we leave the details of the construction of D to the interested reader, noting that we will develop the ring of functions R as a suitable infinite-dimensional example. \square

We next give a companion for Corollary 2.4.

Example 2.5. Let $1 \leq n \leq \infty$. Then there exists an n -dimensional UQLO-ring A which is neither a US-ring nor a ULO-ring. It may be arranged that A is a chained ring and, hence, satisfies Property A.

Proof. Combine [7, Proposition 2.10] with Proposition 2.1 (b). \square

Most of this paper will be devoted to studying a specific ring of functions. We first give a result showing that the $C(-)$ construction is relevant to the above concerns.

Proposition 2.6. $C(W)$ has Property A, for any topological space W .

Proof. Put $B := C(W)$. For any $f \in B$, the annihilator $\text{Ann}(f) := \{g \in B \mid f(w)g(w) = 0 \text{ for all } w \in W\}$ is a radical ideal of B . Indeed, if $b \in B$ and $b^k \in \text{Ann}(f)$ for some $k \in \mathbf{N}$, then $(bf)^k = 0$, whence $bf = 0$ (since bf takes values in a reduced ring, i.e., \mathbf{R}). Thus, if $f_1, \dots, f_n \in B$ and $h := \sum_{i=1}^n f_i^2$, then $\text{Ann}(f_1, \dots, f_n) = \{g \in B \mid f_i(w)^2 g(w)^2 = 0 \text{ for all } w \in W \text{ and all } i = 1, \dots, n\} = \{g \in B \mid h(w)g(w)^2 = 0 \text{ for all } w \in W\} = \text{Ann}(h)$. (The first equation holds since \mathbf{R} is a reduced ring; the second holds since each $f_i(w)^2 g(w)^2$ is non-negative; and the third holds since $\text{Ann}(h)$ is a radical ideal.) \square

We next introduce the ring of functions $R = C(Y)$ and devote the rest of the section to showing that it is a (n infinite-dimensional example of a) US-ring that is not a ULO-ring. In view of Proposition 2.1 and Example 2.5, this will settle the remaining questions about the possible implications among the ULO-ring, UQLO-ring and US-ring concepts.

Let X be a set of cardinality \aleph_1 , and let $Y := X \cup \{\infty\}$, where $\infty \notin X$. Put a topology on Y by defining a subset V of X to be open if and only if either:

- (1) $V \subseteq X$ or
 (2) $\infty \in V$ and V is co-finite in Y (i.e., $Y \setminus V$ is finite).

Thus, Y is the one-point compactification of a discrete space of cardinality \aleph_1 . The following definition will be helpful. Given a function $f : Y \rightarrow \mathbf{R}$, an element $a \in \mathbf{R}$ and a subset U of Y , we say that f *converges to a on U* if for all $\varepsilon > 0$, the set $\{i \in U : |f(i) - a| > \varepsilon\}$, is finite. Now, let $R := C(Y)$ be the ring of continuous real-valued functions defined on Y (with pointwise operations). It is straightforward to verify the following useful characterization of the elements of R . For an arbitrary function $f : Y \rightarrow \mathbf{R}$ with $a := f(\infty)$, we have that f is continuous (i.e., $f \in C(Y)$) if and only if:

- (1) the set $W := W_{a,f} := \{i \in Y : f(i) = a\}$ is co-countable and
 (2) f converges to a on $Y \setminus W$ (since f is identically a on W , this is equivalent to requiring that f converges to a on each subset of Y).

It will also be helpful later to note the following consequence of the fact that Y is compact: if $f \in C(Y)$, then f is a bounded function.

We next collect some facts about the ring R . If $f \in R$, let *the zero set of f* be $Z(f) := \{y \in Y \mid f(y) = 0\}$; let $\text{coz}(f) := Y \setminus Z(f)$, the *cozero set of f* . If $a \in Y$, let $M_a := \{f \in R : f(a) = 0\}$.

Proposition 2.7. *Let R be the ring defined above. Then:*

- (a) $\text{Max}(R) = \{M_a \mid a \in Y\}$; $M_a \neq M_b$ if $a \neq b$ in Y .
 (b) $R = \text{tq}(R)$.
 (c) R is a US-ring.

Proof. (a) This assertion is a consequence of the fact that Y is a compact Hausdorff space: see, for instance, [1, Exercise 26, page 14].

(b) It is enough to show that, if an element $f \in R \setminus Z(R)$, then $Z(f)$ is empty (for then, $1/f \in R$). Suppose not. Note that $Z(f) \neq \{\infty\}$, since $f^{-1}(f(\infty))$ is co-countable. Thus, we can choose $y \in Z(f) \setminus \{\infty\}$. Define a function $g : Y \rightarrow \mathbf{R}$ by $g(y) := 1$ and $g(z) := 0$ for all $z \in Y \setminus \{y\}$. It follows easily from the above characterization of elements of R that g is continuous. Also, it is clear that $gf = 0$. Hence, $f \in Z(R)$, the desired contradiction.

- (c) Combine (b) with Propositions 2.6 and 2.2 (a). □

We also note that $\dim(R) \neq 0$. While this can be inferred directly from the properties of Y , it will also follow from our construction of (some of the) prime ideals of R : see Proposition 2.8. The rest of this section is devoted to showing that R is not a ULO-ring. In fact, we will construct a ring extension T of R such that $R \subseteq T$ does not satisfy LO, see Corollary 2.15.

Note that X is a dense subset of Y . Combining this with the facts that \mathbf{R} is Hausdorff and Y is infinite, we easily see that the ring homomorphism $R = C(Y) \rightarrow T := C(X), f \mapsto f|_X$, is an injection. Therefore, it is harmless to view R as a subring of T (via this injection). Our earlier characterization of the elements of R now leads to the following useful fact. If $f \in T$, then $f \in R = C(Y)$ (in the sense that $f(\infty)$ can be defined so that the extended function $f : Y \rightarrow \mathbf{R}$ is continuous) if and only if there exists $a \in \mathbf{R}$ such that the set $V := \{i \in X : f(i) = a\}$ is co-countable and f converges to a on $X \setminus V$.

As X has the discrete topology, $T = C(X)$ is isomorphic to the von Neumann regular ring $\prod_X \mathbf{R}$ and, hence, is zero-dimensional. It is well known that the prime (i.e., maximal) ideals of T are in bijection with the ultrafilters \mathcal{U} on the set X in the following manner: $\mathcal{U} \leftrightarrow M_{\mathcal{U}}$, where the maximal ideal associated to \mathcal{U} is

$$M_{\mathcal{U}} := \{g \in T : \{x \in X \mid g(x) = 0\} \in \mathcal{U}\}.$$

Next, recall from Proposition 2.7 (c) that R is a US-ring. Therefore, each maximal ideal of R is lain over by at least one prime ideal of T . We next examine the contraction to R of a typical prime ideal $M_{\mathcal{U}}$ of T . In the easy case, \mathcal{U} is the principal ultrafilter based at some $a \in X$ (i.e., \mathcal{U} consists of all the subsets of X that contain a). For this case, it is easy to see that $M_{\mathcal{U}} \cap R = M_a$, where M_a is as defined in Proposition 2.7 (a). We turn next to the harder case.

We next consider the prime ideals of R of the form $P_{\mathcal{U}} := M_{\mathcal{U}} \cap R$ where \mathcal{U} is a free (i.e., non-principal) ultrafilter on X . This case breaks down into two subcases. In the first of these, \mathcal{U} does not contain a countable set; i.e., \mathcal{U} is a uniform ultrafilter. For this subcase, we claim that $P_{\mathcal{U}} = M_{\infty}$.

For a proof, it suffices to show that $M_{\infty} \subseteq P_{\mathcal{U}}$ (i.e., that $M_{\infty} \subseteq M_{\mathcal{U}}$), since $M_{\infty} \in \text{Max}(R)$. We need only show that, if $f \in M_{\infty}$, then

$f \in M_{\mathcal{U}}$, i.e., that $Z(f) \setminus \{\infty\} \in \mathcal{U}$. As $f \in M_{\infty}$, we have $f(\infty) = 0$, and so it follows from the first criterion in our earlier characterization of the elements of R that $\text{coz}(f)$ is a countable subset of X . Therefore, since \mathcal{U} is assumed to be a uniform ultrafilter, $\text{coz}(f) \notin \mathcal{U}$. As \mathcal{U} is an ultrafilter, it follows that $Z(f) \setminus \{\infty\} = X \setminus \text{coz}(f) \in \mathcal{U}$, which proves the above claim.

The remaining subcase concerns the free *non-uniform* ultrafilters \mathcal{U} on X (i.e., the free ultrafilters \mathcal{U} on X that contain a countable set). (By convention, whenever we consider a “non-uniform” ultrafilter, we will also assume that it is free.) We say something about $P_{\mathcal{U}}$ for this subcase in Proposition 2.8. First, the following comments will be helpful. Let $\{a_n\}$ be a sequence of real numbers and let V be a denumerable subset of Y . We will “assign” the given sequence to V , by first choosing an enumeration of the elements of V . Then, if x is the n th element of V , we set $f(x) := a_n$. This creates a function $V \rightarrow \mathbf{R}$. Note further that, if $V \subset X$ and if $\lim_{n \rightarrow \infty} a_n = a$, then by setting $f(y) := a$ for all $y \in Y \setminus V$, the resulting function $f : Y \rightarrow \mathbf{R}$ is continuous, that is, is in R . If, in addition, $a = 0$, then $f \in M_{\infty}$.

