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O-MINIMAL HOMOTOPY AND GENERALIZED (CO)HOMOLOGY

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ABSTRACT. This article explains and extends semialgebraic homotopy theory (developed by Delfs and Knebusch) to o-minimal homotopy theory (over a field). The homotopy category of definable CW-complexes is equivalent to the homotopy category of topological CW-complexes (with continuous mappings). If the theory of the o-minimal expansion of a field is bounded, then these categories are equivalent to the homotopy category of weakly definable spaces. Similar facts hold for decreasing systems of spaces. As a result, generalized homology and cohomology theories on pointed weak polytopes uniquely correspond (up to an isomorphism) to the known topological generalized homology and cohomology theories on pointed CW-complexes.

1. Introduction. In the 1980's, Delfs, Knebusch and others developed "semialgebraic topology" in locally semialgebraic and weakly semialgebraic spaces (see [**7–10, 20**]). In the survey paper [**21**], Knebusch suggested that this theory may be generalized to the o-minimal context. This program was partially undertaken first by Woerheide, who constructed the o-minimal singular homology theory in [**31**], and later by Edmundo, who developed and applied the singular homology and cohomology theories over o-minimal structures (see for example [**13**]). For homotopy theory, Berarducci and Otero worked with the o-minimal fundamental group and transfer methods in o-minimal geometry ([**5, 6**]). During the period this paper was written, several authors wrote about different types of homology and cohomology (see [**14, 15**], for example).

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Still the semialgebraic homotopy theory contained in [**10, 20**] was not extended to the case of spaces over o-minimal expansions of fields. For the question why, the author may only guess that people in the field wanted to avoid generalized topology. (Notice the failure of Baro and Otero [**4**] to give precise definitions and to present the theory clearly, see below.)

The aim of extending a whole theory, not a single theorem or even tens or hundreds of facts, may be sometimes achieved by careful choice of the definitions and explaining the differences that appear. This can be done in the case of the semialgebraic homotopy theory of Delfs and Knebusch.

First, the spaces of our interest (with their morphisms) over each of the considered structures form several categories that are best described as full subcategories of some ambient category. The choice of a good ambient category is very important. In [**10**] the task was done using sheaf theory, but Knebusch in [**20**] has already simplified the definitions by using what is called "function sheaves" (involving a simple set-theoretic definition). Notice that the usual sheaf theory is not necessary to understand locally semialgebraic spaces. Thus, the extension of the theory should be done through extension of the basic definitions from [**20**]. Another argument for this is the fact that locally definable spaces do not suffice; we need to speak about weakly definable spaces to get a satisfactory homotopy theory.

Second, some proofs of [**10**] need modification. The mapping spaces from III.3 are specific for the semialgebraic case. This is modified in the present paper. Moreover, Lemma II.4.3 from [**10**] (and related facts) need to be modified since one needs to add the Third comparison theorem (the o-minimal *expansion case*). This was done in [**3**] by the use of "normal triangulations" from [**2**] (the problem appears on the definable sets level).

And, third, we need to distinguish between theories that are *bounded* (definition in the present paper) and others that are not bounded. The theory RCF itself is bounded, and some proofs of [**20**] (related to IV.9-10) do not work in the general setting of an o-minimal expansion of a (real closed) field. The question arises if the corresponding facts are true.

After considering these remarks, one can see that the two volumes [**10, 20**] are a source of thousands of facts and their proofs about locally definable spaces and weakly definable spaces. It is usually done just by changing the word "semialgebraic" into the word "definable." The intention of the author of the present paper is not to re-write about 600 pages with this simple change, but to give enough understanding of the theory to the reader. Some examples and facts from [**10, 20**] are restated to make this understanding easy. (The above remarks apply to so-called "geometric" theory. The so-called "abstract" theory, contained in Appendix A of [**10**], is not considered in the present paper.)

It is convenient to understand that the semialgebraic homotopy theory of Delfs and Knebusch is basically the usual homotopy theory re-done in the presence of the generalized topology. The constructions of homotopy theory may be carried out in the semialgebraic context. Thus, it is not surprising that these constructions may also be done in the context of o-minimal expansions of fields. The use of the generalized topology may be extended far beyond the above context (see [**25, 26**] for details).

