ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 2, 2013

NATURAL BOUNDARIES OF A FAMILY OF DIRICHLET SERIES

ALEXANDRU ZAHARESCU AND MOHAMMAD ZAKI

ABSTRACT. We study the natural boundary of the Dirichlet series

$$F_{q,b,H,\alpha}(s) = \sum_{\substack{m,k\geq 1 \\ mk\equiv b \pmod q}} H(\alpha \log(m+k)) \frac{\Lambda(m)\Lambda(k)}{(m+k)^s},$$

where Λ is the classical Von Mangoldt function, H is a smooth periodic function with period 1, $\alpha > 0$ is a real number and $b, q > 0$ are integers with $(b, q) = 1$.

1. Introduction. Let $\psi_k(s)$ be a Dirichlet series given by $\psi_k(s) = \sum_{m=1}^{\infty} a_k(m)/n^s$, $1 \leq k \leq r$, and consider the multiple Dirichlet series

$$\Psi_r(s_1, \dots, s_r|\psi_1, \dots, \psi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \frac{a_2(m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}}$$

Matsumoto and Tanigawa [**14**] showed that $\Psi_r(s_1, \dots, s_r \mid \psi_1, \dots, \psi_r)$ can be continued meromorphically to $\mathbf{C}^r$ if, for each $k \in \{1, 2, \dots, r\}$, $\psi_k(s) = \sum_{m=1}^{\infty} a_k(m)/n^s$ is absolutely convergent for $\sigma = \Re(s) > \alpha_k > 0$, can be continued meromorphically to the whole plane $\mathbf{C}$, holomorphic except for a possible pole of order at most 1 at $s = \alpha_k$, and of polynomial order in any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$. They also described explicitly the location of the singularities. In particular, if all $\psi_k(s)$ are entire, then $\Psi_r(s_1, \dots, s_r \mid \psi_1, \dots, \psi_r)$ is also entire. One

2010 AMS *Mathematics subject classification.* Primary 11M06, 11M20, 11M26.
Keywords and phrases. Zeta and L-functions, Dirichlet series, analytic continuation.
The first author is supported in part by the NSF grant number DMS-0901621.
Received by the editors on January 15, 2010, and in revised form on August 27, 2010.
DOI:10.1216/RMJ-2013-43-2-661

of the tools employed in the proof of this result is the Mellin-Barnes formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\lambda^{-z}dz \tag{1.1}$$

where $s, \lambda \in \mathbf{C}$, $\lambda \neq 0$, $|\arg \lambda| < \pi$, $\Re(s) > 0$, $0 < c < \Re(s)$, and the path of integration is the vertical line from $c - i\infty$ to $c + i\infty$. Egami and Matsumoto [**4**] also point out that, if one assumes that $\psi_k(s)$, $1 \leq k \leq r$, have finitely many poles, then $\Psi_r(s_1, \dots, s_r \mid \psi_1, \dots, \psi_r)$ can be continued meromorphically to $\mathbf{C}^r$. If $\psi_k(s)$, $1 \leq k \leq r$, have infinitely many poles, then the behavior of $\Psi_r(s_1, \dots, s_r \mid \psi_1, \dots, \psi_r)$ might be quite different. Let $\Lambda(n)$ be the Von Mangoldt function, and let

$$M(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Define

$$\phi_2(s) = \Psi(0, s; M, M) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)}{(k+m)^s} = \sum_{n=1}^{\infty} \frac{G_2(n)}{n^s},$$

where

$$G_2(n) = \sum_{k+m=n} \Lambda(k)\Lambda(m).$$

It is easy to see that $\phi_2(s)$ converges absolutely for $\Re(s) > 2$. Fujii [**6**] showed that, if we assume RH, then

$$\sum_{n \leq x} G_2(n) = \frac{1}{2}x^2 - H(x) + O((x \log x)^{4/3}),$$

where $H(x) = 2\sum_{\rho} x^{1+\rho}/\rho(1+\rho)$. For more results on $G_2(n)$, the reader is referred to [**6, 9, 15**]. In [**4**], Egami and Matsumoto proved the following result.

Assuming RH, *$\phi_2(s)$ has a meromorphic continuation to $\Re(s) > 1$, and holomorphic except for the simple pole at $s = 2$ with residue 1, and*

$s = 1 + \rho$ with residue $-2\eta(\rho)/\rho$ for every nontrivial zero ρ of $\zeta(s)$, where $\eta(\rho)$ is the multiplicity of ρ.

