

JONES'S POLYNOMIAL FOR LINKS IN THE HANDLEBODY

KHALED BATAINEH AND MUSTAFA HAJI

ABSTRACT. In [3] Hoste and Przytycki defined a two variable polynomial invariant for 1-trivial dichromatic links by a method similar to that of Kauffman in defining the Jones polynomial. In this paper we view the invariant of Hoste and Przytycki as an invariant for knots and links in the solid torus, and we give a state sum formula for this invariant. Then we define Jones's polynomial for links in the handlebody. In particular, we give formulas for the Jones's polynomial for knots and links in the handlebody with two handles. Moreover, we give a state sum formula for this invariant.

1. Introduction. Knots and links in three manifolds have been studied by different authors and from different points of view. For example, Kalfagianni in [5] studied finite type invariants for knots in general three manifolds, and Bataineh in [2] studied invariants for knots in the solid torus of types one and two. In this paper we study a polynomial invariant for knots in the handlebody.

In 1984, Vaughan Jones gave his revolutionary new polynomial invariant for knots and links [4]. This polynomial opened a wide area of applications to many branches of mathematics and physics. The Jones's polynomial succeeded in distinguishing many different knots and links that could not be distinguished by other invariants including the well-known Alexander invariant.

Louis Kauffman defined the Jones polynomial in a very natural way using his well-known bracket polynomial [6]. Moreover, Kauffman gave a formula (a state sum formula) that makes the calculation of the Jones's polynomial easier by looking at what is called the states of a given diagram of a link. See also [7].

Hoste and Przytycki in [3] defined what can be called the Jones's polynomial invariant for 1-trivial dichromatic links by a method similar

Keywords and phrases. Knots and links, Jones's polynomial, Kauffman's bracket polynomial, handlebody.

Received by the editors on December 17, 2009, and in revised form on July 13, 2010.

DOI:10.1216/RMJ-2013-43-3-737 Copyright ©2013 Rocky Mountain Mathematics Consortium

to that of Kauffman. In this paper we view the invariant of Hoste and Przytycki as an invariant for knots and links in the solid torus, and we give a state sum formula for this invariant. Then we define the Jones polynomial for knots and links in the handlebody. In particular, we give the formulas for the Jones polynomial for knots and links in the handlebody with two handles, and we give a state sum formula for this invariant.

In Section 1 we introduce the concepts and terminology that we will use in the later sections. We give the definition of the Jones's polynomial as defined by Kauffman, and we define knots and links in the solid torus and the handlebody.

In Section 2 we introduce a polynomial invariant for links in the solid torus; the Y polynomial. This invariant, which has two variables, is easy to calculate. We give two ways to calculate this invariant, one is by using the rules by which the invariant is defined and the other is a state sum formula that makes the calculations much easier. We also give an example of two knots that are distinguished by this polynomial invariant, but not by Aicardi's polynomial invariant defined in [1].

In Section 3 we generalize the Y polynomial invariant to the handlebody with two handles. This invariant has four variables rather than two in the case of the solid torus. We again give two ways to calculate this invariant.

2. Basic concepts and terminology. In this section, we give Jones's polynomial as defined by Kauffman [6] in terms of his bracket polynomial. Then a simple change of the variable will give the original Jones's polynomial. We will be mainly following the methodology of [8] in defining the Jones's polynomial. We also give a formula, called a *state sum formula*, of Kauffman that makes the calculation of Kauffman's bracket polynomial easier. We also introduce the terminology for knots and links in the handlebody. Particularly, we study knots and links in the handlebody with only one handle; the solid torus.

2.1. Jones's polynomial and Kauffman's state sum formula.

Definition 1. The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials

with integer coefficients in an indeterminate A . It maps a diagram D of a link L to $\langle D \rangle \in \mathbf{Z}[A, A^{-1}]$, and it is characterized by three rules:

- (i) $\langle \text{crossing} \rangle = A \langle \text{right-hand crossing} \rangle + A^{-1} \langle \text{left-hand crossing} \rangle$,
- (ii) $\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle$,
- (iii) $\langle \bigcirc \rangle = 1$.