The author considers the main result of this paper to be the following: the semialgebraic homotopy theory of Delfs and Knebusch is now explained and extended to the o-minimal homotopy theory (over a field). The extension part includes the Comparison theorems (especially Theorems 36 and 51), a definable version of the Whitehead theorem (Theorem 55) and equivalence of the homotopy categories (Corollaries 38, 57, 60, 61). A majority of the examples and Theorems 9 and 10 contribute to the explanation part. Of independent interest are: a characterization of real analytic manifolds as locally definable manifolds (Theorem 21) and the definable version of a Bertini or a Lefschetz theorem (Theorem 28), see [**1**].

As a result of the homotopy approach, deeper than the homology and cohomology ones, we get the generalized homology and cohomology theories (including the standard singular theories) for so-called pointed weak polytopes, and these theories appear, if T is bounded, to be "the same" as their topological counterparts.

The categories of locally and weakly definable spaces over o-minimal expansions of real closed fields, introduced here, with their subspaces (locally definable subsets and weakly definable subsets) are far generalizations of *analytic-geometric categories* of van den Dries and Miller ([**12**]). In particular, paracompact locally definable manifolds are generalizations of both definable manifolds over o-minimal expansions of fields and real analytic manifolds.

For basic properties of o-minimal structures, see the book [**11**] and the survey paper [**12**]. *Assume that R is an* o*-minimal expansion of a real closed field.*

2. Spaces over o-minimal structures. As o-minimal structures have natural topology, it is quite natural that algebraic topology for such structures should be developed. (This paper deals only with the case of o-minimal expansions of fields.) Unfortunately, there are obstacles to the above when one is doing traditional topology: if R is not (an expansion of) the (ordered) field of real numbers $\mathbf{R}$, then R is not locally compact and is totally disconnected. Moreover, even for $\mathbf{R}$, not every family of open definable sets has a definable union, and continuous definable functions do not form a sheaf.

A good idea to overcome that in the case of o-minimal pure (ordered) fields was given by Delfs and Knebusch in [**10**]: it is the concept of a generalized topological space. This idea serves well also in our setting.

A *generalized topological space* is a set M together with a family of subsets $\mathring{\mathcal{T}}(M)$ of M, called *open sets*, and a family of open families Cov_M, called *admissible* (*open coverings*), such that:

(A1) $\varnothing, M \in \mathring{\mathcal{T}}(M)$ (the empty set and the whole space are open);

(A2) if $U_1, U_2 \in \mathring{\mathcal{T}}(M)$, then $U_1 \cup U_2, U_1 \cap U_2 \in \mathring{\mathcal{T}}(M)$ (finite unions and finite intersections of open sets are open);

(A3) if $\{U_i\}_{i\in I} \subseteq \mathring{\mathcal{T}}(M)$ and I is finite, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ (finite families of open sets are admissible);

(A4) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ then $\cup_{i\in I} U_i \in \mathring{\mathcal{T}}(M)$ (the union of an admissible family is open);

(A5) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$, $V \subseteq \cup_{i\in I} U_i$, and $V \in \mathring{\mathcal{T}}(M)$, then $\{V \cap U_i\}_{i\in I} \in \mathrm{Cov}_M$ (the traces of an admissible family on an open subset of the union of the family form an admissible family);

(A6) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ and for each $i \in I$, there is $\{V_{ij}\}_{j\in J_i} \in \mathrm{Cov}_M$ such that $\cup_{j\in J_i} V_{ij} = U_i$, then $\{V_{ij}\}_{i\in I, j\in J_i} \in \mathrm{Cov}_M$ (members of all admissible coverings of members of an admissible family form together an admissible family);

(A7) if $\{U_i\}_{i\in I} \subseteq \overset{\circ}{\mathcal{T}}(M)$, $\{V_j\}_{j\in J} \in \mathrm{Cov}_M$, $\cup_{j\in J} V_j = \cup_{i\in I} U_i$, and for all $j \in J$ there exists $i \in I : V_j \subseteq U_i$, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$ (a coarsening, with the same union, of an admissible family is admissible);

(A8) if $\{U_i\}_{i\in I} \in \mathrm{Cov}_M$, $V \subseteq \cup_{i\in I} U_i$ and $V \cap U_i \in \overset{\circ}{\mathcal{T}}(M)$ for each i, then $V \in \overset{\circ}{\mathcal{T}}(M)$ (if a subset of the union of an admissible family has open traces with members of the family, then the subset is open).