By applying Perron's formula, we find that

$$\sum_{n \leq x} G_2(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \phi_2(s) \frac{x^s}{s} \, ds + O(T^{-1} x^{2+\varepsilon}), \quad c > 2.$$

By shifting the path of integration to $\Re(s) = 1 + \varepsilon$, one finds that $1/2x^2 - H(x)$ equals the sum of the residues. So, it is reasonable to say that the properties of $H(x)$ are closely related to the behavior of $\phi_2(s)$.

Let $\mathcal{I}$ denote the set of imaginary parts of nontrivial zeros of $\zeta(s)$. It is a well-known conjecture that the elements of $\mathcal{I}$ are linearly independent over the rationals. The following is a special case of this conjecture.

(**Condition A**). *If $\gamma_j \in \mathcal{I}$ $(1 \leq j \leq 4)$, and $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ $(\neq 0)$, then (γ_3, γ_4) equals (γ_1, γ_2) or (γ_2, γ_1).*

Fujii [**7**] studied additive properties of the zeros of $\zeta(s)$, and proved that the set

$$\{\gamma_1 + \gamma_2 : \gamma_1, \gamma_2 \in \mathcal{I}, \gamma_1 > 0, \gamma_2 > 0\}$$

is uniformly distributed modulo 1. In [**4**], the following hypothesis is introduced.

(**Condition B**). *There exists a constant α, with $0 < \alpha < \pi/2$, such that if $\gamma_j \in \mathcal{I}$ $(1 \leq j \leq 4)$, $\gamma_1 + \gamma_2 \neq 0$, and (γ_3, γ_4) is neither equal to (γ_1, γ_2) nor to (γ_2, γ_1), then*

$$|(\gamma_1 + \gamma_2) - (\gamma_3 + \gamma_4)| \geq \exp\left(-\alpha\left(|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|\right)\right).$$

It is proved in [**4**] that, under RH, and (B), $\Re(s) = 1$ is the natural boundary of $\phi_2(s)$. A number of authors including Kluyver [**10**], Landau [**12**] and Landau and Walfisz [**13**] investigated the analytic continuation and the natural boundary of the series $\sum_p p^{-s}$, $s \in \mathbf{C}$, where p runs over the primes. Estermann [**5**] gave a criterion for a certain class of Dirichlet series which have Euler products to indicate when the series can be continued to the whole plane, and when it has a natural boundary. The natural boundaries of Euler products in more general situations were studied by Dahlquist [**3**] and Kurokawa [**11**]. Bhowmik, Essouabri and Lichtin [**2**] recently discussed a multi-variable generalization of their works.

In [**18**], the authors considered the function $F_{q,b,H,\alpha}(s)$ defined by

$$F_{q,b,H,\alpha}(s) = \sum_{\substack{m,k\geq 1\\ mk\equiv b \pmod q}} H(\alpha \log(m+k)) \frac{\Lambda(m)\Lambda(k)}{(m+k)^s},$$

for any positive integer q any integer b relatively prime to q, any smooth periodic function H with period 1, and any real number $\alpha > 0$. They proved the following result:

Let b and $q > 0$ be integers with $(b,q) = 1$. Let $H \in C^2(\mathbf{R})$ be periodic with period 1, and $\alpha > 0$. Assuming GRH*, $F_{q,b,H,\alpha}(s)$ has an analytic continuation to the half plane $\Re(s) > 3/2$, except for simple poles at $s = 2 + 2\pi i\alpha n$.*

In this paper, we study the natural boundary of $F_{q,b,H,\alpha}(s)$ for various choices of q, b, H, and α.

2. Natural boundary. Assume that H has period 1, $H \in C^3(\mathbf{R})$, and

$$H(t) = \sum_{n\in\mathbf{Z}} c_n e(nt), \qquad c_n = \int_0^1 H(t)e(-nt)\, dt.$$

Here, we have that $c_n \ll_H 1/n^3$. In this section our first objective is to show that $\Re s = 3/2$ is the natural boundary of

$$F_{H,\alpha}(s) = \sum_{m_1\geq 1}\sum_{m_2\geq 1} H(\alpha\log(m_1+m_2)) \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s}$$

for most values of $\alpha > 0$. In [**18**], it is shown that $F_{H,\alpha}(s)$ is meromorphic for $\Re(s) > 3/2$ and equals $\sum_{n\in\mathbf{Z}} c_n\phi_2(s - 2\pi i\alpha n)$. $F_{H,\alpha}(s)$ has a singularity at each point of the form $3/2 + i(\gamma + 2\pi\alpha n)$ where $\gamma \in I$, i.e., $\zeta((1/2) + i\gamma) = 0$, and $n \in \mathbf{Z}$.