Note that there exists a non-uniform ultrafilter on X . (While this fact can be extracted from [3, Theorem 7.1], here is a more direct proof. Choose a denumerable subset W of X and a free ultrafilter \mathcal{F} on W ; then $\mathcal{U} := \{Z \subseteq X \mid Z \text{ contains some element of } \mathcal{F}\}$ is a non-uniform ultrafilter on X .) It follows from the first assertion in the next result that $\dim(R) \neq 0$.

Proposition 2.8. *Let \mathcal{U} be a non-uniform ultrafilter on X . Then $P_{\mathcal{U}} := M_{\mathcal{U}} \cap R \subset M_{\infty}$. Moreover, $P_{\mathcal{U}}$ is not contained in M_a for any $a \in X$.*

Proof. Let $f \in M_{\mathcal{U}} \cap R$ and $b := f(\infty)$. Since f is continuous, the set $W := \{i \in X \mid f(i) \neq b\}$ is countable and f converges to b on W . If $b \neq 0$, it would follow that $Z(f)$ is finite. Then $f \in M_{\mathcal{U}} \cap R$ would force $Z(f) \in \mathcal{U}$, but the finiteness of $Z(f)$ would contradict that \mathcal{U} is free. Hence, $b = 0$, and so $f \in M_{\infty}$. This proves that $M_{\mathcal{U}} \cap R \subseteq M_{\infty}$.

To prove the first assertion, it remains to show that $M_{\mathcal{U}} \cap R$ is not all of M_{∞} . We will do this by using the above comments to construct a function $g \in M_{\infty} \setminus M_{\mathcal{U}}$. By hypothesis, we can pick a denumerable set $V \in \mathcal{U}$. As above, we can define $g \in M_{\infty} \subset R$ by setting $g(j) := 0$

for all $j \in Y \setminus V$ and “assigning” the sequence $\{1/n\}$ to the elements of V . However, $g \notin M_{\mathcal{U}}$, that is, $X \setminus \text{coz}(g) = Z(g) \setminus \{\infty\} \notin \mathcal{U}$ (the point being that $\text{coz}(g) \supseteq V$ entails $\text{coz}(g) \in \mathcal{U}$).

It remains to prove the final assertion of the proposition. Let $a \in X$. We need only find a function in $P_{\mathcal{U}} \setminus M_a$. Define $h \in R$ by $h(a) = 1$ and $h(i) = 0$ for all $i \neq a$. Of course, $h \notin M_a$. On the other hand, $h \in P_{\mathcal{U}}$; that is, $h \in M_{\mathcal{U}}$. Indeed, $\{a\} \notin \mathcal{U}$ since \mathcal{U} is free, and so the complementary set $\{x \in X \mid h(x) = 0\} \in \mathcal{U}$, as required. \square

We next give a technical result on non-uniform ultrafilters. While the result is probably known, we include a proof for the sake of completeness.

Proposition 2.9. *If \mathcal{U}_1 and \mathcal{U}_2 are distinct non-uniform ultrafilters on X , then there exist countable subsets V and W of X such that $V \in \mathcal{U}_1 \setminus \mathcal{U}_2$ and $W \in \mathcal{U}_2 \setminus \mathcal{U}_1$.*

Proof. Suppose not. Then (relabeling if necessary) we can assume that every countable element of \mathcal{U}_1 is an element of \mathcal{U}_2 . Since the ultrafilters are distinct, there exists $H \in \mathcal{U}_1 \setminus \mathcal{U}_2$. Thus, $K := X \setminus H \in \mathcal{U}_2 \setminus \mathcal{U}_1$. Let L be a countable element of \mathcal{U}_1 . By the definition of an ultrafilter, $H \cap L \in \mathcal{U}_1$. Clearly, $H \cap L$ is also a countable set. Then, by our assumption, $H \cap L$ is in \mathcal{U}_2 . Hence, $\emptyset = (H \cap L) \cap K \in \mathcal{U}_2$, which is absurd. \square

Corollary 2.10. *Let \mathcal{U}_1 and \mathcal{U}_2 be distinct non-uniform ultrafilters on X . Then $P_{\mathcal{U}_1}$ and $P_{\mathcal{U}_2}$ are incomparable prime ideals of R .*

Proof. Let V and W be the countable subsets of X given by Proposition 2.9. As before, enumerate the elements of V and the elements of W , and then obtain f (respectively, g) in R by defining $f(i) = 1/n$ where i is the n th element of V (respectively, $g(j) = 1/n$ where j is the n th element of W) and 0 elsewhere. To conclude, note that $f \in P_{\mathcal{U}_2} \setminus P_{\mathcal{U}_1}$ while $g \in P_{\mathcal{U}_1} \setminus P_{\mathcal{U}_2}$. \square

The above work shows that the image of the canonical map from $\text{Spec}(T)$ to $\text{Spec}(R)$ is the union of the set $\text{Max}(R) = \{M_a \mid a \in Y\}$

with the set $\{P_{\mathcal{U}} \mid \mathcal{U} \text{ a non-uniform ultrafilter on } X\}$. Additionally, these ideals $P_{\mathcal{U}}$ are pairwise incomparable, and each of them is contained in a unique maximal ideal of R , namely M_{∞} . We also know that each minimal prime ideal of R belongs to the image of the map $\text{Spec}(T) \rightarrow \text{Spec}(R)$ (see, for instance, [11, Exercise 1, page 41]). We can now conclude that $\text{Min}(R)$ coincides with the set of minimal elements of the image of $\text{Spec}(T) \rightarrow \text{Spec}(R)$; that is, $\text{Min}(R)$ is the union of $\{P_{\mathcal{U}} \mid \mathcal{U} \text{ a non-uniform ultrafilter on } X\}$ with $\{M_a \mid a \in X\}$. The next obvious question is: apart from M_{∞} and the minimal prime ideals of R , does R have *any* other prime ideals? In other words, are *any* prime ideals of R *not* lain over from T ?

We devote the rest of this section to answering the above question. We next describe prime ideals of R contained between $P_{\mathcal{U}}$ and M_{∞} , where \mathcal{U} is a non-uniform ultrafilter on X . This description, which is very reminiscent of what was done in [12], is determined by the growth of certain functions $X \rightarrow \{r \in \mathbf{R} \mid r \geq 0\}$.

Let f and g be arbitrary non-negative real-valued functions defined on X , and let \mathcal{U} be an ultrafilter on X . We say that $g \leq f \pmod{\mathcal{U}}$ if there exists a $V \in \mathcal{U}$ such that $g(i) \leq f(i)$ for all $i \in V$. If there is no chance of confusion, we will suppress the “ \mathcal{U} ” and simply write $g \leq f$.

A very useful fact is that, for any f, g, \mathcal{U} as above, g and f must be comparable $\pmod{\mathcal{U}}$. To prove this, partition X into two sets, $W := \{i \in X \mid g(i) \leq f(i)\}$ and $X \setminus W = \{i \in X \mid g(i) > f(i)\}$. Since exactly one of $W, X \setminus W$ is in \mathcal{U} , the assertion follows easily.

Let f, g, \mathcal{U} as above. If $g \leq f$ and there do not exist positive integers n and M such that $f^n \leq Mg$, then we write $g \ll f$. Using the result of the preceding paragraph, one can show that \ll is a transitive relation.

Let \mathcal{U} be a non-uniform ultrafilter on X , and let $f \in M_{\infty}$ be a non-negative function such that $\text{coz}(f) \in \mathcal{U}$. (So, $f \notin P_{\mathcal{U}}$.) We now use f and \mathcal{U} to define two subsets of M_{∞} that each contain $P_{\mathcal{U}}$.

$$P_{\mathcal{U},f} =: \{g \in M_{\infty} \mid |g| \ll f\}.$$

$$P_{\mathcal{U}}^f =: \{g \in M_{\infty} \mid \exists n > 0, M > 0 \text{ in } \mathbf{N} \text{ with } |g|^n \leq Mf\}.$$

Observe that, if $g \in P_{\mathcal{U}}$ (so, $Z(g) \in \mathcal{U}$), then $g \ll f$. (The point is that $V := \text{coz}(f) \cap Z(g) \in \mathcal{U}$ and $f^n(z) > Mg(z)$ for any positive integers n and M and all $z \in V$.) In view of Proposition 2.8, the upshot is that

$P_U \subseteq P_{U,f}$. On the other hand, it is easy to check that $P_{U,f} \subseteq P_U^f$ and that $f \in P_U^f \setminus P_{U,f}$. In summary,

$$P_U \subseteq P_{U,f} \subset P_U^f \subseteq M_\infty.$$

Theorem 2.11. *Let U be a non-uniform ultrafilter on X , and let $f \in M_\infty$ be a non-negative function such that $\text{coz}(f) \in U$. Then both $P_{U,f}$ and P_U^f are prime ideals of R .*

Proof. We consider $P_{U,f}$ first. To show that $P_{U,f}$ is closed under sums, we let $g, h \in P_{U,f}$ and will show that $g + h \in P_{U,f}$. We have that $|g| \ll f$ and $|h| \ll f$ and must show that $|g + h| \ll f$. Without loss of generality, $|g| \leq |h|$. If $f \leq |g + h|$, then $f \leq |g| + |h| \leq 2|h|$, which contradicts the fact that $|h| \ll f$. Thus, $|g + h| \leq f$. Hence, to show that $|g + h| \ll f$, it will suffice to prove that there do not exist integers $n, M > 0$ such that $f^n \leq M|g + h|$. If such n, M exist, then $f^n \leq 2M|h|$, another contradiction to the fact that $h \ll f$. Therefore, $P_{U,f}$ is closed under sums.