Generalized topological spaces may be identified with certain Grothendieck sites, where the underlying category is a full, closed on finite (in particular: empty) products and coproducts subcategory of the category of subsets $\mathcal{P}(M)$ of a given set M with inclusions as morphisms, and the Grothendieck topology is subcanonical, contains all finite jointly surjective families and satisfies some regularity condition. (See [**22**] for the definition of a Grothendieck site. Considering such an identification we should remember the ambient category $\mathcal{P}(M)$.) More precisely, axioms (A1), (A2) and (A3) contain a stronger version of the identity axiom of the Grothendieck topology. It is natural, since in model theory and in geometry we love finite unions, finite intersections and finite coverings. Axiom (A4) may be called *co-subcanonicality.* Together with subcanonicality, it ensures that admissible coverings are coverings in the traditional sense. (Subcanonicality is imposed by the notation of [**10**]. Axiom (A4), weaker than (A8), justifies the notation $\mathrm{Cov}_M(U)$ of [**10**].) The next are: (A5) the stability axiom of the Grothendieck topology, followed by the transitivity axiom (A6). Finally, (A7) is the saturation property of the Grothendieck topology (usually the Grothendieck topology of a site is required to be saturated), and the last axiom (A8) may be called the *regularity axiom.* Both saturation and regularity have a smoothing character. Saturation may be achieved by modifying any generalized topological space, and regularity by modifying a locally definable space (see [**10**, I.1, pages 3, 9]). The reader should be warned that (in general) the closure operator does not exist for generalized topology.

A *strictly continuous mapping* between generalized topological spaces is such a mapping that the preimage of an (open) admissible covering is admissible, which implies that the preimage of an open set is open. (So strictly continuous mappings may be seen as morphisms of sites.) Inductive limits exist in the category GTS of generalized topological spaces and their strictly continuous mappings (see [**10**, I.2]).

Generalized topological spaces help to introduce further notions of interest that are generalizations of corresponding semialgebraic notions (we follow here [**20**]).

A *function sheaf of rings over* R on a generalized topological space M is a sheaf F of rings on M (here the sheaf property is assumed only for admissible coverings) such that for each U open in M the ring $F(U)$ is a subring of the ring of all functions from U into R, and the restrictions of the sheaf are the set-theoretical restrictions of mappings. A *function ringed space over* R is a pair (M, O_M), where M is a generalized topological space and O_M is a function sheaf of rings over R. We will say about *spaces* (over R) for short. An *open subspace* of a space over R is an open subset of its generalized topological space together with the function sheaf of the space restricted to this open set. A *morphism* $f : (M, O_M) \to (N, O_N)$ of function ringed spaces over R is a strictly continuous mapping $f : M \to N$ such that for each open subset V of N the set-theoretical substitution $h \mapsto h \circ f$ gives a morphism of rings $f_V^{\#} : O_N(V) \to O_M(f^{-1}(V))$. (We could express this by saying that $f^{\#} : O_N \to f_* O_M$ is the morphism of sheaves of rings on N over R induced by f. However, if we define for function sheaves

$$(f_* O_M)(V) = \{h : V \to R \mid h \circ f \in O_M(f^{-1}(V))\},$$

then each $f_V^{\#} : O_N(V) \to f_* O_M(V)$ becomes just an inclusion.) Inductive limits exist in the category Space(R) of spaces over any R and their morphisms (cf. [**10**, I.2] and [**25**]). Notice that our category of spaces over R, being a generalization (by passing from the semialgebraic to the general o-minimal case) of the category of spaces from [**20**], does not use the general sheaf theory for generalized topological spaces (as does [**10**]), but only a bit of a simpler "function sheaf theory."