For $\alpha > 0$ and $A \subseteq \mathbf{Z}$, let $D_{A,\alpha}$ denote the set

$$D_{A,\alpha} = \left\{\gamma + 2\pi\alpha n : n \in A, \zeta\left(\frac{1}{2} + i\gamma\right) = 0\right\}.$$

Theorem 1. *Assume Riemann hypothesis. Let $\alpha > 0$ be a real number such that there exist only finitely many tuples (n, n', γ, γ') with $\gamma \neq \gamma'$ for which $\gamma + 2\pi\alpha n = \gamma' + 2\pi\alpha n'$. Let $H \in C^3(\mathbf{R})$ be periodic with period 1. Also assume that infinitely many of the Fourier coefficients of H are nonzero. Then the line $\Re s = 3/2$ is the natural boundary of*

$$\begin{aligned} F_{H,\alpha}(s) &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} H(\alpha \log(m_1 + m_2)) \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1 + m_2)^s} \\ &= \sum_{m \in \mathbf{Z}} c_m \phi_2(s - 2\pi\alpha i m). \end{aligned}$$

Remark 1. If E denotes the set of all $\alpha > 0$ which does not satisfy the hypothesis of the theorem above, then E is countable. Also, observe that, if $\alpha > 0$ satisfies the hypothesis, then there exists an $N \in \mathbf{N}$ such that, for all $n > N$ and all $m \in \mathbf{Z}$, $1/2 + i(\kappa - 2\pi\alpha m)$, $\kappa = \gamma + 2\pi\alpha n$, is not a zero of the Riemann zeta function $\zeta(s)$. We fix the least such N and denote it by N_α.

Remark 2. If $\alpha > 0$, and $A \subseteq \mathbf{Z}$ is an infinite set, then the set $D_{A,\alpha}$ is dense. This follows immediately from the classical result of Littlewood which states that the distance between consecutive imaginary parts of zeros of $\zeta(s)$ tends to 0.

2.1. Proof of Theorem 1. Assume that $\alpha > 0$ satisfies the hypothesis of Theorem 1. $F_{H,\alpha}(s)$ is holomorphic in $\Re s > 3/2$, except for simple poles at $s = 2 + 2\pi i\alpha n$ for each $n \in \mathbf{Z}$. Also, for each $n \in \mathbf{Z}$ and $\gamma \in I$, $F_{H,\alpha}(s)$ has a singularity at $(3/2) + i(\gamma + 2\pi\alpha n)$ (see [**18**]).

We have proved earlier that the set

$$D_\alpha = \{\gamma + 2\pi\alpha n : n \in \mathbf{Z}, \gamma \in I\}.$$

is dense in $\mathbf{R}$.

By the definition of α, there exists an N_α such that, for all $n > N_\alpha$, for all $\gamma \in I$ and for all integers m, we have that $(1/2_+ i(\kappa - 2\pi\alpha m)$ with $\kappa = \gamma + 2\pi\alpha n$, is not a zero of $\zeta(s)$. Fix such an n so that $c_n \neq 0$. Recall that $\phi_2(s)$ has a simple pole at $s = (3/2) + i\gamma$ with residue $-2n(\rho)/\rho$. So, $\phi_2(s - 2\pi\alpha i n)$ has a simple pole at $s = (3/2) + i\kappa$. Fix $0 < \eta < 1$. Let

$$\phi_2(z - 2\pi\alpha i n) = \frac{a_{-1}}{z - ((3/2) + i\kappa)} + \sum_{k=0}^{\infty} a_k \left(z - \left(\frac{3}{2} + i\kappa\right)\right)^k$$

be the Laurent series expansion of $\phi_2(z - 2\pi\alpha i n)$ at $z = (3/2) + i\kappa$. Then,

$$\phi_2\left(\eta + \frac{3}{2} + i\kappa\right) = \frac{a_{-1}}{\eta} + \sum_{k=0}^{\infty} a_k \eta^k. \tag{2.1}$$

Define

$$h_n(s) = \sum_{\substack{m \in \mathbf{Z} \\ m \neq n}} c_m \phi_2(s - 2\pi\alpha i m).$$

We want to investigate the behavior of $F_{H,\alpha}(s)$ as s tends to $(3/2) + i\kappa$ horizontally. Put $s = \eta + (3/2) + i\kappa$ into $h_n(s)$. We get

$$h_n\left(\eta + \frac{3}{2} + i\kappa\right) = \sum_{\substack{m \in \mathbf{Z} \\ m \neq n}} c_m \phi_2\left(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m)\right).$$

By (3.9) in [**4**], we have that, for $\delta > 0$, and $\Re s > 1$,

$$\begin{aligned}\phi_2(s) = {} & \frac{M(s-1)}{s-1} - \log(2\pi) M(s) \\ & + \frac{1}{2\pi i} \int_{(-\delta)} \frac{\Gamma(s-z)\Gamma(z)M(z)M(s-z)}{\Gamma(s)}\, dz \\ & - \sum_{\rho \in I} \frac{\Gamma(s-\rho)\Gamma(\rho)M(s-\rho)}{\Gamma(s)}.\end{aligned} \tag{2.2}$$