In this definition, \bigcirc is the diagram of the unknot with no crossings, and $D \cup \bigcirc$ is a diagram consisting of the diagram D together with an extra closed curve \bigcirc that contains no crossings at all, neither with itself nor with D . In (i), the formula refers to three diagrams that are exactly the same except near a point where they differ in the way indicated. In the case when we have an oriented link D and we want to calculate the bracket polynomial for this link, we will denote by $|D|$ the nonoriented diagram that is obtained from D by forgetting the orientations of all components. A crossing in an oriented diagram D has a sign of $+1$ or -1 according to the right-hand rule. The sum of signs of the crossings for a given diagram D is called the *writhe* of D and is denoted by $w(D)$.

Theorem 1. *Let D be a an oriented link diagram. Then the X polynomial defined by*

$$X(D) = (-A)^{-3w(D)} \langle |D| \rangle$$

is an invariant of links.

Definition 2. The Jones's polynomial $V(L)$ of an oriented link L is the Laurent polynomial in the indeterminate q . With integer coefficients, defined by

$$V(L) = \left((-A)^{-3w(D)} \langle |D| \rangle \right)_{q^{-1/4}=A} \in \mathbf{Z}[q^{-1}, q].$$

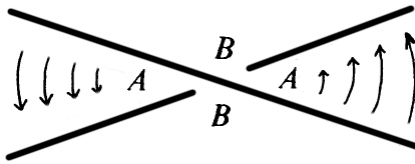


FIGURE 1. Regions of a crossing.

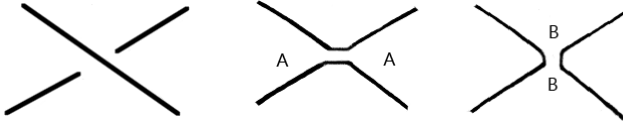


FIGURE 2. An *A*-split and a *B*-split of a crossing.

We have seen that the X polynomial for an oriented link diagram D can be calculated using the three rules in the definition of the bracket polynomial. An alternative and a more direct way to calculate the X polynomial is now introduced.

Consider a link diagram. Each crossing divides the plane into four regions. We label two of them with an A and two of them with a B by the following rule. The regions labeled by A are those swept out when we rotate the upper strand in the crossing counterclockwise, and the regions labeled by B are the other two regions. See Figure 1.

We see that the A regions correspond to a smoothing in the crossing being considered that “opens the A -channel” while B regions correspond to another smoothing in that crossing that opens the B -channel (see Figure 2). Thus, we can associate, at each crossing of the link diagram, two types of splitting: an *A-split* and a *B-split*.

Now, given a link diagram D with n crossings. A *state* S of D is a choice of how to smooth all of the n crossings in the link diagram D . Each state is a set of nonoverlapping circles, and we have 2^n states. The following theorem provides us with a formula for calculating $\langle D \rangle$ called a *state sum formula*.

Theorem 2. *The bracket polynomial of the link diagram D is given by*

$$\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1}$$

where the sum runs over all possible states S , such that $|S|$ is the total number of unknots in state S , $a(S)$ is the number of A -splits in state S and $b(S)$ is the number of B -splits in state S .

2.2. Knots and links in the handlebody. Let ST be the closed solid torus defined by

$$ST = \{(x, y, z) \in \mathbf{R}^3 : 1 \leq x^2 + y^2 \leq 9, -1 \leq z \leq 1\}.$$

Let HB^n be the closed handlebody with n handles defined by

$$HB^n = \cup_{i=0}^{n-1} \{(x, y, z) \in \mathbf{R}^3 : 1 \leq (x - 5i)^2 + y^2 \leq 9, -1 \leq z \leq 1\}.$$

Let P^n be the projection to the xy -plane of the handlebody HB^n . So P^n is given by

$$P^n = \cup_{i=0}^{n-1} \{(x, y) \in \mathbf{R}^2 : 1 \leq (x - 5i)^2 + y^2 \leq 9\}.$$

Note that ST is just HB^1 . We will denote links in the solid torus by LST and links in the handlebody by LHB^n .

Definition 3. A knot K in the handlebody HB^n is the image of a piecewise-linear one-to-one mapping $f : S^1 \rightarrow HB^n$ such that $f(S^1) \subseteq \text{int}(HB^n)$.

As in the definition of knots in \mathbf{R}^3 , we might consider a knot in HB^n to be oriented with an orientation induced from that on S^1 .

Definition 4. Two knots K_1 and K_2 in HB^n are said to be *isotopy equivalent* if there exists an orientation-preserving homeomorphism $\phi : HB^n \rightarrow HB^n$ such that $\phi(K_1) = K_2$, ϕ is the identity function on the boundary of HB^n , and ϕ preserves the orientation on the knots (if the knots are oriented).