Next, to show that $P_{U,f}$ is closed under multiplication by elements of R , we let $g \in P_{U,f}$ and $r \in R$ and will show that $rg \in P_{U,f}$. We have that $|g| \ll f$. Pick a positive integer N such that $|r(i)| \leq N$ for all $i \in Y$. (Such an N exists because each element of R is a bounded function.) Then $|r| \leq N$, where “ N ” here denotes the constant function that takes only the value N . If $f \leq |rg|$, then $f \leq N|g|$, which contradicts the fact that $|g| \ll f$. Thus, $|rg| \leq f$. Hence, to show that $|rg| \ll f$, it will suffice to prove that there do not exist integers $n, M > 0$ such that $f^n \leq M|rg|$. If such n, M exist, then $f^n \leq M|rg| \leq MN|g|$, a contradiction to the fact that $g \ll f$. Therefore, $P_{U,f}$ is closed under scalar multiplication by R , and so $P_{U,f}$ is an ideal of R . Of course, it is a proper ideal because it is contained in M_∞ .

Finally, we show that $P_{U,f} \in \text{Spec}(R)$. Suppose $r, s \in R$ satisfy $rs \in P_{U,f}$. If there exist positive integers n_1, M_1, n_2, M_2 such that $f^{n_1} \leq M_1|r|$ and $f^{n_2} \leq M_2|s|$, then $f^{n_1+n_2} \leq M_1M_2|rs|$, which contradicts the fact that $|rs| \ll f$. So, without loss of generality, there do not exist integers $n, M > 0$ such that $f^n \leq M|r|$. Since $|r|$ and f

are comparable (mod \mathcal{U}), we must have $|r| \leq f$. Therefore $|r| \ll f$, proving that $r \in P_{\mathcal{U},f}$, and so $P_{\mathcal{U},f} \in \text{Spec}(R)$.

Next, we show that $P_{\mathcal{U}}^f \in \text{Spec}(R)$. Of course, $P_{\mathcal{U}}^f \neq R$ since $P_{\mathcal{U}}^f \subseteq M_{\infty}$. Let $g, h \in P_{\mathcal{U}}^f$. There exist positive integers n_1, n_2, M_1 and M_2 such that $|g|^{n_1} \leq M_1 f$ pointwise on some $V_1 \in \mathcal{U}$ and $|h|^{n_2} \leq M_2 f$ pointwise on some $V_2 \in \mathcal{U}$. Note that $g, h \in P_{\mathcal{U}}^f \subseteq M_{\infty}$. In particular, $g(\infty) = 0 = h(\infty)$. So, by continuity (and the fact that \mathcal{U} is an ultrafilter), there exists an element of \mathcal{U} on which g and h each take values in the interval $[0, 1)$, whence $|h|^n \leq |h|^{n_i}$ if $n \geq n_i$. Thus, there is no harm in assuming that $n_1 = n = n_2$. Also, without loss of generality, $|h| \leq |g|$, and so $|h|^j \leq |g|^j$ for all $j = 1, \dots, n$. Then on $V_1 \cap V_2 \in \mathcal{U}$, we have

$$|g + h|^n \leq \sum_{i=0}^n \binom{n}{i} |g|^{n-i} |h|^i \leq (n + 1)S|g|^n \leq (n + 1)SM_1 f,$$

where S is the maximum of the above binomial coefficients. Thus $g + h \in P_{\mathcal{U}}^f$, which shows that $P_{\mathcal{U}}^f$ is closed under sums. Next, to prove that $P_{\mathcal{U}}^f$ is closed under scalar multiplication from R , observe that, if $|g|^n \leq Mf$ and $|r| \leq N$, then $|rg|^n \leq N^n Mf$.

Finally, we must show that, if $r, s \in R$ such that $rs \in P_{\mathcal{U}}^f$, then either r or s is in $P_{\mathcal{U}}^f$. Suppose $|rs|^n \leq Mf$ pointwise on $V \in \mathcal{U}$. Let

$$W_1 = \{i \in V : |r(i)|^n \leq \sqrt{Mf}\}$$

and

$$W_2 = \{i \in V : |s(i)|^n \leq \sqrt{Mf}\}.$$

If $W_1 \in \mathcal{U}$, then $|r|^{2n} \leq Mf$ pointwise on W_1 and so $r \in P_{\mathcal{U}}^f$; similarly, if $W_2 \in \mathcal{U}$, then $s \in P_{\mathcal{U}}^f$. In the remaining case, both $X \setminus W_1$ and $X \setminus W_2$ are elements of \mathcal{U} . In this case, $Z_1 := (X \setminus W_1) \cap (X \setminus W_2) \in \mathcal{U}$. Clearly $|rs|^n > Mf$ pointwise on Z_1 . But this contradicts the fact that \mathcal{U} is a filter, since the set $Z_2 := \{i \in V : |r(i)s(i)|^n \leq Mf(i)\}$ is also an element of \mathcal{U} and $Z_1 \cap Z_2 = \emptyset$. The proof is complete. \square

We show next that the proper inclusion of prime ideals of the form $P_{\mathcal{U}}^f$ can be characterized in terms of \ll . Proposition 2.12 will be used significantly in the next two sections.

Proposition 2.12. *Let \mathcal{U} be a non-uniform ultrafilter on X , and let $f, g \in M_\infty \setminus P_{\mathcal{U}}$ be such that f and g are each non-negative functions and $g \ll f$. Then:*

- (a) *If $g \leq f \pmod{U}$, then $P_{\mathcal{U}}^g \subseteq P_{\mathcal{U}}^f$.*
- (b) *$g \ll f$ if and only if $P_{\mathcal{U}}^g \subset P_{\mathcal{U}}^f$.*

Proof. Since (a) is clear, we turn to (b). Suppose first that $g \ll f$. As $g \leq f$, (a) gives that $P_{\mathcal{U}}^g \subseteq P_{\mathcal{U}}^f$. Moreover, $g \ll f$ forces $f \notin P_{\mathcal{U}}^g$, and so $f \in P_{\mathcal{U}}^f \setminus P_{\mathcal{U}}^g$, thus proving the “only if” assertion. For the converse, suppose that $P_{\mathcal{U}}^g \subset P_{\mathcal{U}}^f$. By (a), it cannot be the case that $f \leq g$, and so $g \leq f$. It remains to show that $g \ll f$, namely, that there cannot exist $n, M \in \mathbf{N}$ such that $f^n \leq Mg$. We will show that, if this fails and $h \in P_{\mathcal{U}}^f$, then $h \in P_{\mathcal{U}}^g$ (which would contradict the hypothesis that $P_{\mathcal{U}}^g \subset P_{\mathcal{U}}^f$). Pick $n', M' \in \mathbf{N}$ such that $|h|^{n'} \leq M'f$. Then

$$|h|^{n'n} = (|h|^{n'})^n \leq (M'f)^n = M'^n f^n \leq M'^n Mg,$$

whence $h \in P_{\mathcal{U}}^g$, and we have the desired contradiction. □

The key to showing that $\dim(R) = \infty$ is contained in the following lemma.

Lemma 2.13. *Let V be a denumerable subset of X ; for convenience, identify V with \mathbf{N} . Let $\{a_i\}$ and $\{b_i\}$ each be sequences of positive real numbers that converge to ∞ . As in the discussion prior to Proposition 2.8, consider the function f (respectively, g) from Y to \mathbf{R} obtained by “assigning” the sequence $\{1/2^{a_i}\}$ (respectively, $\{1/2^{b_i}\}$) to V . (This means that f (respectively, g) sends the n th element of V to $1/2^{a_n}$ (respectively, $1/2^{b_n}$) and each element of $Y \setminus V$ to $\lim_{i \rightarrow \infty} 1/2^{a_i} = 0$ (respectively, to $\lim_{i \rightarrow \infty} 1/2^{b_i} = 0$)).) Then:*

- (a) *f, g in R . In fact, $f, g \in M_\infty$.*
- (b) *If, in addition, $b_i \leq a_i$ for all i and the sequence $\{a_i/b_i\}$ converges to ∞ , then $f \ll g$ relative to any non-uniform ultrafilter \mathcal{U} on X such that $V \in \mathcal{U}$.*

Proof. (a) It is enough to consider f . As $\lim_{n \rightarrow \infty} 1/2^{a_n} = 0$, the assertions follow from the discussion prior to Proposition 2.8.