Plugging in $s = \eta + (3/2) + i(\kappa - 2\pi\alpha m)$ in equation (2.2), we get

$$\begin{aligned}(2.3)\quad &\phi_2\left(\eta + \frac{3}{2} + i\,(\kappa - 2\pi\alpha m)\right)\\ &= \frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)} - \log(2\pi) M\left(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m)\right)\\ &\quad + \frac{1}{2\pi i}\int_{(-\delta)} \frac{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - z)}{1}\\ &\qquad\qquad \times \frac{\Gamma(z)M(z)M(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - z)}{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m))}\,dz\\ &\quad - \sum_{\rho\in I} \frac{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - \rho)}{1}\\ &\qquad\qquad \times \frac{\Gamma(\rho)M\left(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - \rho\right)}{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m))}.\end{aligned}$$

We now consider the first term in (2.3):

$$\frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)}.$$

We also recall that, uniformly for $s = \sigma + it$, $-1 \le \sigma \le 2$,

$$(2.4)\qquad M(s) = -\frac{\zeta'(s)}{\zeta(s)} = -\sum_{\rho} \frac{1}{s-\rho} + O(\log|t| + 2),$$

where the sum is limited to those ρ for which $|\gamma - t| < 1$ ([**17**, Theorem 9.6 (A)]). Now consider the sum

$$\sum_{\substack{m\in\mathbf{Z}\\ m\neq n}} c_m \frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)}.$$

Fix a positive integer L, and divide this sum into two sums according to $|m| \le L$ and $|m| > L$.

First consider the sum

$$\sum_{\substack{m\in\mathbf{Z}\\ m\neq n\\ |m|>L}} c_m \frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)}.$$

Note that, if we put $s = (1/2) + \eta + i(\kappa - 2\pi\alpha m)$ into (2.4), then we get

$$\begin{aligned} M\left(\frac{1}{2} + \eta + i(\kappa - 2\pi\alpha m)\right) &= -\frac{\zeta'((1/2) + \eta + i(\kappa - 2\pi\alpha m))}{\zeta((1/2) + \eta + i(\kappa - 2\pi\alpha m))} \\ &= -\sum_{\rho} \frac{1}{(1/2) + \eta + i(\kappa - 2\pi\alpha m) - \rho} \\ &\quad + O\bigg(\log(|\kappa - 2\pi\alpha m| + 2)\bigg) \\ &= O\left(\frac{1}{\eta}\log(|\kappa - 2\pi\alpha m| + 2)\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{\substack{m\in\mathbf{Z}\\ m\neq n\\ |m|>L}} c_m \frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)} \\ &\qquad = O\left(\sum_{\substack{m\in\mathbf{Z}\\ m\neq n\\ |m|>L}} |c_m| \, \frac{(1/\eta)\log(|\kappa - 2\pi\alpha m| + 2)}{|\kappa - 2\pi\alpha m| + 2}\right) \\ &\qquad = O\left(\frac{1}{\eta} \sum_{\substack{m\in\mathbf{Z}\\ |m|>L}} \frac{1}{m^2} \, \frac{\log(|\kappa - 2\pi\alpha m| + 2)}{|\kappa - 2\pi\alpha m| + 2}\right) \\ &\qquad = O_{\gamma,\alpha,n}\left(\frac{1}{\eta} \sum_{\substack{m\in\mathbf{Z}\\ |m|>L}} \frac{1}{m^2}\right) \\ &\qquad = O_{\gamma,\alpha,n}\left(\frac{1}{\eta L}\right). \end{aligned}$$

Next, consider the sum

$$\sum_{\substack{m\in\mathbf{Z}\\ m\neq n\\ |m|<L}} c_m \frac{M(\eta + (1/2) + i(\kappa - 2\pi\alpha m))}{\eta + (1/2) + i(\kappa - 2\pi\alpha m)}.$$

Note that α is such that $(1/2)+i(\kappa-2\pi\alpha m)$ is not a zero of $\zeta(s)$ for any $|m|<L$. For $z=\eta+(1/2)+i(\kappa-2\pi\alpha m)$, and for any zero ρ of $\zeta(s)$,

$$\frac{1}{|z-\rho|}\leq\frac{1}{|(1/2)+i(\kappa-2\pi\alpha m)-\rho|}.$$

Thus, we find that

$$\sum_{\substack{m\in\mathbf{Z}\\ m\neq n\\ |m|\leq L}} c_m\frac{M(\eta+(1/2)+i(\kappa-2\pi\alpha m))}{\eta+(1/2)+i(\kappa-2\pi\alpha m)}=O_{\gamma,\alpha,n,L}(1).$$