The following basic example is a very special case of a space over R (cf. [**11**]).

Fundamental Example 1. Each definable subset D of R^n has a natural structure of a function ringed space over R. Its open sets in the sense of the generalized topology are (relatively) open definable subsets, admissible coverings are such open coverings that already finitely many open sets cover the union, and on each open definable subset $O \subseteq D$ we take the ring $\mathcal{DC}_D(O)$ of all continuous definable R-valued functions on O. *Definable sets will be identified with such function ringed spaces.* Notice that the topological closure of a definable set is definable, so the topological closure operator restricted to the class of definable subsets of a definable set D can be treated as the closure operator in the generalized topological sense.

We start to reintroduce the theory of locally definable spaces by generalizing the definitions from [**10**].

An *affine definable space* over R is a space over R isomorphic to a definable subset of some R^n. (Notice that morphisms of affine definable spaces are given by continuous definable maps between definable subsets of affine spaces.)

The following example, not explicitly studied before, shows that it is important to consider affine definable spaces as definable sets "embedded" into their ambient affine spaces.

Example 2 ("Bad boy"). Consider the semialgebraic (that is definable in the ordered field structure) space S^1_{angle} over $\mathbf{R}$ on the underlying subset S^1 of $\mathbf{R}^2$ obtained by taking the generalized topology from the usual affine definable circle $S^1 \subseteq \mathbf{R}^2$ and declare the structure sheaf to contain the continuous semialgebraic functions of the angle θ (having period 2π). The two semialgebraic spaces are different. The usual circle S^1 is an "affine model" of S^1_{angle}: there exists an isomorphism of semialgebraic spaces over $\mathbf{R}$ (whose formula is not semialgebraic, since it involves a trigonometric function) transforming the "non-embedded circle" S^1_{angle} into the "embedded circle" S^1.

A *definable space* over R is a space over R that has a finite open covering by affine definable spaces. Definable spaces were introduced by van den Dries in [**11**, Chapter 10]. They admit clear notions of a definable subset and of an open subset. The definable subsets of a definable space form a Boolean algebra generated by the open definable subsets; "definable" here means "constructible from the generalized topology." A *locally definable space* over R is a space over R that has an admissible covering by affine definable open subspaces. (So definable spaces are examples of locally definable spaces.) Each locally definable space is an inductive limit of a directed system of definable spaces in the category of spaces over a given R (cf. [**10**, I.2.3]). The *dimension* of a locally definable space is defined as usual (cf. [**10**, page 37]), and may be infinite. *Morphisms* of affine definable spaces, definable spaces and locally definable spaces over R are their morphisms as spaces over R. So affine definable spaces, definable spaces and locally definable spaces form full subcategories $\mathrm{ADS}(R)$, $\mathrm{DS}(R)$, and $\mathrm{LDS}(R)$ of the category $\mathrm{Space}(R)$ of spaces (over R).

A *locally definable subset* of a locally definable space is a subset having definable intersections with all definable open subspaces. Such subsets are also considered to be *subspaces*, the locally definable space of such a set is formed as an inductive limit of definable subspaces of the definable open spaces forming the ambient space (cf. [**10**, I.3, page 28]). A locally definable subset of a locally definable space is called *definable* if as a subspace it is a definable space. (The definable subsets of a definable space are exactly the definable subsets of these spaces as locally definable ones.)

On locally definable spaces we often consider a topology in the traditional sense, called the *strong topology* (cf. [**10**, page 31]), taking the open sets from the generalized topology as the basis of the topology. Nevertheless, we will usually work in the generalized topology. This allows us, in many cases, to omit the word "definably" applied to topological notions (as in "definably connected"). On a definable space the generalized topology generates both the strong topology and the definable (i.e., "constructible") subsets. Similarly, the locally definable subsets of a locally definable space are exactly the sets "locally constructible" from the generalized topology, where "locally" means "when restricted to an open definable subspace." The closure operator of the strong topology restricted to the class of locally definable subsets may be treated as the closure operator of the generalized topology.