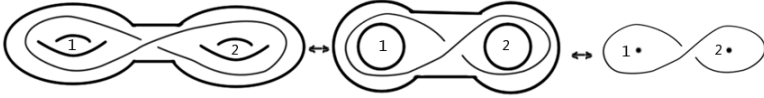


FIGURE 3. A knot in the handlebody with two handles and double punctured diagram.

A *knot diagram* for a knot K in HB^n is a diagram of K in P^n . So a knot diagram in HB^2 can be viewed as a diagram in P^2 , as in Figure 3.

Note that, in this research, we work on ordered handlebodies, that is, each hole in a given handlebody has a number, and this is one reason why we have a specific formula for our handlebody.

Definition 5. A *link* L in the handlebody HB^n is a finite collection of knots in HB^n that do not intersect each other.

Definition 6. Two links $L = \{K_1, K_2, \dots, K_u\}$ and $L' = \{K'_1, K'_2, \dots, K'_v\}$ in HB^n are said to be *isotopy equivalent* if the following conditions hold:

- (1) $u = v$, that is, L and L' have the same number of components;
- (2) There exists an orientation-preserving homeomorphism $\phi : HB^n \rightarrow HB^n$ such that $\phi(K_i) = K'_j$ for $i = \{1, 2, \dots, u\}$, ϕ is the identity function on the boundary of HB^n and ϕ preserves the orientation on the knots (if the knots are oriented).

Note that taking ϕ to be the identity function on the boundary of HB^n implies keeping the numbering of the holes unchanged.

A *link diagram* for a link L in HB^n is a usual diagram of L in P^n .

Definition 7. A *1-trivial dichromatic link* in S^3 is a link having at least two components, one of which is unknotted and labeled, or colored, “1,” while all other components are colored “2.”

A *link diagram* of a 1-trivial dichromatic link L in S^3 is a usual diagram of the components colored 2 in the punctured plane P^1 . The puncture represents the trivial component, since the unknotted

component can be isotoped to be the z -axis along with the point at infinity, which projects to a point. A diagram of a 1-trivial dichromatic link is usually referred to as a *punctured diagram*.

3. Jones's polynomial for links in the solid torus. In this section we introduce a polynomial invariant for links in the solid torus. This invariant, which we denote by Y , is the same as the invariant for 1-trivial dichromatic links in \mathbf{R}^3 that was defined by Hoste and Przytycki [3]. The Y invariant, which has two variables, is easy to calculate and has some characteristics that are inherited from the Jones's polynomial. We also give a state sum formula for the bracket polynomial involved in the Y invariant. An example of two knots that are distinguished by this polynomial invariant, but not by Aicardi's polynomial invariant defined in [1], is also given.

The following theorem is due to Hoste and Przytycki [3].

Theorem 3. *Two punctured diagrams represent equivalent 1-trivial dichromatic links if and only if one can be transformed into the other by a finite sequence of the usual three Reidemeister moves in the punctured plane P^1 , preceded by possibly flipping over one of the diagrams.*

Note that the following corollary follows immediately.

Corollary 1. *Two links L_1 and L_2 in the solid torus ST are isotopy equivalent if and only if a diagram of one can be transformed into a diagram of the other by a finite sequence of the usual three Reidemeister moves in the punctured plane P^1 .*

We will call the knot that circles once around the hole in the solid torus and has no crossings the *dottedcircle*, see Figure 4.

Now we give the theorem that introduces the Y invariant of Hoste and Przytycki [3], but as an invariant of links in the solid torus instead

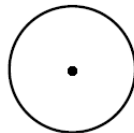


FIGURE 4. The dottedcircle.

of 1-trivial dichromatic links in \mathbf{R}^3 , and we use our own terminology in naming the formulas.

Theorem 4. *Let D be a diagram of a link L in the solid torus, and let $\langle D \rangle$ be determined by the following formulas.*

(I) *The smoothing formulas:*

$$(1) \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle = A \left\langle \begin{array}{c}) \\ (\end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle$$

$$(2) \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle = A \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c}) \\ (\end{array} \right\rangle$$

(II) *The reduction formulas:*

$$(1) \left\langle |D| \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle |D| \rangle$$

$$(2) \left\langle |D| \cup \odot \right\rangle = (-A^2 - A^{-2}) t \langle |D| \rangle.$$

(III) *The finishing formulas:*

$$(1) \left\langle \bigcirc \right\rangle = 1$$

$$(2) \left\langle |D| \cup \odot \right\rangle = t.$$

Then $Y(L) = (-A^3)^{-w(D)} \langle |D| \rangle$ is a Laurent polynomial invariant in $\mathbf{Z}[A, A^{-1}, t]$.