Putting this all together, we get

$$\sum_{\substack{m\in\mathbf{Z}\\ m\neq n}} c_m\frac{M(\eta+(1/2)+i(\kappa-2\pi\alpha m))}{\eta+(1/2)+i(\kappa-2\pi\alpha m)}=O_{\gamma,\alpha,n,L}(1)+O\left(\frac{1}{\eta L}\right).$$

Let $\varepsilon>0$ be given. Choose $L=\lfloor 1/\varepsilon\rfloor$. Then, we have that the sum above is

$$O\left(\frac{\varepsilon}{\eta}\right)+O_{\varepsilon,\gamma,\alpha,n}(1).$$

Thus, with α, γ and n fixed,

$$\left|\sum_{\substack{m\in\mathbf{Z}\\ m\neq n}} c_m\frac{M(\eta+(1/2)+i(\kappa-2\pi\alpha m))}{\eta+(1/2)+i(\kappa-2\pi\alpha m)}\right|\leq C(\varepsilon)+\frac{\varepsilon\beta}{\eta},$$

where $C(\varepsilon)$ is a constant independent of η and β is a constant independent of both ε and η.

We remark here that a similar argument was applied to get rid of Condition (B) of [**4**], see Bhowmik [**1**].

In what follows, we will make use of the following well-known estimate on $\Gamma(s)$.

$$|s|^{\sigma-(1/2)}e^{-\pi/2|t|}e^{-\sigma/3}\ll\Gamma(s)\ll|s|^{\sigma-(1/2)}e^{-\pi/2|t|},$$
$$\text{for } s=\sigma+it \text{ with } 0<\sigma<|t|.$$

Here the implied constants are absolute and independent of σ and t. This implies that

$$\sup_{\substack{t\in\mathbf{R}\\ m\in\mathbf{Z}}} \Gamma\left(\eta+\frac{3}{2}+\delta+i(\kappa-2\pi\alpha m-t)\right) = \sup_{u\in\mathbf{R}} \Gamma\left(\eta+\frac{3}{2}+\delta+iu\right) = O\left(1\right).$$

Also, $\sup_{t\in\mathbf{R}} \Gamma(-\delta+it) = O(1)$, since $\Gamma(-\delta+it) = 1/(-\delta+it)\Gamma(1-\delta+it)$.

Next, consider the integral in (2.3) which we denote by $Z_m(s)$:

$$\frac{1}{2\pi i}\int_{(-\delta)} \frac{\Gamma(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-z)}{1} \frac{\Gamma(z)M(z)M(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-z)}{\Gamma(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m))}\,dz$$

Note that $M(-\delta+it) = O(\log|t|+2)$, and $M(\eta+(3/2)+i(\kappa-2\pi\alpha m)-z) = O(1)$. So the integral is

$$\ll \int_{-\infty}^{\infty} \frac{\log(|t|+2)|\Gamma(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)+\delta-it)\Gamma(-\delta+it)|}{|\Gamma(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m))|}\,dt.$$

For each $m\in\mathbf{Z}$, let us denote $L_m = [\gamma+2\pi\alpha(n-m)]/2 = [\kappa-2\pi\alpha m]/2$. Let m_0 be the largest integer such that $L_{m_0} > 0$. Then $L_{m_0+k} \le 0$ for all $k\in\mathbf{Z}$. Also, for all $m\in\mathbf{Z}$, $|L_m - L_{m+1}| = \pi\alpha$. So, $|L_{m_0+k}| > \pi\alpha k$ for all $k\in\mathbf{Z}$. By applying Stirling's formula, the integral is

$$\begin{aligned}
&\ll \int_{-\infty}^{\infty} \frac{\log(|t|+2)|\Gamma(\eta+(3/2)+\delta+i(2L_m-t))||\Gamma(-\delta+it)|}{|\Gamma(\eta+\frac{3}{2}+i2L_m)|}\,dt\\
&\ll \exp(\pi l_m)(2l_m)^{-\eta-1}\\
&\qquad\times\int_{-\infty}^{\infty}\exp\left(\frac{-\pi}{2}(|t|+|2L_m-t|\right)\log(|t|+2)|\\
&\qquad\qquad(|t|+1)^{-\delta-(1/2)}(|t-2L_m|+1)^{1+\eta+\delta}dt,
\end{aligned}$$

where $l_m = |L_m|$.