The following new example gives some understanding of the variety of locally definable spaces even in the semialgebraic case. They are obtained by "partial localization," which generalizes passing to the "localization" M_{loc} of a locally complete locally semialgebraic space M (see [**10**, I.2.6]).

Example 3. Consider any o-minimal expansion $\mathbf{R}_{\mathcal{S}}$ of the field $\mathbf{R}$. Take the admissible union (see [**25**]) of (embedded) real line open intervals $(-\infty, n)$ over all natural n, which implies that this family is assumed to be admissible. The space described is definable "on the left-hand side," but only locally definable "on the right-hand side." The definable subsets are the finite unions of intervals (of any kind) that are bounded from above. The locally definable subsets are locally finite unions of intervals that have only finitely many connected components on the negative half-line. The structure sheaf consists of functions that are continuously definable on each of the intervals $(-\infty, n)$. This space will be called $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},+}$. Analogously we define the space $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},-}$ to be the admissible union of the family $(-n, +\infty)$, for $n \in \mathbf{N}$. (By taking the admissible union of the family $(-n, n)$ for $n \in \mathbf{N}$, we would get the usual "localization" $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc}}$ of the real line $\mathbf{R}_{\mathcal{S}}$.)

As in [**10**], we have

Example 4 (cf. [**10**, I.2.4]. Any "direct (generalized) topological sum" of definable spaces (in the category of spaces over a given R) is a locally definable space.

We call a subset K of a generalized topological space M *small* if for each admissible covering $\mathcal{U}$ of any open U, the set $K \cap U$ is covered by finitely many members of $\mathcal{U}$. (We say that $\mathcal{U}$ is *essentially finite* on K in such a situation.) Just from the definitions, we get (as in the semialgebraic case):

Facts 5. Each definable space is small. Each subset of a definable space is also small. Every small open subspace of a locally definable space is definable. Each small set of a locally definable space is contained in a small open set. In particular "small open" means exactly "definable open," but "small" does not imply "definable."

One can easily check that: any locally definable space is topologically Hausdorff if and only if it is *Hausdorff* in the generalized topological sense. Similarly, a locally definable space is topologically regular if and only if it is *regular* in the generalized topological sense: any single point, assumed always to be closed, and any closed subspace not containing the point can be separated by disjoint open subspaces.

Clearly, each affine definable space is regular. Of great importance for the theory of definable spaces is the following:

Theorem 6 (Robson [**28**], van den Dries [**11**]). *Each regular definable space is affine.*

Remark 7. Even if we define locally definable spaces with the use of structure sheaves, a locally definable space is determined by its generalized topology when we assume silently that the structure of each affine subspace is understood, since it has an admissible covering of regular small open subspaces, which are affine definable spaces. The main purpose of introducing function ringed spaces was to define morphisms.

A practical way of defining and denoting a locally definable space is to write it as the *admissible union* ($\overset{a}{\cup}$) of its admissible covering by open definable (often affine) subspaces, not just the union set (even if considered with a topology). Such a notation is defined in [**25**] and used in [**26**]. One can also just specify an admissible covering of the space by known open subspaces.

The author considers an attempt to encode the generalized topology under the notion of "equivalent atlases" a little bit risky. We have the following important example, which is again obtained by the "localization" process known from [**10**].

Example 8. Take an Archimedean R. Consider three locally definable spaces X_1, X_2 and X_3 on the same open interval $(0,1)$ given, respectively, by admissible families of open definable sets $\mathcal{U}_1 = \{(1/n, 1-1/n) : n \geq 3\}$, $\mathcal{U}_2 = \{(0,1)\}$, $\mathcal{U}_3 = \mathcal{U}_1 \cup \mathcal{U}_2$. Then $X_1 \neq X_2 = X_3$. Such a space X_1 is the "localized" unit interval $(0,1)_{\mathrm{loc}}$.

We would have two non-equivalent atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ that combine to a third atlas $\mathcal{U}_3$, and the combined atlas $\mathcal{U}_3$ would be equivalent to $\mathcal{U}_2$, but not equivalent to $\mathcal{U}_1$.