(b) Note that $f \leq g$ (in fact, pointwise on Y). Suppose that $g^n \leq Mf$ for some $n > 0$ and $M > 0$. Since $V \in \mathcal{U}$ and \mathcal{U} is an ultrafilter, we infer that $1/(2^{b_i})^n \leq M/2^{a_i}$ for infinitely many $i \in \mathbf{N}$. Hence, $2^{a_i - nb_i} \leq M$ for infinitely many $i \in \mathbf{N}$, and so $a_i \leq \log_2 M + nb_i$ for infinitely many $i \in \mathbf{N}$. Since $\log_2 M$ is a constant and $\{b_i\}$ converges to ∞ , we then have $a_i \leq b_i + nb_i = (1 + n)b_i$ for infinitely many $i \in \mathbf{N}$. But this contradicts the assumption that $\lim_{i \rightarrow \infty} a_i/b_i = \infty$. Thus, $f \ll g$. \square

Theorem 2.14. *Fix a denumerable subset V of X . Let \mathcal{U} be any non-uniform ultrafilter \mathcal{U} on X such that $V \in \mathcal{U}$. For each positive real number r , let a_r be the sequence of real numbers whose i th term is $(a_r)_i := i^r$, and let $f_r \in M_\infty$ be constructed from a_r and V just as f was constructed from $\{a_i\}$ and V in Lemma 2.13. Then $P_{\mathcal{U}}^{f_{r_1}} \subset P_{\mathcal{U}}^{f_{r_2}}$ whenever $0 < r_2 < r_1$ in \mathbf{R} . Hence, there is a chain \mathcal{C} of prime ideals in R such that the cardinality of \mathcal{C} is \mathfrak{c} (the cardinality of \mathbf{R}) and each $P \in \mathcal{C}$ contains the minimal prime $P_{\mathcal{U}}$ of R and is contained in the maximal ideal M_∞ of R . In particular, $\dim(R) = \infty$.*

Proof. For any positive real number r , consider the sequence of positive real numbers whose i th term is $(a_r)_i := i^r$. Note that $\lim_{i \rightarrow \infty} i^r = \infty$. Hence, by Lemma 2.13 (a), $f_r \in M_\infty \subset R$. Also, the above comments show that $f_r \notin P_{\mathcal{U}}$. Moreover, since $\text{coz}(f_r) = V \in \mathcal{U}$, Theorem 2.11 shows that both $P_{\mathcal{U}, f_r}$ and $P_{\mathcal{U}}^{f_r}$ are prime ideals of R containing $P_{\mathcal{U}}$ and contained in M_∞ .

Note also that, if $r_1 > r_2 > 0$ in \mathbf{R} , then $i^{r_2} < i^{r_1}$ for each $i \in \mathbf{N}$ and $\lim_{i \rightarrow \infty} i^{r_1}/i^{r_2} = \lim_{i \rightarrow \infty} i^{r_1 - r_2} = \infty$. Hence, by Lemma 2.13 (b), $f_{r_1} \ll f_{r_2}$. Therefore, by Proposition 2.12 (b), $P_{\mathcal{U}}^{f_{r_1}} \subset P_{\mathcal{U}}^{f_{r_2}}$. Thus, one way to build the desired chain is to let $\mathcal{C} := \{P_{\mathcal{U}}^{f_r} \mid r > 0 \text{ in } \mathbf{R}\}$. \square

We can now prove a particularly striking property of ring R .

Corollary 2.15. *The ring extension $R \subset T$ does not satisfy LO, and so R is neither a ULO-ring nor a UQLO-ring.*

Proof. The discussion following Corollary 2.10 showed, in particular, that the image of the canonical map $\text{Spec}(T) \rightarrow \text{Spec}(R)$ is $\text{Max}(R) \cup$

$\text{Min}(R)$. Thus, to prove that $R \subset T$ does not satisfy LO, it is enough to show that $\dim(R) \geq 2$. Theorem 2.14 showed even more, namely, that $\dim(R) = \infty$. This proves the first assertion. The second assertion then follows from the definition of a ULO-ring. As for the final assertion, combine the second assertion with Propositions 2.7 (c) and 2.1 (b). \square

Combining Theorem 2.14, Proposition 2.7 (c) and Corollary 2.15, we see that R is an infinite-dimensional US-ring which is not a ULO-ring (or even a UQLO-ring). To close the section (and to provide a counterpoint to the example given in Corollary 2.4 for $n = \infty$), we show next that R is not of the form $A + B$. The proof of Proposition 2.16 uses nothing that was given after Proposition 2.7 (a).

Proposition 2.16. *R cannot be built via the $A + B$ construction.*

Proof. Suppose, on the contrary, that R has the form $A + B$. It will be convenient in this proof to say that a prime ideal of a ring E is a *minimax* prime ideal of E if it is both a maximal ideal and a minimal prime ideal of E . Recall from the discussion following Corollary 2.10 that $\{M_a \mid a \in X\}$ is an uncountably infinite set of minimax prime ideals of R . Next, it follows easily from Lemma 2.3 (fortified by the proof of [10, Theorem 26.2]) that the minimax prime ideals of $A + B$ are of two kinds, as follows. Each $P \in \text{Max}(D) \cap \mathcal{P}$ gives rise to denumerably many minimax prime ideals of $A + B$ which do not contain B , and this is how all the minimax prime ideals of $A + B$ which do not contain B arise. On the other hand, if \mathfrak{J} denotes the intersection of the elements of \mathcal{P} , the minimax prime ideals of $A + B$ which do contain B are in bijection with those maximal ideals P of D which are minimal with respect to containing \mathfrak{J} . Moreover, $A \cong D/\mathfrak{J}$, by the first isomorphism theorem. Thus, if we could show that $\text{Max}(A)$ is finite, it would follow that $A + B$ has at most denumerably many minimax prime ideals (contradicting the above information about the number of minimax prime ideals of R). We will show, in fact, that A is quasilocal. As $R/B = (A + B)/B \cong A$, it will suffice to prove that the only maximal ideal of R that contains B is M_∞ .

Let K be the ideal of R that is generated by the set of primitive idempotents of R . (Recall that an idempotent element e of a ring E

is called a *primitive idempotent* (of E) if e is not a nontrivial sum of orthogonal idempotents of E , equivalently, if the ring Ee has no nontrivial idempotents.) We claim that $K \subseteq B$, i.e., that each primitive idempotent of R is contained in B .

One consequence of the $A + B$ construction is that each element s of $A + B$ is of the form $s = a + b$, where $a \in A$, $b \in B$, either $a = 0$ or a is nonzero in infinitely many coordinates, and b is nonzero in only finitely many coordinates. Consider an arbitrary element of $R = A + B$, say $s = a + b$, where a, b are as in the preceding sentence. Recall that $A + B$ is constructed as a subring of a certain product, $\prod D_i$. Of course, s is idempotent if and only if each of its coordinates is idempotent. To prove the above claim, it suffices to show that, if s is idempotent and $a \neq 0$, then s is not a primitive idempotent. Note that infinitely many coordinates of a are nonzero idempotents. Therefore, since b has only finitely many nonzero coordinates, we can find a coordinate j such that a_j is a nonzero idempotent and $b_j = 0$. Define $c \in D_j \subseteq B \subseteq A + B$ to be the element of $\prod D_i$ whose only nonzero coordinate is $c_j := a_j$. Observe that $(a - c) + b$ and c are nonzero orthogonal idempotents of R . (Perhaps the key calculation is that, at any coordinate $i \neq j$, we have that $(a_i - c_i) + b_i = a_i + b_i$ is the i th coordinate of s and hence is idempotent.) Since the sum of $(a - c) + b$ and c is $a + b = s$, it follows that s is not a primitive idempotent of R . This proves the above claim.

Recall that it suffices to prove that the only maximal ideal of R that contains B is M_∞ . As $K \subseteq B$, it therefore suffices to prove that the only maximal ideal of R that contains K is M_∞ . By Proposition 2.7 (b), we need only show that, if $a \in X$, there exists a $g \in K \setminus M_a$. To that end, consider the function $f : Y \rightarrow \mathbf{R}$ defined by $f(a) := 1$ and $f(y) := 0$ for all $y \in Y \setminus \{a\}$. By our earlier criteria characterizing continuity, it is easy to see that $f \in R$. Of course, $f^2 = f$. Moreover, f is a primitive idempotent of R , for it is clear that the ring of functions $Y \rightarrow \mathbf{R}$ that vanish on $Y \setminus \{a\}$ has no nontrivial idempotents. Finally, it is clear that $f \notin M_a$, and so taking $g := f$ completes the proof. \square

3. Factor domains of $C(Y)$. We begin the section by showing that, modulo any prime ideal, the ring R satisfies a certain divisibility property.

Proposition 3.1. *Let $P \in \text{Spec}(R)$ and $D := R/P$. Then any two elements of D may be labeled d and e such that, if d does not divide e (in D), then $d \mid e^2$ and $e \mid d^3$.*

Proof. If $Q \subseteq P$ in $\text{Spec}(R)$ and R/Q has the asserted property, then so does R/P , since R/P is (isomorphic to) a factor domain of R/Q . So, without loss of generality, $P \in \text{Min}(R)$.

Either $P = M_a$ for some $a \in X$ or $P = P_{\mathcal{U}}$ for some non-uniform ultrafilter \mathcal{U} on X . (Compare with Proposition 2.8.) In the first case, $P \in \text{Max}(R)$, and so $D = R/P$ is a field and the assertion is obvious. Hence, without loss of generality, $P = P_{\mathcal{U}}$ for some non-uniform ultrafilter \mathcal{U} on X . Note that, modulo P , each element of $R \setminus M_{\infty}$ is a unit, since $(R/P, M_{\infty}/P)$ is quasilocal by Proposition 2.8.