Let

$$T_2(s) = \int_{-\infty}^{\infty} \exp\left(\frac{-\pi}{2}(|t| + |2L_m - t|\right) \times \log(|t|+2)|(|t|+1)^{-\delta-(1/2)}(|t-2L_m|+1)^{1+\eta+\delta}dt.$$

Assume that $m < m_0$. Then $L_m > 0$. We break up the above integral as $J_1 + J_2 + J_3$, where J_1 is the integral over the interval $[-\infty, 0]$, J_2 is the integral over the interval $[2l_m, \infty]$ and J_3 is the integral over the interval $[0, 2l_m]$.

Next we get an estimate of J_3.

$$\begin{aligned} J_3 &= \exp(-\pi l_m) \int_0^{2l_m} \log(t+2)|(t+1)^{-\delta-(1/2)}(2l_m - t + 1)^{1+\eta+\delta}dt \\ &= \exp(-\pi l_m) \int_0^{l_m} \log(t+2)|(t+1)^{-\delta-(1/2)}(2l_m - t + 1)^{1+\eta+\delta}dt \\ &\quad + \exp(-\pi l_m) \int_{l_m}^{2l_m} \log(t+2)|(|t|+1)^{-\delta-(1/2)}(2l_m - t+1)^{1+\eta+\delta}dt \\ &\ll \exp(-\pi l_m) \times \int_0^{l_m} \log(t+2)|(t+1)^{-\delta-(1/2)}(2l_m+1)^{1+\eta+\delta}dt \\ &\quad + \exp(-\pi l_m) \int_{l_m}^{2l_m} \log(l_m+2)|(l_m+1)^{-\delta-(1/2)}(2l_m - t+1)^{1+\eta+\delta}dt \\ &\ll \exp(-\pi l_m) l_m^{\eta+(3/2)} \log(l_m+2). \end{aligned}$$

Similarly, one can easily show that

$$J_1 \ll \exp(-\pi l_m)(l_m)^{\eta+1+\delta},$$

and, also

$$J_2 \ll \exp(-\pi l_m)(2l_m)^{-\delta-(1/2)} \log(2l_m).$$

Thus,

$$T_2(s) \ll \exp(-\pi l_m) l_m^{\eta+1+\delta}(\log l_m + 2).$$

Now assume that $m > m_0$. Then $L_m < 0$. By changing variables we get

$$T_2(s) = \int_{-\infty}^{\infty} \exp\left(\frac{-\pi}{2}(|t| + |2L_m + t|\right) \times \log(|t|+2)|(|t|+1)^{-\delta-(1/2)}(|-t-2L_m|+1)^{1+\eta+\delta}dt.$$

Since $L_m = -l_m$, we have that

$$T_2(s) = \int_{-\infty}^{\infty} \exp\left(\frac{-\pi}{2}(|t| + |-2l_m + t|\right) \times \log(|t|+2)|(|t|+1)^{-\delta-(1/2)}(|-t+2l_m|+1)^{1+\eta+\delta} dt$$
$$= \int_{-\infty}^{\infty} \exp\left(\frac{-\pi}{2}(|t| + |2l_m - t|\right) \times \log(|t|+2)|(|t|+1)^{-\delta-(1/2)}(|t-2l_m|+1)^{1+\eta+\delta} dt.$$

As before, we can show that

$$T_2(s) \ll \exp(-\pi l_m) l_m^{\eta+1+\delta}(\log l_m + 2).$$

Hence,

$$Z_m(s) \ll l_m^{\frac{1}{2}}(\log l_m + 2)^2.$$

Next, observe that the second term in (2.3) is $O(1)$. Lastly, we consider the sum

$$B_m(s) = \sum_{\rho \in I} \frac{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - \rho)}{1} \frac{\Gamma(\rho) M(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m) - \rho)}{\Gamma(\eta + \frac{3}{2} + i(\kappa - 2\pi\alpha m))}.$$

This sum is

$$\ll \frac{1}{\Gamma(\eta + (3/2) + i2L_m)} \times \sum_{\rho = \frac{1}{2} + i\gamma' \in I} \log(|2L_m - \gamma'| + 2)\Gamma\left(\eta + \frac{3}{2} + i(2L_m - \rho\right)\Gamma(\rho).$$