Notice that the recent paper [**4**] by Baro and Otero can easily mislead the reader. Their definition of a locally definable space is not equivalent to that of [**10**]. They define a locally definable space as a set with a concrete atlas, call some atlases equivalent (which is not studied later), and in Theorems 3.9 and 3.10 say that a set with only a topology is a locally definable space. The reader gets the impression that they consider only the usual topology and do not see the essential use of the generalized topology (see the proof of (iii) of their Proposition 2.9). Their notion of an "ld-homeomorphism" is never defined, and the reader may wrongly guess that an ld-homeomorphism is just a locally definable homeomorphism (see Remark 2.11). Their "locally finite generalized simplicial complex" is given a locally definable space structure "star by star," so it is not necessarily "embedded" into the ambient affine space. This may mislead the reader when reading their version of the Triangulation theorem (Fact 2.10) and some proofs. Their Example 3.1 is highly imprecise, since it depends upon the choice of the covering of M by definable subsets M_i. The same symbol M denotes both a locally definable space and just a subset of R^n (and this is continued in their Example 3.3). Formula $\mathrm{Fin}(\mathbf{R}) = \mathbf{R}$ (see page 492) again suggests to the reader the nonexistence of the generalized topology (never mentioned explicitly). It's worth noting that, if R does not have any saturation (as in the important case of the field of real numbers $\mathbf{R}$), then the usual topology does not determine the generalized topology.

We will say that an object N of $\mathrm{LDS}(R)$ *comes from* R^k if the underlying topological space of N is equal to the standard topological space of R^k and for each $x \in R^k$ both N and the affine space R^k induce on an open box B containing x the same definable open subspace. The following two original theorems show the variety of locally definable spaces "living" on the same topological space.

Theorem 9. *For each* o*-minimal expansion* $\mathbf{R}_{\mathcal{S}}$ *of the field of real numbers* $\mathbf{R}$:

1) *there are exactly four different objects of* $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ *that come from* $\mathbf{R}^1_{\mathcal{S}}$ *and are admissible unions of embedded open intervals, namely*: $\mathbf{R}_{\mathcal{S}}$, $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc}}$, $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},+}$, $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},-}$; *there are uncountably many objects of* $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ *that come from* $\mathbf{R}^1_{\mathcal{S}}$ *and are not embedded*;

2) *there are uncountably many different objects of* $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ *that come from* $\mathbf{R}^2_{\mathcal{S}}$ *and are admissible unions of embedded open definable sets.*

Proof. 1) Assume N is an object of $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ coming from $\mathbf{R}^1_{\mathcal{S}}$. Each open subset of N is a countable union of open intervals. Each open definable set is a finite union of open intervals, since it has a finite number of connected components. There is an admissible covering $\mathcal{U}$ of the real line by open intervals that are affine definable spaces. If such an interval is relatively compact, then it is an embedded subspace of $\mathbf{R}_{\mathcal{S}}$. On the other hand, there exists an uncountable family of non-embedded infinite open intervals. Choose $r > 0$ and find a non-linear, locally linear, strictly increasing function $\phi_r : (0, +\infty) \to (0, +\infty)$ approximating the function $x \mapsto x^r$. By transporting the structure of a locally definable space through such a mapping, we get a new object of $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ coming from $\mathbf{R}^1_{\mathcal{S}}$ for each r.

If there are no infinite intervals in $\mathcal{U}$, then the open family $\{(-n, n)\}_{n\in\mathbf{N}}$ is admissible, and $N = (\mathbf{R})_{\mathcal{S}})_{\mathrm{loc}}$. If both $+\infty$ and $-\infty$ are ends of embedded intervals from $\mathcal{U}$, then N is a finite union of embedded intervals; thus, it is isomorphic to the affine space $\mathbf{R}_{\mathcal{S}}$. Similarly, the other two cases of admissible unions of embedded intervals give the spaces $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},+}$, $(\mathbf{R}_{\mathcal{S}})_{\mathrm{loc},-}$.