Now we give the state sum formula for this bracket polynomial, which gives an alternative way of calculating this invariant, and which is mostly easier to apply.

Theorem 5. *Let D be a diagram of a link L in the solid torus. Then the bracket polynomial of diagram D of the link L is given by*

$$\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1} t^{|T|}$$

where the sum runs over all possible states S of the link diagram, $|S|$ is the total number of circles and dotted circles in state S , $|T|$ is the

number of the dotted circles, $a(S)$ is the number of A -splits in state S and $b(S)$ is the number of B -splits in state S .

Proof. Assume that the link diagram D has n crossings. Pick a crossing in D . By using the smoothing formulas, the bracket polynomial of D can be determined by the bracket polynomials of two diagrams D_1 and D_2 , each of which has one fewer crossing than D . The diagrams D_1 and D_2 are obtained by applying an A -split and a B -split on that crossing in D . Now use the smoothing formulas again to determine the bracket polynomials of each of D_1 and D_2 in terms of the bracket polynomials of four diagrams, each of which has two fewer crossings than D . Continuing with the same procedure, we eventually get the bracket polynomial for D in terms of the bracket polynomials for 2^n diagrams, each of which has no crossings. Hence, we have 2^n states in the solid torus. Note that each state in the solid torus is a set of nonoverlapping circles and dotted circles.

Now let S be a given state; then, by using the reduction formulas and the finishing formulas we get directly

$$\langle S \rangle = (-A^2 - A^{-2})^{|S|-1} t^{|T|}.$$

Note that each time we split a crossing, the polynomials of the resultant diagrams were multiplied by either an A or A^{-1} , depending on whether the split was an A -split or a B -split. So the polynomial of S is multiplied by $A^{a(S)-b(S)}$. Hence, the total contribution to the bracket polynomial by state S is $A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1} t^{|T|}$.

So the bracket polynomial of diagram D of link L will be the sum over all possible states of these contributions. We write this as

$$\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1} t^{|T|}.$$

This completes the proof. □

Example 1. The Y polynomial for a given knot is calculated below in two different ways of calculating the bracket polynomial. At first we do not use the state sum formula.

$$\begin{aligned} \langle \textcircled{\textcircled{\circ}} \rangle &= A \langle \textcircled{\textcircled{\circ}} \rangle + A^{-1} \langle \textcircled{\textcircled{\circ}} \rangle \\ &= A(1) + A^{-1}((-A^2 - A^{-2})t^2) = A - At^2 - A^{-3}t^2. \end{aligned}$$

Now, using the state sum formula, we get the same answer.

For the state $\textcircled{\circ}$, we have $a(S) = 1, b(S) = 0, |S| = 1, |T| = 0$.

For the state $\textcircled{\ominus}$, we have $a(S) = 0, b(S) = 1, |S| = 2, |T| = 2$.

Therefore,

$$\begin{aligned} \langle \textcircled{\circ} \rangle &= A^{1-0}(-A^2 - A^2)^{1-1}t^0 + A^{0-1}(-A^2 - A^2)^{2-1}t^2 \\ &= A + (-A^2 - A^2)t^2 = A - At^2 - A^{-3}t^2. \end{aligned}$$

Since $w(D) = -1$, by using the definition of the Y polynomial, we get

$$Y(\textcircled{\circ}) = (-A^3)^{-(-1)}(A - At^2 - A^{-3}t^2) = -A^4 + A^4t^2 + t^2.$$

3.1. Aicardi’s invariant $[S(K)](t)$ for knots in the solid torus.

Aicardi’s invariant $[S(K)](t)$ is a Laurent polynomial invariant with integer coefficients defined for any oriented knot K in the solid torus by:

$$[S(K)](t) = \frac{1}{2} \sum_p e(p)[t^{i_1(p)} + t^{i_2(p)}],$$

where the sum runs over all crossings p of a diagram of a knot K . The integers $i_1(p)$ and $i_2(p)$ are defined to be the winding numbers of the two lobes resulting from replacing the crossing p by a double point. The integer $e(p)$ is defined to be the sign of the crossing p if both $i_1(p)$ and $i_2(p)$ are nonzero, and $e(p)$ is defined to be zero otherwise.