Note that, by Stirling's formula, we have

$$B_m(s) \ll \exp(\pi l_m)(2l_m)^{-\eta-1} \sum_{\rho = \frac{1}{2} + i\gamma' \in I} \exp\left(\frac{-\pi}{2}(|\gamma'| + |2l_m - \gamma'|\right) \times \log(|2l_m - \gamma'| + 2)(|2l_m - \gamma'| + 1)^{\eta+(1/2)}$$
$$= \exp(\pi l_m)(2l_m)^{-\eta-1}(C_1 + C_2 + C_3),$$

where

$$C_1 = \sum_{\substack{\rho=(1/2)+i\gamma'\in I \\ \gamma'<0}} \exp\left(\frac{-\pi}{2}(|\gamma'|+|2l_m-\gamma'|)\right) \times \log(|2l_m-\gamma'|+2)(|2l_m-\gamma'|+1)^{\eta+(1/2)},$$

$$C_2 = \sum_{\substack{\rho=(1/2)+i\gamma'\in I \\ \gamma'>2l_m}} \exp\left(\frac{-\pi}{2}(|\gamma'|+|2l_m-\gamma'|)\right) \times \log(|2l_m-\gamma'|+2)(|2l_m-\gamma'|+1)^{\eta+(1/2)},$$

and

$$C_3 = \sum_{\substack{\rho=(1/2)+i\gamma'\in I \\ 0<\gamma'\le 2l_m}} \exp\left(\frac{-\pi}{2}(|\gamma'|+|2l_m-\gamma'|)\right) \times \log(|2l_m-\gamma'|+2)(|2l_m-\gamma'|+1)^{\eta+(1/2)}.$$

Note that

$$C_1 = \exp(-\pi l_m) \sum_{\substack{\rho=(1/2)+i\gamma'\in I \\ \gamma'>0}} \exp(-\pi\gamma') \times \log(2l_m+\gamma'+2)(2l_m+\gamma'+1)^{\eta+(1/2)}.$$

Divide C_1 into two sums according to $0<\gamma'\le 2l_m$ and $\gamma'>2l_m$. Applying partial summation to the sum over $0<\gamma'\le 2l_m$, we have that it is $O((2l_m)^{\eta+(1/2)})\log 2l_m$. Also, the sum over $\gamma'>2l_m$ is of exponential decay uniformly in m, and one can show that

$$C_1 \ll \exp(-\pi l_m)((2l_m)^{\eta+(1/2)})\log 2l_m.$$

By partial summation, we have that

$$\begin{aligned} C_2 &= \exp(-\pi l_m) \sum_{\substack{\rho=(1/2)+i\gamma'\in I \\ \gamma'>2l_m}} \exp(-\pi\gamma') \times \log(\gamma'-2l_m+2)(\gamma'-2l_m+1)^{\eta+(1/2)} \\ &\ll \exp(-\pi l_m)\int_{2l_m}^{\infty} \exp(-\pi t) \times t\log t \log(t-2l_m+2)(t-2l_m+1)^{\eta+(1/2)} \\ &\ll \exp(-\pi l_m)((2l_m)^{\eta+(3/2)})(\log 2l_m)^2. \end{aligned}$$

We can similarly show that

$$C_3 \ll \exp(-\pi l_m)((2l_m)^{\eta+(3/2)})(\log 2l_m)^2.$$

We conclude that

$$B_m(s) \ll l_m^{1/2}(\log l_m + 2)^2 \ll_{\alpha,\gamma,\eta} m.$$

Since $H \in C^3(\mathbf{R})$, it follows that $\sum_{m\in\mathbf{Z}, m\neq n} |c_m B_m(s)|$ is convergent. We deduce that

$$h_n(\eta + \frac{3}{2} + i\kappa) \ll C(\varepsilon) + \frac{\varepsilon\beta}{\eta} + D,$$

where $C(\varepsilon)$ is a constant independent of η, β is a constant independent of both ε and η, and D is independent of η. Since ε can be chosen arbitrarily, it is now clear that $F_{H,\alpha}(\eta + (3/2) + i\kappa) \to \infty$ as $\eta \to 0^+$. Hence, $(3/2) + i\kappa$ is indeed a singularity of $F_{H,\alpha}(s)$. This completes the proof of Theorem 1.

Theorem 2. *Assume the Riemann hypothesis. Let $\alpha > 0$ be a real number such that there exist only finitely many tuples (n, n', γ, γ') with $\gamma \neq \gamma'$ for which $\gamma + 2\pi\alpha n = \gamma' + 2\pi\alpha n'$. Let $H \in C^3(\mathbf{R})$ be periodic with period 1. Also assume that infinitely many of the Fourier coefficients of H are nonzero. Then the line $\Re s = 3/2$ is the natural boundary of*

$$F_{q,b,H,\alpha}(s) = \sum_{\substack{m,k\geq 1 \\ mk\equiv b \pmod q}} H(\alpha\log(m+k))\frac{\Lambda(m)\Lambda(k)}{(m+k)^s},$$