Let $d, e \in D$. Then $d = f + P$ and $e = g + P$ for some $f, g \in R$. The assertion is clear if d or e is either 0 or a unit of D . Hence, without loss of generality, f and g are elements of $M_{\infty} \setminus P$. As $P = P_{\mathcal{U}}$ and \mathcal{U} is an ultrafilter, we have $\text{coz}(f), \text{coz}(g) \in \mathcal{U}$, and so $\text{coz}(f) \cap \text{coz}(g) \in \mathcal{U}$. Moreover, it follows via continuity that f and g each converge to 0 on $\text{coz}(f) \cap \text{coz}(g)$. It follows from the definition of an ultrafilter that we can find an element U' (respectively, U'') of \mathcal{U} such that f (respectively, g) is either pointwise strictly positive or pointwise strictly negative on U' (respectively, on U''). Therefore, by possibly replacing f and/or g with its negative (which has the harmless effect of replacing d or e with its negative) and considering $U' \cap U''$, we can assume that both f and g are strictly positive on some $U \in \mathcal{U}$. Also, as noted earlier, we may assume (possibly by interchanging f and g) that $0 < g(i) \leq f(i)$ for all $i \in U$. (Note that this is the *only* point in the proof that the original labels d and e may be switched.) Thus, the real-valued function g/f is defined and bounded on U .

There are two cases to consider. Suppose first that $g \ll f$. Then it cannot happen that $f^2 \leq g$, and so $g \leq f^2$ (without loss of generality, pointwise on U). By replacing U with $U \cap W$ where W is a countable element of the non-uniform ultrafilter \mathcal{U} , we can also assume that U is denumerable. Note that g/f^2 is bounded on U . Thus, $g/f = (g/f^2)f$ is the product of a bounded function with a function that converges to 0 (on U), and so g/f converges (to 0 on U). Let $h : Y \rightarrow \mathbf{R}$ be the function defined by $h(i) := g(i)/f(i)$ for $i \in U$ and $h(j) := 0$ if

$i \in Y \setminus U$. Then $h \in R$. (Note that we get the continuity of h by using the criterion from the preceding section, and that is why we needed U to be countable.) Moreover, $g - fh$ is 0 pointwise on U . In particular, $g - fh \in M_{\mathcal{U}} \cap R = P_{\mathcal{U}} = P$. In other words, f divides g modulo P , i.e., $d \mid e$ (in D). We will soon use the above technique again.

In the remaining case, ($g \leq f$ but) $g \not\ll f$. As above, we can assume that U is countable and g/f is bounded on U . Hence, $g^2/f = (g/f)g$ is the product of a bounded function with a function that converges to 0 on an element of \mathcal{U} . Therefore, g^2/f converges to 0. Then, arguing as before, there exists $h' \in R$ such that $g^2 - h'f \in P$. In other words, f divides g^2 modulo P , i.e., $d \mid e^2$.

There are now two subcases. Modulo the ultrafilter \mathcal{U} , either $g \leq f^2$ or $f^2 \leq g$. In the first of these subcases, g/f^2 is bounded and then we see as above that g/f converges to 0 on a countable element of \mathcal{U} , in which case f in fact divides g modulo P , that is, $d \mid e$. We turn to the final subcase, where we have that f^2/g is bounded. Then $f^3/g = (f^2/g)f$ converges to 0 on a countable element of \mathcal{U} , which implies as above that g divides f^3 modulo P , that is, that $e \mid d^3$. \square

A ring A is called *divided* if, for each $P \in \text{Spec}(A)$ and $a \in A$, either $a \in P$ or $P \subseteq Aa$. Note that a domain A is a divided ring if and only if A is a divided domain in the sense of [4], i.e., if and only if $P = PA_P$ for all $P \in \text{Spec}(A)$. Recall from [2, Proposition 2] that a ring A is a divided ring if and only if, for any elements $a, b \in A$, either $a \mid b$ or there exists an $n = n(a, b) \in \mathbf{N}$ such that $b \mid a^n$. Combing these facts with Proposition 3.1 immediately yields the following corollary.

Corollary 3.2. *R/P is a divided domain for each prime ideal P of R .*

Corollary 3.3. *Let \mathcal{U} be a non-uniform ultrafilter on X . Then the set of prime ideals of R that are contained between $P_{\mathcal{U}}$ and M_{∞} is linearly ordered by inclusion.*

Proof. By Corollary 3.2, $R/P_{\mathcal{U}}$ is a divided domain. But it is easy to see that, in any divided domain, the set of prime ideals is linearly ordered by inclusion. Therefore, the assertion follows from a standard homomorphism theorem. \square

We will use Corollary 3.3 in Section 4 to give two descriptions of the prime ideals contained between $P_{\mathcal{U}}$ and M_{∞} in terms of prime ideals of the form $P_{\mathcal{U},f}$ and $P_{\mathcal{U}}^f$.

We next pursue some consequences of the method of proof of Proposition 3.1.

Remark 3.4. Let \mathcal{U} be a non-uniform ultrafilter on X . Consider (typically non-negative) $f, g \in M_{\infty} \setminus P_{\mathcal{U}}$.

(a) It follows from the proof of Proposition 3.1 that, if $g \in P_{\mathcal{U}}^f$, then modulo $P_{\mathcal{U}}$, f divides some power of g . Moreover, in that proof where $g \leq f^2$ but $g \not\ll f$, we saw that modulo $P_{\mathcal{U}}$, f in fact divides g . However, we also know that, in that case, there exist $n, M > 0$ such that $f^n \leq Mg$. Then it would follow as in the above proof that modulo $P_{\mathcal{U}}$, g divides some power of f . Therefore, if $g \leq f$ but $g \not\ll f$, then a prime ideal P that contains $P_{\mathcal{U}}$ will contain f if and only if P contains g . On the other hand, if $g \ll f$, then every prime ideal of R that contains $P_{\mathcal{U}}$ and f must contain g .

(b) If $g \in P_{\mathcal{U}}^f$, then $f \not\ll g$. Thus, if $g \in P_{\mathcal{U}}^f \setminus P_{\mathcal{U},f}$ (and we have seen that such g exist), then $g \not\ll f$ and $f \not\ll g$.

Proposition 3.5. *Let \mathcal{U} be a non-uniform ultrafilter on X , and let $f \in M_{\infty} \setminus P_{\mathcal{U}}$ be a non-negative function. Then there are no prime ideals of R contained strictly between $P_{\mathcal{U},f}$ and $P_{\mathcal{U}}^f$.*

Proof. Suppose $g, h \in P_{\mathcal{U}}^f \setminus P_{\mathcal{U},f}$. Since \mathcal{U} is an ultrafilter, we may assume that g and h are non-negative modulo \mathcal{U} (for either g or h could be replaced by its negative). We claim that $g \not\ll h$ and $h \not\ll g$.

If the claim fails, we can suppose, without loss of generality, that $g \ll h$. As $h \in P_{\mathcal{U}}^f$, there exist (integers) $n, M > 0$ such that $|h|^n \leq Mf$. Since $g \ll h$, there cannot exist $t, M' > 0$ such that $|h|^{nt} \leq M'g$. Hence, $g \ll |h|^n$. Note that, if $f \leq g$ (modulo \mathcal{U}), then $|h|^n \leq Mf \leq Mg$, whence $|h|^n \leq Mg$, contradicting $g \ll |h|^n$. Therefore, $g \leq f$. Now, as $g \notin P_{\mathcal{U},f}$, it is not the case that $g \ll f$, and so there exist $m, K > 0$ such that $f^m = |f|^m \leq Kg$. Thus,

$$|h|^{nm} = (|h|^n)^m \leq (Mf)^m = M^m f^m \leq M^m Kg,$$

contradicting $g \ll h$. This proves the above claim.

Without loss of generality, $g \leq h$. Thus, by Remark 3.4 (a), any prime ideal of R that contains $P_{\mathcal{U}}$ will contain h if and only if it contains g . The assertion follows, for if there is a prime ideal Q of R such that $P_{\mathcal{U},f} \subset Q \subset P_{\mathcal{U}}^f$, then you could have taken $g \in Q \setminus P_{\mathcal{U},f}$ and $h \in P_{\mathcal{U}}^f \setminus Q$. \square

Corollary 3.6. *Let \mathcal{U} be a non-uniform ultrafilter on X , and let $f, g \in M_{\infty} \setminus P_{\mathcal{U}}$ be such that f and g are each non-negative functions and $g \ll f$. Then $P_{\mathcal{U}}^g \subseteq P_{\mathcal{U},f}$.*

Proof. By Proposition 2.12 (b), $P_{\mathcal{U}}^g \subset P_{\mathcal{U}}^f$. By Corollary 3.3 (and Theorem 2.11), the prime ideals $P_{\mathcal{U}}^g$ and $P_{\mathcal{U},f}$ are comparable. In view of Proposition 3.5, the assertion follows. \square

We have seen that each factor domain R/P of R is a divided domain. It is reasonable to wonder if these rings are, in fact, valuation domains. We will show in Corollary 3.9 that, in general, they are not, even if $P \in \text{Min}(R)$. We will show, moreover, that the answer to the question “Can R/P be a valuation domain for some $P \in \text{Min}(R) \setminus \text{Max}(R)$ ” is axiom dependent. First, we need to collect some facts about ultrafilters.