Aicardi shows that S is an isotopy invariant by showing that it is invariant under the three Reidemeister moves. For the proof, see [1].

Next, we give an example of two knots in the solid torus that are distinguished by the Y polynomial, but not by Aicardi’s polynomial.

Example 2. Consider the following two knots B and B' , respectively.



One can easily see that Aicardi’s invariant does not distinguish these two knots, since they differ only at a crossing that does not contribute

to Aicardi's sum formula. Calculation of Aicardi's invariant yields $[S(B)](t) = t^1 + t^{-1} = [S(B')](t)$. However, calculation of the Y invariant of these two knots yields:

$$\begin{aligned}
 Y(B) &= A^{-8} - A^{-8}t^2 + t^2, \\
 Y(B') &= A^{-16}t^2 - A^{-8}t^2 + A^{-4} + A^{-12} - A^{-8}.
 \end{aligned}$$

4. Jones's polynomial for links in the handlebody. In this section we go further with the work in the previous section to generalize the Y polynomial invariant to links in the handlebody. For simplicity, we give the formulas in the double solid torus HB^2 , and the invariant will have four variables. Moreover, we give the state sum formula for the generalized invariant.

We start with the following theorem in which we prove that isotopy equivalence of knots in the handlebody is characterized by the usual three Reidemeister moves on diagrams of the knots.

Theorem 6. *Two knots K_1 and K_2 in the handlebody are isotopy equivalent if we can get from a diagram of one to a diagram of the other by a finite sequence of the usual three Reidemeister moves.*

Proof. Let K_1 and K_2 be two isotopy equivalent knots in HB^n . Let $\phi : HB^n \rightarrow HB^n$ be an orientation-preserving homeomorphism as in the definition. Let $N = \mathbf{R}^3 - \text{int}(HB^n)$. Let $I : N \rightarrow N$ be the identity function. Let $\psi : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be defined by

$$\psi(x) = \left\{ \begin{array}{ll} \phi(x) & \text{if } x \in HB^n \\ I(x) & \text{if } x \in N \end{array} \right\}.$$

Note that ψ is a well-defined function because ϕ is the identity map on the boundary of the handlebody. Also, by the Gluing lemma, ψ is an orientation-preserving homeomorphism of \mathbf{R}^3 that preserves the orientation of the knots such that $\psi(K_1) = K_2$. Now we are in a position like that in the proof of Theorem 1 in [3]. Note that ϕ is ambient isotopic to the identity map since ϕ is the identity on the boundary of HB^n . Hence, ψ is ambient isotopic relative to N to the identity map. So we get an isotopy of \mathbf{R}^3 relative to N which takes K_1

to K_2 . Projecting this isotopy into the xy -plane, we see a diagram D_1 of K_1 in $P^n = \cup_{i=0}^{n-1} \{(x, y) \in \mathbf{R}^2 : 1 \leq (x - 5i)^2 + y^2 \leq 9\}$ taken to a diagram D_2 of K_2 in P^n . The usual proof that this can be accomplished by a finite sequence of Reidemeister moves can now be employed. \square

In this section, we need to define some knots that will be used in the main theorem. We will call the knot that circles once around the first hole in the handlebody and has no crossings *dottedcircle*₁, the knot that circles once around the second hole in the handlebody and has no crossings *dottedcircle*₂, the knot that circles once around both holes in the handlebody and has no crossings *double-dottedcircle*. See Figure 5.