Proof of Theorem 2. For a Dirichlet character χ modulo the prime q, let

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s} = \sum_{n=1}^{\infty}\frac{G_2(n,\chi)}{n^s},$$

where

$$G_2(n,\chi) = \sum_{k+m=n}\Lambda(k)\Lambda(m)\chi(k)\chi(m).$$

Let us denote

$$\begin{aligned} F_{\chi,H,\alpha}(s) &= \sum_{m,k\ge 1} H(\alpha\log(m+k))\frac{\Lambda(m)\Lambda(k)\chi(m)\chi(k)}{(m+k)^s} \\ &= \sum_{n\in\mathbf{Z}} c_n\phi_2(s-2\pi i\alpha n,\chi). \end{aligned}$$

It is shown in [**18**] that, if χ is a nonprincipal character modulo q, then $F_{\chi,H,\alpha}(s)$ is holomorphic in $\Re s > 1$. Also, if χ_0 is the principal character modulo q, then $F_{\chi_0,H,\alpha}(s) - F_{H,\alpha}(s)$ is holomorphic in $\Re s > 1$. Now note that

$$\begin{aligned} F_{q,b,H,\alpha}(s) &= \sum_{\substack{m,k\ge 1\\ mk\equiv b \pmod q}} \frac{\Lambda(m)\Lambda(k)}{(m+k)^s} H(\alpha\log(m+k)) \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(b) F_{\chi,H,\alpha}(s). \end{aligned}$$

Theorem 2 now follows from Theorem 1.

Acknowledgments. The authors are grateful to the referee for many useful comments and suggestions.

REFERENCES

1. G. Bhowmik, *Analytic continuation of some zeta functions. Algebraic and analytic aspects of zeta functions and L-functions*, Math. Sci. J. Mem. **21**, Math. Soc. Tokyo, Japan, 2010.

2. G. Bhowmik, D. Essouabri and B. Lichtin, *Meromorphic continuation of multivariable Euler products*, Forum Math. **19** (2007), 1111–1139.

3. G. Dahlquist, *On the analytic continuation of Eulerian products*, Ark. Mat. **1** (1952), 533–555.

4. S. Egami and K. Matsumoto, *Convolutions of the von Mangoldt function and related Dirichlet series*, Number Theory Appl. **2**, World Sci. Publ., Hackensack, NJ, 2007.

5. T. Estermann, *On certain functions represented by Dirichlet series*, Proc. London Math. Soc. **27** (1928), 435–448.

6. A. Fujii, *An additive problem of prime numbers.* II, Proc. Japan Acad. Math. Sci. **67** (1991), 248–252.

7. ———, *An additive theory of the zeros of the Riemann zeta function*, Comment. Math. Univ. St. Paul. **45** (1996), 49–116.

8. A. Fujii, *Some problems of Diophantine approximation in the theory of the Riemann zeta function* (III), Comment. Math. Univ. St. Paul. **42** (1993), 161–187.

9. G.H. Hardy and J.E. Littlewood, *Some problems of "partitio numerorum* (V): *A further contribution to the study of Goldbach's problem*, Proc. Lond. Math. Soc. **22** (1924), 46–56.

10. J.C. Kluyver, *Benaderingsformules betreffende de priemgetallen beneden eene gegeven grens*, Konin. Akad. Wet. Amsterdam, Versl. Gew. Verg. Wis. Natuur. Afd. **8** (1900), 672–682.

11. N. Kurokawa, *On the meromorphy of Euler products* (I), Proc. Lond. Math. Soc. **53** (1986), 1–47; (II), ibid., 209–236.

12. E. Landau, *Üder die Multiplikation Dirichlet'scher Reihen*, Rend. Circ. Mat. Palermo **24** (1907), 81–160.

13. E. Landau and A. Walfisz, *Üder die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen*, ibid. **44** (1920), 82–86.

14. K. Matsumoto and Y. Tanigawa, *The analytic continuation and the order estimate of multiple Dirichlet series*, J. Théorie des Nombres de Bordeaux **15** (2003), 267–274.

15. H.L. Montgomery and R.C. Vaughan, *Error terms in additive prime number theory*, Quart. J. Math. Oxford **24** (1973), 207–216.

16. H. Rademacher, *Fourier analysis in number theory*, Symposium on harmonic analysis and related integral transforms: Final technical report, Cornell Univ., Ithica, NY (1956), 25 pages; also in: II. *Rademacher, Collected works*, 434–458.

17. E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press, New York, 1951.

18. A. Zaharescu and M. Zaki, *Analytic continuation of a family of Dirichlet series*, Comment. Math. Univ. St. Paul. **58** (2009), 105–118.

19. J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Society **282** (1999), 1275–1283.

Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, Bucharest 014700, Romania and Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL, 61801
Email address: zaharesc@math.uiuc.edu

Department of Mathematics, Ohio Northern University, 525 S Main Street, Ada, OH, 45810
Email address: m-zaki@onu.edu