Remark 3.7. (a) Let C be a fixed denumerable set and \mathcal{U} a free ultrafilter on C . Recall that a function $f: C \rightarrow \mathbf{R}$ is said to *converge on \mathcal{U}* if there is a real number L such that, for each $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $|f(n) - L| < \varepsilon$ for all $n \in U$. Let us say that f *converges strongly on \mathcal{U}* if there exists a $U \in \mathcal{U}$ such that f converges to some a on U (essentially as defined in Section 2, where the domain of f was Y). It is known that, if $f: C \rightarrow \mathbf{R}$ is any bounded function, then f converges on \mathcal{U} . If every bounded function converges strongly on \mathcal{U} , then \mathcal{U} is said to be a *P -point* (this is also known as \mathcal{U} being “weakly selective”).

Without any set-theoretic assumptions, there are free ultrafilters that are not P -points. (See, for example, [9, 13]). Rudin [16] proved that, if the continuum hypothesis holds, then in fact there are P -points. Some other set-theoretic hypotheses are also known to imply the existence of P -points. However, Shelah (cf. [17]) has proved that it is consistent with ZFC that there are no P -points.

Thus, it is true in ZFC that there are free ultrafilters \mathcal{U} on C such that every bounded function $f: C \rightarrow \mathbf{R}$ converges on \mathcal{U} , but some bounded functions from C to \mathbf{R} do not converge strongly on \mathcal{U} . On the other hand, it is independent of the axioms of ZFC that there exists an ultrafilter \mathcal{U} such that every bounded function $f: C \rightarrow \mathbf{R}$ converges strongly on \mathcal{U} . Finally, we note that, if D is a set of cardinality \aleph_1 and \mathcal{V} is a non-uniform ultrafilter on D , then the previous comments may be applied to any denumerable set $C \in \mathcal{V}$.

(b) We need to examine more closely the question of when, given two elements $f, g \in R$, one divides the other modulo a minimal prime ideal of the form $P_{\mathcal{U}}$. Note that the canonical image of f in $R/P_{\mathcal{U}}$ divides the canonical image of g if and only if there exists an $h \in R$ such that $g - fh \in P_{\mathcal{U}}$. Now, suppose that f divides g modulo $P_{\mathcal{U}}$ and that $g \notin P_{\mathcal{U}}$. Therefore, on some $U \in \mathcal{U}$, $g = fh$. Thus, g , and hence f , are each pointwise nonzero on this element of the ultrafilter U . Hence, for the restriction of h to U , we have $h = g/f$. Since $h \in R$, h converges on U . (In detail, since $f, g \in R$, we know that f (respectively, g) converges on any subset W of X to $f(\infty)$ (respectively, to $g(\infty)$), and so g/f converges on U to $g(\infty)/f(\infty)$.) Thus, in the language of part (a), we can say that g/f converges strongly on \mathcal{U} . The above argument is reversible. Therefore, the canonical image of f divides the canonical image of g in $R/P_{\mathcal{U}}$ if and only if g/f converges strongly on \mathcal{U} .

(c) Let \mathcal{U} be a non-uniform free ultrafilter on X . Then U is a P -point if the restriction of \mathcal{U} to some (and hence every) countable $U \in \mathcal{U}$ is a P -point.

For a proof, suppose first that \mathcal{U} is a P -point. Let f be a bounded continuous real-valued function on X . Fix a denumerable set $U \in \mathcal{U}$. Note that f converges on some $W \in \mathcal{U}$. Then the restriction of f to W converges on $U \cap W$, and so ultrafilter obtained by restricting \mathcal{U} to U is a P -point. Conversely, suppose that the restriction of \mathcal{U} to U is a P -point for each denumerable $U \in \mathcal{U}$, and pick one such U . Let g be a bounded continuous real-valued function on X . Then the restriction of g to U is bounded on U . Since the restriction of \mathcal{U} to U is a P -point, there exists a V in this ultrafilter such that the restriction of g to U converges on V . As V must also be in the original ultrafilter, U is a P -point.

Proposition 3.8. *Let \mathcal{U} be a non-uniform ultrafilter on X . Then $R/P_{\mathcal{U}}$ is a valuation domain if and only if \mathcal{U} is a P -point.*

Proof. Suppose first that \mathcal{U} is a P -point. Let $f, g \in R$. To show that $R/P_{\mathcal{U}}$ is a valuation domain, it is enough to show that, possibly after interchanging f and g , there exists an $h \in R$ such that $g - fh \in P_{\mathcal{U}}$. As in the proof of Proposition 3.1, we can reduce to the case that $f, g \in M_{\infty} \setminus P_{\mathcal{U}}$ and, possibly after interchanging f and g , we can pick $U \in \mathcal{U}$ such that $0 < g(i) \leq f(i)$ for all $i \in U$. Then the restriction of g/f to U is a well-defined bounded function. Since \mathcal{U} is a P -point, g/f converges strongly on \mathcal{U} , and so there exists a denumerable set $W \subseteq U$ such that $W \in \mathcal{U}$ and g/f converges on W to, say, a . Define a function $h : Y \rightarrow \mathbf{R}$ via $h(i) := g(i)/f(i)$ for all $i \in W$ and $h(j) := a$ for all $j \in Y \setminus W$. Then $h \in R$ by the criterion for continuity in Section 2. Moreover, $g - fh$ is zero on W , whence $g - fh \in M_{\mathcal{U}} \cap R = P_{\mathcal{U}}$, as required.

We next prove the contrapositive of the “only if” assertion. Suppose that \mathcal{U} is not a P -point. Since \mathcal{U} is not a P -point, there exists a bounded real-valued function f on X such that the restriction of f to any subset of X that is in \mathcal{U} does not converge. Fix a denumerable set $U \in \mathcal{U}$. View the restriction of f to U as a sequence “on U .” By the choice of f , this bounded sequence on U , call it $\{a_n\}$, does not converge on any subset of U that is in \mathcal{U} . We may assume that $\{a_n\}$ is a strictly positive sequence that is bounded away from 0 on U . (In detail, choose $M > 0$ such that $f(x) > -M$ for all $x \in X$ and replace f with $f + M + 1$, thus replacing $\{a_n\}$ with the sequence $\{a_n + M + 1\}$ which is strictly positive, bounded away from 0 on U and does not converge on any subset of U that is in \mathcal{U} .) Then $\{1/a_n\}$ is also a well-defined bounded sequence on U that does not converge on any element of \mathcal{U} .

Next, we will construct two elements of R such that neither of their canonical images divides the other in $R/P_{\mathcal{U}}$ (thus showing that $R/P_{\mathcal{U}}$ is not a valuation domain). Define functions $g, f : Y \rightarrow \mathbf{R}$ as follows. Take g to be $(1/n)a_n$ on U (i.e., at the n th element of U) and 0 elsewhere on Y ; and take f to be $1/n$ on U and 0 elsewhere on Y . Since a_n is bounded on U , it is easy to see that g and f each converge to 0 on U , and so the criterion for continuity from Section 2 yields that $g, f \in R$. However, on U , g/f is given by a_n , which does not converge on any subset of U that is in \mathcal{U} .

Suppose that $g = rf + h$, where $r \in R$ and $h \in P_{\mathcal{U}}$. As $h \in M_{\mathcal{U}}$, there exists a $W \in \mathcal{U}$ such that h is pointwise identically 0 on W . On the other hand, since $r \in R$, r converges to $r(\infty)$ on W . It follows that, on $U \cap W \in \mathcal{U}$, g/f converges to $r(\infty)$, a contradiction. Thus, the image of f cannot divide the image of g . Similarly, since f/g is given on U by $1/a_n$, which does not converge on any element of \mathcal{U} , we see that the canonical image of g does not divide that of f , as required. \square

Corollary 3.9. *Not every factor domain of R is a valuation domain. Moreover, it is consistent with ZFC that there does not exist any non-uniform ultrafilter \mathcal{U} on X such that $R/P_{\mathcal{U}}$ is a valuation domain.*

Proof. As noted in Remark 3.7, there exist non-uniform free ultrafilters on X that are not P -points, and so the first assertion follows from Proposition 3.8. To prove the second assertion, combine Proposition 3.8 with the result of Shelah that was noted in Remark 3.7, namely, that it is consistent with ZFC that there are no P -points. \square

Corollary 3.10. *If one assumes the continuum hypothesis, then there exists a non-uniform ultrafilter \mathcal{U} on X such that $R/P_{\mathcal{U}}$ is a valuation domain.*

Proof. As noted in Remark 3.7, if the continuum hypothesis holds, then there exist non-uniform ultrafilters on X that are P -points. Hence, the assertion follows from Proposition 3.8. \square

4. Describing the prime ideals of $C(Y)$. In any ring, the union or intersection of any chain of prime ideals is a prime ideal [11, Theorem 9]. While the precise nature of $\text{Spec}(R)$ remains unknown, we will show, in the converse direction, that each prime ideal of R is both a union and an intersection of prime ideals of the types constructed in Section 2. We begin the section by showing that R_P is a domain for all prime ideals $P \neq M_{\infty}$.