FIGURE 5. (a) Dottedcircle₁ (b) Dottedcircle₂ (c) Double-dottedcircle

4.1. The main theorem.

Theorem 7. *Let D be a diagram of a link L in the handlebody with two handles, and let $\langle D \rangle$ be determined by the following formulas:*

(I) *The smoothing formulas:*

$$(1) \langle \diagup \diagdown \rangle = A \langle \rangle \langle \rangle + A^{-1} \langle \diagdown \diagup \rangle$$

$$(2) \langle \diagdown \diagup \rangle = A \langle \diagup \diagdown \rangle + A^{-1} \langle \rangle \langle \rangle$$

(II) *The reduction formulas:*

$$(1) \langle |D| \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle |D| \rangle$$

$$(2) \langle |D| \cup \bigcirc_{\cdot 1} \rangle = (-A^2 - A^{-2}) t_1 \langle |D| \rangle$$

$$(3) \left\langle |D| \cup \begin{array}{c} \bigcirc \\ \cdot_2 \end{array} \right\rangle = (-A^2 - A^{-2})t_2 \langle |D| \rangle$$

$$(4) \left\langle |D| \cup \begin{array}{c} \text{---} \bigcirc \text{---} \\ \cdot_1 \quad \cdot_2 \end{array} \right\rangle = (-A^2 - A^{-2})s \langle |D| \rangle.$$

(III) *The finishing formulas:*

$$(1) \left\langle \bigcirc \right\rangle = 1$$

$$(2) \left\langle \begin{array}{c} \bigcirc \\ \cdot_1 \end{array} \right\rangle = t_1$$

$$(3) \left\langle \begin{array}{c} \bigcirc \\ \cdot_2 \end{array} \right\rangle = t_2$$

$$(4) \left\langle \begin{array}{c} \text{---} \bigcirc \text{---} \\ \cdot_1 \quad \cdot_2 \end{array} \right\rangle = s.$$

Then $Y(L) = (-A^3)^{-w(D)} \langle |D| \rangle$ is a Laurent polynomial invariant in $\mathbf{Z}[A, A^{-1}, t_1, t_2, s]$.

Proof. Note that the term “ $(-A^3)^{-w(D)}$ ” depends on the writhe of the link diagram D , and the writhe is invariant under the second and third Reidemeister’s moves. So all we have to do, in order to prove that the Y is invariant under the second and the third Reidemeister’s moves, is to prove that the bracket polynomial $\langle \rangle$ is invariant under

these two moves respectively as follows:

$$\begin{aligned}
 \langle \text{Diagram 1} \rangle &= A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle \\
 &= A^2 \langle \text{Diagram 4} \rangle + \langle \text{Diagram 5} \rangle + \langle \text{Diagram 6} \rangle + A^{-2} \langle \text{Diagram 7} \rangle \\
 &= A^2 \langle \text{Diagram 8} \rangle + (-A^2 - A^{-2}) \langle \text{Diagram 9} \rangle + \langle \text{Diagram 10} \rangle + A^{-2} \langle \text{Diagram 11} \rangle \\
 &= \langle \text{Diagram 12} \rangle.
 \end{aligned}$$

For the third Reidemeister's move,

$$\begin{aligned}
 \langle \text{Diagram 13} \rangle &= A \langle \text{Diagram 14} \rangle + A^{-1} \langle \text{Diagram 15} \rangle \\
 &= A \langle \text{Diagram 16} \rangle + A^{-1} \langle \text{Diagram 17} \rangle \\
 &= \langle \text{Diagram 18} \rangle
 \end{aligned}$$

For the invariance of Y under the first Reidemeister's move, assume without loss of generality, that we have a link diagram D in the handlebody, and that D' is D with a first Reidemeister's move applied on D with a positive crossing. Then

$$\begin{aligned}
 Y(L) &= (-A^3)^{-w(D')} \langle |D'| \rangle \\
 &= (-A^3)^{-(w(D)+1)} \langle |D'| \rangle \\
 &= \left[(-A^3)^{-(w(D)+1)} \right] \left[(-A^3) \langle |D| \rangle \right] \\
 &= (-A^3)^{-w(D)} \langle |D| \rangle = Y(L).
 \end{aligned}$$

Note that the order in which we apply the formulas of the bracket to a diagram of some link is irrelevant, and this follows from the nice algebraic properties of addition and multiplication of Laurent polynomials such as commutativity, associativity and distributivity. \square

4.2. The state sum formula for the Y polynomial. In the following theorem, we repeat almost the same justification to get the state sum formula for a link in the handlebody with two handles.

Theorem 8. *Let D be a diagram of a link L in the handlebody with two handles HB^2 . Then the bracket polynomial of diagram D of the link L is given by*

$$\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1} t_1^{|T_1|} .t_2^{|T_2|} .s^{|H|},$$

where the sum runs over all possible states S of the link diagram, $|S|$ is the total number of circles, dottedcircle₁s, dottedcircle₂s and double-dottedcircles in the state S . $|T_1|$ is the number of the dottedcircle₁s, $|T_2|$ is the number of the dottedcircle₂s, $|H|$ is the number of double-dottedcircles, $a(S)$ is the number of A -split in state S and $b(S)$ is the number of B -split in state S .