Lemma 4.1. *If \mathcal{U}_1 and \mathcal{U}_2 are distinct non-uniform ultrafilters on X , then $P_{\mathcal{U}_1} + P_{\mathcal{U}_2} = M_{\infty}$.*

Proof. By Proposition 2.8, it suffices to prove that, if $f \in M_\infty$, then $f \in P_{\mathcal{U}_1} + P_{\mathcal{U}_2}$. If $Z(f)$ is in \mathcal{U}_1 (respectively, \mathcal{U}_2), then $f \in P_{\mathcal{U}_1}$ (respectively, $f \in P_{\mathcal{U}_2}$); in either of these cases, $f \in P_{\mathcal{U}_1} + P_{\mathcal{U}_2}$. Hence, without loss of generality, $Z(f)$ is in neither \mathcal{U}_1 nor \mathcal{U}_2 . Thus, $V := \text{coz}(f) \in \mathcal{U}_1 \cap \mathcal{U}_2$. Since $f \in M_\infty$, V is countable and the restriction of f to V converges to $f(\infty) = 0$. As $\mathcal{U}_1 \neq \mathcal{U}_2$, there exists a set $W \subset X$ such that $W \in \mathcal{U}_1 \setminus \mathcal{U}_2$. Therefore, $Z := X \setminus W \in \mathcal{U}_2 \setminus \mathcal{U}_1$. Then $V_1 := V \cap W \in \mathcal{U}_1$, $V_2 := V \cap Z \in \mathcal{U}_2$, $V_1 \cup V_2 = V$, and $V_1 \cap V_2 = \emptyset$. So, $V_2 \notin \mathcal{U}_1$ and $V_1 \notin \mathcal{U}_2$. Define two real-valued functions f_1 and f_2 on X by taking f_1 to be f on V_2 and 0 elsewhere, while f_2 is taken to be f on V_1 and 0 elsewhere. Observe that $f_1, f_2 \in R$. Clearly, $f_1 + f_2 = f$. Since $Z(f_1) = X \setminus V_2 \in \mathcal{U}_1$ and, similarly, $Z(f_2) \in \mathcal{U}_2$, we have $f_1 \in P_{\mathcal{U}_1}$ and $f_2 \in P_{\mathcal{U}_2}$, which proves the result. \square

Corollary 4.2. *If $P \in \text{Spec}(R) \setminus \{M_\infty\}$, then $\{Q \in \text{Spec}(R) \mid Q \subseteq P\}$ is linearly ordered by inclusion, and so R_P is a domain.*

Proof. If $P = M_a$ for some $a \in X$, then P is both a maximal and a minimal prime ideal of R , and so the first assertion is clear. As for the second assertion, note that R_P is then a reduced ring (since R is reduced) with a unique prime ideal, that is, a field (and hence a domain). Thus, we may assume henceforth that P is not of the form M_a , $a \in X$. By Lemma 4.1, P contains exactly one minimal prime ideal P_0 of R , and P_0 is necessarily of the form $P_{\mathcal{U}}$ for some non-uniform ultrafilter \mathcal{U} . Therefore, the first assertion follows from Corollary 3.3. For the second assertion, note that R_P is a reduced ring with a unique minimal prime ideal. \square

Note that Corollary 4.2 is best-possible. Indeed, R_{M_∞} is not a domain, in view of Proposition 2.8 and the fact that there is more than one non-uniform ultrafilter on X .

As noted in Corollary 4.2, if \mathcal{U} is a non-uniform ultrafilter on X , the prime ideals contained between $P_{\mathcal{U}}$ and M_∞ are linearly ordered (by inclusion). We use this fact to describe these prime ideals in Theorem 4.4. First, we need the following lemma.

Lemma 4.3. *Let \mathcal{U} be a non-uniform ultrafilter on X , and let $f \in M_\infty \setminus P_{\mathcal{U}}$. Then there exists a non-negative function $h \in M_\infty \setminus P_{\mathcal{U}}$*

such that, for any prime ideal P of R which contains P_U , $f \in P$ if and only if $h \in P$.

Proof. Consider $U := \text{coz}(f)$. Since $f \in M_\infty$, f converges to 0 on U . Moreover, $U \in \mathcal{U}$ since $f \notin P_U$. Let $U_1 := \{u \in U \mid f(u) > 0\}$ and $U_2 := \{u \in U \mid f(u) < 0\} = U \setminus U_1$. As either U_1 or U_2 is an element of \mathcal{U} , while f and $-f$ are members of exactly the same prime ideals, we can assume (possibly by replacing f with $-f$) that $U_1 \in \mathcal{U}$. Define a function $g : Y \rightarrow \mathbf{R}$ via $g(i) := f(i)$ if $i \in U_2$ and $g(j) = 0$ otherwise. Using the criterion from Section 2, we see that g is continuous, i.e., $g \in R$. As $U_1 \subseteq Z(g)$ and $U_1 \in \mathcal{U}$, we have $g \in P_U$. Then $h := f - g$ has the asserted property (since g is in each relevant P). \square

Theorem 4.4. *Let \mathcal{U} be a non-uniform ultrafilter on X , and let P be any prime ideal contained strictly between P_U and M_∞ . Then*

$$P = \bigcup_{f \in P} P_U^f = \bigcap_{g \in M_\infty \setminus P} P_{U,g},$$

where the items $f \in P$ and $g \in M_\infty \setminus P$ are assumed also to be non-negative.

Proof. For the first equality, it is enough to consider elements $f \in P \setminus P_U$. Without loss of generality (i.e., by Lemma 4.3), f is non-negative. Since $f \in P_U^f$ for any such f , we see that P is contained in the union. For the reverse containment, we need only show that if $g \in P_U^f$, then $g \in P$. Without loss of generality, g is non-negative. It was noted in Remark 3.4 (a) that, modulo P_U , f divides some power of every element of P_U^f . Thus, some power of g is in P . Since P is a prime ideal, it follows that $g \in P$, thus proving the first equality.

Next, we show that P is equal to the stated intersection. Let Q denote that intersection. Suppose that $g \in M_\infty \setminus P$ and that g is non-negative. Then, since the prime ideals of R contained between P_U and M_∞ are linearly ordered (by Corollary 3.3), we have $P \subset P_U^g$. (The inclusion is strict since $g \in P_U^g \setminus P$.) Combining Corollary 3.3 with Proposition 3.5, we can now infer that $P \subseteq P_{U,g}$. Therefore, $P \subseteq Q$.

If the assertion fails, there exists $h \in Q \setminus P$. Without loss of generality, h is non-negative. As $h \in M_\infty \setminus P$, the definition of Q gives $Q \subseteq P_{U,h}$, whence $h \in P_{U,h}$, the desired contradiction. \square

Let \mathcal{U} be a non-uniform ultrafilter on X . We do not know if there are any prime ideals between $P_{\mathcal{U}}$ and M_{∞} other than those of the form $P_{\mathcal{U},f}$ and $P_{\mathcal{U}}^f$. However, we are able to show in Proposition 4.6 and Corollary 4.7 that the union of any strictly ascending countable chain of prime ideals of the form $P_{\mathcal{U}}^f$ must be of the form P_h for some $h \in M_{\infty}$. First, we need the following proposition.

Lemma 4.5. *Let \mathcal{U} be a non-uniform ultrafilter on X . Consider any sequence $\{f_k\}_{k=1,2,\dots}$ of non-negative functions contained in M_{∞} . Then there exists an $f \in M_{\infty}$ such that $f_k \leq f \pmod{\mathcal{U}}$ for all k .*

Proof. Pick a countable element U of \mathcal{U} , and suppose that we have enumerated the elements of U so that we can write $U = \{1, 2, 3, \dots\}$. Since each f_k is continuous and sends ∞ to 0, we can inductively construct a strictly increasing (hence unbounded) sequence of positive integers $n_2 < n_3 < n_4 < \dots$ such that $n > n_k$ implies $f_1(n) < 1/k, f_2(n) < 1/k, \dots, f_k(n) < 1/k$. (In other words, each of the first k functions f_i is less than $1/k$ at all arguments after n_k .) Now define a function $f : X \rightarrow \mathbf{R}$ via

$$f(i) = \begin{cases} 0 & \text{for } i \notin U, \\ f_1(i) & \text{for } i = 1, 2, \dots, n_2, \\ \max\{f_1(i), f_2(i)\} & \text{for } i = n_2 + 1, n_2 + 1, \dots, n_3, \\ \max\{f_1(i), f_2(i), f_3(i)\} & \text{for } i = n_3 + 1, n_3 + 2, \dots, n_4, \\ \vdots & \end{cases}$$

If $i > n_k$, then $f(i) < 1/k$. (To see this, we can suppose $n_j + 1 \leq i \leq n_{j+1}$; note that $f(i) = \max\{f_t(i) \mid 1 \leq t \leq j\}$.) It follows that f is continuous, and so $f \in R$. As $f(\infty) = 0$, we have $f \in M_{\infty}$. Also, for $i \in U$ and for a given k , we have $f(i) \geq f_k(i)$ except for finitely many i 's. More precisely, if $i > n_k$, then $f(i) \geq f_k(i)$. Thus for all k , $f_k \leq f \pmod{\mathcal{U}}$. \square

Recall the definition of the ring T . It is the ring of continuous real-valued functions on X , a set of cardinality \aleph_1 with the discrete topology. For the moment, we want to consider the ordered field $\overline{T} := T/M$ and

its subring $\overline{R} := R/P_{\mathcal{U}}$, where $M = M_{\mathcal{U}}$ is the maximal ideal of T determined by some free ultrafilter \mathcal{U} on X . A natural ordering on T (modulo \mathcal{U}) can be obtained just as it was on R and, in fact, extends that earlier order. Then the ordering on \overline{T} is derived from the ordering on T modulo \mathcal{U} . (More precisely, we can well-define an ordering on \overline{T} as follows: if $f, g \in T$, then $f + M \leq g + M$ if and only if $f \leq g \pmod{\mathcal{U}}$.) Similarly, the notion of strong inequality (i.e., \ll) also carries over to \overline{T} . If \overline{A} and \overline{B} be subsets of \overline{T} , we say that $\overline{A} < \overline{B}$ (respectively, $\overline{A} \ll \overline{B}$) if, for each $a \in \overline{A}$ and $b \in \overline{B}$, $a < b$ (respectively, $a \ll b$).

It was shown by Erdős, Gillman, and Henriksen [8, Theorem 3.4] that \overline{T} is an η_1 -set. This means that, for any countable subsets \overline{A} and \overline{B} of \overline{T} such that $\overline{A} < \overline{B}$, there exists a $\overline{h} \in \overline{T}$ such that $\overline{A} < \overline{h} < \overline{B}$. Moreover, as noted in [8], the construction of \overline{h} is independent of \overline{B} . In particular, if \overline{B}' is another countable subset of \overline{T} such that $\overline{A} < \overline{B}'$, then for the same \overline{h} , we have $\overline{A} < \overline{h} < \overline{B}'$. This construction also works for strong inequalities in \overline{T} . Specifically, suppose that \overline{A} and \overline{B} are subsets of \overline{T} , with $\overline{A} = \{\overline{f}_1 \ll \overline{f}_2 \ll \dots \ll \overline{f}_i \ll \dots\}$ and $\overline{B} = \{\dots \ll \overline{g}_i \ll \dots \ll \overline{g}_2 \ll \overline{g}_1\}$ such that $\overline{A} \ll \overline{B}$. By [8, Theorem 3.4], choose $\overline{h} \in \overline{T}$ satisfying $\overline{A} < \overline{h} < \overline{B}$. Then $\overline{f}_i \ll \overline{f}_{i+1} < \overline{h} < \overline{g}_{j+1} \ll \overline{g}_j$, and so $\overline{A} \ll \overline{h} \ll \overline{B}$. Now if A and B are subsets of R with $A \ll B \pmod{\mathcal{U}}$, then by passing to $\overline{R} \subset \overline{T}$, we can find $\overline{h} \in \overline{T}$ between the two sets. We can then pull this element back to some $h \in T$ with $A \ll h \ll B$. It is not clear whether h is also in \overline{R} . However, under the circumstances that we need, it will be. We are now ready to show our next result.

Proposition 4.6. *Let \mathcal{U} be a non-uniform ultrafilter on X . Let $A = \{f_i\}_{i \in \mathbb{N}}$ be a denumerable subset of M_{∞} , where $0 < f_i \ll f_{i+1} \pmod{\mathcal{U}}$ for each i . Then there exists an $h \in M_{\infty}$ such that $\cup P_{\mathcal{U}}^{f_i} = P_{\mathcal{U},h}$.*

Proof. We will suppress the ultrafilter \mathcal{U} when writing inequalities or prime ideals. Consider the prime ideal $P := \cup P^{f_i} \subseteq M_{\infty}$, and let f be as in Lemma 4.5. Then $f_k \leq f$ for all k . As $f_k \ll f_{k+1}$, we conclude from the construction of f that $f_k \ll f$ for all k . (Indeed, if there exist $n, M > 0$ such that $f^n \leq Mf_k$, then $(f_{k+1})^n \leq Mf_k$, a contradiction.) Let $A := \{f_k\}_{k=1,2,\dots}$ and $B := \{f^k\}_{k=1,2,\dots}$. Since $f \in M_{\infty}$, $\text{coz}(z)$ is countable and f converges to 0 on $\text{coz}(f)$. Thus $\text{mod } \mathcal{U}$, $f^{k+1} \leq f = f^1$ for all k . Moreover $A < B$ (since $\text{mod } \mathcal{U}$,

$f_k \ll f$ implies that $f_k \leq f \leq f^n$ for all k and n). Then, using [8, Theorem 3.4] applied to \overline{T} , we can find $h \in T$ such that $A \ll h \leq B$.

Viewing the elements f_k, h and f^j in T , we see that modulo \mathcal{U} (in the obvious sense), we have $0 < f_k \leq h \leq f^j$. Since $f \in M_\infty$, we can conclude that the set of inputs where h is nonzero is an element of \mathcal{U} , and that on this set, h converges to 0. Hence, $h \in M_\infty$. (Note that we are abusing notation, as $M_\infty \subseteq R$, but the meaning is clear here, namely, that $h \in T$ and $h(\infty) = 0$.) As $f_i \ll h$ for all i , Corollary 3.6 gives us that $P^{f_i} \subset P_h$ for all i , whence $P \subseteq P_h$. If the assertion fails, choose $g \in P_h \setminus P$. By Lemma 4.3, g can be assumed non-negative. Since $g \notin P^{f_k}$, we conclude that $f_k \ll g$. (In detail, if there exist $n, K > 0$ such that $g^n \leq K f_k$, note that K is a unit of R , so that $(1/K)g^n \leq f_k$, whence $(1/K)g^n \in P^{f_k}$ and $g \in P^{f_k}$, a contradiction.) Therefore, $A \ll g$, and so $A \ll \{g^n\}_{n=1,2,\dots}$. But, as noted prior to the statement of the proposition, it follows from the construction of h in [8, Theorem 3.4] that $A \ll h \ll \{g^n\}_{n=1,2,\dots}$. This gives the desired contradiction, for $g \in P_h$ implies that $h \not\ll |g| = g$. \square

We return to the ring R for the next result.

Corollary 4.7. *Let \mathcal{U} be a non-uniform ultrafilter on X . Let*

$$P_{\mathcal{U}}^{f_1} \subset P_{\mathcal{U}}^{f_2} \subset P_{\mathcal{U}}^{f_3} \subset \dots$$

be a strictly ascending denumerable chain of prime ideals of R . Then there exists $h \in M_\infty$ such that $\cup P_{\mathcal{U}}^{f_i} = P_h$.

Proof. By Proposition 2.12 (b), $f_1 \ll f_2 \ll \dots$, and so an application of Proposition 4.6 completes the proof. \square

We close with some observations and questions. In Section 2, we showed that R is not a ULO ring by showing that $R \subseteq T$ does not satisfy LO, where T is the ring of real-valued functions on the discrete space of cardinality \aleph_1 . Note that T is, in fact, the complete ring of quotients of R . To see this, observe that T is a direct product of fields and that the ideal of T consisting of those functions with finite support is contained in R (for if f is such a function, then one can extend its domain to Y by defining $f(\infty)$ to be 0). Moreover, this is the unique

ideal of R that is minimal with respect to being dense in R (“dense” in the sense that its annihilator is zero). In [6, Remark 2.18 (b)], we suggested that it might be possible to show that a ring A is a ULO-ring if (and only if) the canonical embedding of A into its complete ring of quotients satisfies LO. The example given by $R \subseteq T$ represents one more (small) piece of evidence in that direction.

By Proposition 2.16, R is not of the form $A + B$. However, the referee pointed out the following. Consider the more general $C + D$ construction for rings (as opposed to the specific $A + B$ construction from [10] that was examined in Section 2). Namely, if S is an arbitrary ring, C a subring of S and D an ideal of S , then the additive group $C + D$ is also a subring of S . Now let R and T be as usual. Let C be the diagonal in T , that is, the subring consisting of constant functions, and let $D \subset T$ again be the ideal of functions which have finite support. Then $C + D$ is a (von Neumann regular) subring of R . Moreover, it follows from the Stone-Weierstrass theorem that $C + D$ is in fact dense in R when R is supplied with the sup norm. Since $R = C(Y)$ is complete with the sup norm, R is the norm-completion of $C + D$.

Finally, note that, if \mathcal{U} is a non-uniform ultrafilter on X , we have described each of the prime ideals of R contained between $P_{\mathcal{U}}$ and M_{∞} as being either a union of prime ideals of the form $P_{\mathcal{U}}^f$ or an intersection of prime ideals of the form $P_{\mathcal{U},f}$. However, we do not know whether the prime ideals of the form $P_{\mathcal{U}}^f$ or $P_{\mathcal{U},f}$ are the only prime ideals between $P_{\mathcal{U}}$ and M_{∞} . In particular, we would ask the following, given a chain \mathcal{C} of functions in M_{∞} , ordered by \ll : does \mathcal{C} have a co-final countable subchain? A positive answer to this question, when combined with Theorem 4.4 and Corollary 4.7, would imply that R has no prime ideals other than those of the form $P_{\mathcal{U}}, P_{\mathcal{U},f}, P_{\mathcal{U}}^f$ and $M_a, a \in Y$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996

Email address: dobbs@math.utk.edu

DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA 22030

Email address: rlevy@gmu.edu

DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA 22030

Email address: jshapiro@gmu.edu