Proof. Assume that the link diagram D has n crossings. Pick a crossing in D . By using the smoothing formulas, the bracket polynomial of D can be determined by the bracket polynomials of two diagrams D_1 and D_2 , each of which has one fewer crossing than D . The diagrams D_1 and D_2 are obtained by applying an A -split and a B -split on that crossing in D . Now use the smoothing formulas again to determine the bracket polynomials of each of D_1 and D_2 in terms of the bracket polynomials of four diagrams, each of which has two fewer crossings than D . Continuing with the same procedure, we eventually get the bracket polynomial for D in terms of the bracket polynomials for 2^n diagrams, each of which has no crossings. Hence, we have 2^n states in the handlebody with two handles.

Now let S be a given state. Then, by using the reduction and finishing formulas, we get directly

$$\langle S \rangle = (-A^2 - A^{-2})^{|S|-1} t_1^{|T_1|} .t_1^{|T_2|} .s^{|H|}.$$

Note that, each time we split a crossing, the polynomials of the resultant diagrams were multiplied by either an A or A^{-1} , depending on whether the split was an A -split or a B -split. So the polynomial of S is multiplied by $A^{a(S)-b(S)}$. Hence, the total contribution to the bracket polynomial by state S is $A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}t_1^{|T_1|}t_2^{|T_2|}s^{|H|}$.

So the bracket polynomial of diagram D of the link L will be the sum over all possible states of these contributions. We write this as

$$\langle D \rangle = \sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}t_1^{|T_1|}t_2^{|T_2|}s^{|H|}.$$

This completes the proof. \square

We can give the state sum formula for the bracket polynomial of a link diagram D in the handlebody with n handles HB^n . For example, we give it for the handlebody with 3 handles HB^3 as follows:

$$\langle D \rangle = \sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}t_1^{|T_1|}t_2^{|T_2|}t_3^{|T_3|}s_1^{|H_{12}|}s_2^{|H_{13}|}s_3^{|H_{23}|}q^{|G|}.$$

$|T_1|$ is the number of the dottedcircle_{1S}, $|T_2|$ is the number of the dottedcircle_{2S}, $|T_3|$ is the number of the dottedcircle_{3S}, $|H_{12}|$ is the number of the double-dottedcircle_{12S}, $|H_{13}|$ is the number of the double-dottedcircle_{13S}, $|H_{23}|$ is the number of the double-dottedcircle_{23S}, $|G|$ is the number of the triple-dottedcircles, $a(S)$ is the number of A -splits in state S , and $b(S)$ is the number of B -splits in state S .

Finally, note that one can write down the smoothing, reduction and finishing formulas for the general case in HB^n . The smoothing formulas are the same in HB^n as S^3 . There will be 2^n reduction formulas, the first of which is

$$\langle |D| \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle |D| \rangle,$$

and, for the other $2^n - 1$ reduction formulas, the expression $(-A^2 - A^{-2})\langle |D| \rangle$ is multiplied by some variable. These $2^n - 1$ variables are in one-to-one correspondence with the nonempty subsets of the set of n punctures. At last, there will be 2^n finishing formulas.

REFERENCES

1. F. Aicardi, *Topological invariants of knots and framed knots in the solid torus*, C.R. Acad. Sci. Paris **321** (1995), 81–86.
2. K. Bataineh, *Finite type invariants for knots in the solid torus*, Ph.D. thesis, New Mexico State University, 2003.
3. J. Hoste and J. Przytycki, *An invariant of dichromatic links*, Pros. Amer. Math. Soc. **105** (1989), 1003–1007.
4. V. Jones, *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc., 1985.
5. E. Kalfagianni, *Finite type invariants for knots in three manifolds*, Topology **37** (1998), 673–707.
6. L. Kauffman, *New invariants in the theory of knots*, Amer. Math. Month. **95** (1988), 195–242.
7. ———, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
8. W. Raymond, *An introduction to knot theory*, Springer-Verlag, New York, 1997.

JORDAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, IRBID, JORDAN
Email address: khaledb@just.edu.jo

JORDAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, IRBID, JORDAN
Email address: mustafahajj@gmail.com