2) Choose a slope $a \in \mathbf{R}$, and consider the space N_a defined by the admissible covering $\{U_{a,n}\}_{n\in\mathbf{N}}$, where $U_{a,n} = \{(x, y) \in \mathbf{R}^2 : y < ax + n\}$ are embedded open definable sets. All N_a, for $a \in \mathbf{R}$, are different objects of $\mathrm{LDS}(\mathbf{R}_{\mathcal{S}})$ coming from $\mathbf{R}^2_{\mathcal{S}}$. $\square$

We remind the reader (from non-standard analysis) that each non-Archimedean R is partitioned into many *galaxies* (two elements $x, y \in R$ are in the same galaxy if their "distance" $|x-y|$ is bounded from above by a natural number).

Theorem 10. *For any* o*-minimal expansion* R *of a field not isomorphic to* $\mathbf{R}$*, there are already uncountably many different objects of* $\mathrm{LDS}(R)$ *that come from the line* R^1 *and are admissible unions of embedded open intervals.*

Proof. Case 1: *R contains* $\mathbf{R}$. The set of galaxies of R is uncountable.

For any galaxy G of R, take $x \in G$ and consider the space N_G defined as the disjoint generalized topological union of the following: all the galaxies $G' > G$, treating each one as a locally definable space (see Remark 14), and the space N'_G given by the admissible covering $\{(-\infty, x+n)\}_{n\in\mathbf{N}}$, which is the union of all galaxies $G'' \leq G$ "partially localized" (only) at the end of G. All of N_G are different objects of $\mathrm{LDS}(R)$ and come from R^1.

Case 2: *R does not contain* $\mathbf{R}$. Consider a maximal archimedean subfield R_{ar} of R. This field embeds into $\mathbf{R}$. There exist uncountably many irrational cuts of R, determined by elements $r \in \mathbf{R} \setminus R_{ar}$. For each such r, consider the space N_r over R defined by the admissible covering

$$\{(-\infty, s)\}_{s<r} \cup \{(s, +\infty)\}_{s>r},$$

where $s \in R$. This space consists of two connected components given by the conditions $x < r$ and $x > r$. All of N_r, $r \in \mathbf{R} \setminus R_{ar}$, are different objects of $\mathrm{LDS}(R)$ and come from R^1. $\square$

There exist more general sets that are called in [**16**, Definition 7.1 (a)], "locally definable." We will call them local subsets. (A subset Y of a space X is a *local subset* if, for each point y only of Y, there is an open definable neighborhood U of y in X such that $U \cap Y$ is definable.) They can be given a locally definable space structure, but their properties are not nice: they are closed only on finite intersection and are not closed under complement or even finite union.

The locally definable space on such $Y \subseteq X$ may be introduced by the following admissible covering

$$\mathcal{U}_Y = \{Y \cap U_i \mid U_i \text{ is a definable open subset of } X \text{ and } Y \cap U_i \text{ is definable}\}.$$

(The above definition does not depend on any arbitrary choice of an admissible covering, contrary to Examples 3.1 and 3.3 of [**4**].)

Local subsets are *not* (as such) called subspaces! Their use often does not recognize the space structure given above (even if they are definable sets), since we mainly want to study "locally definable functions" on them (see [**16**, Definition 7.1 (b)]). (Consider a function "locally definable" if its domain and codomain are local subsets of some objects of LDS(R), and all function germs of this function at points of its domain are definable. A function germ f_x at x is called definable if some definable neighborhood of x is mapped by f into a definable neighborhood of $f(x)$ and the obtained restriction of f is a definable mapping.)

The following examples make the above considerations more clear.

Example 11. The semialgebraic set $(-1,1)_{\mathbf{R}}$ inherits an affine semialgebraic space structure from $\mathbf{R}$. Nevertheless, when speaking about "locally semialgebraic functions" into $\mathbf{R}$ (in the sense of [**16**, Definition 7.1 (b)]) we want to treat it as the "localized" open interval $(-1,1)_{\mathrm{loc}}$, which is not a semialgebraic space. Define, for example, functions $w : \mathbf{R} \to \mathbf{R}$ and $u : (-1,1) \to \mathbf{R}$ by the formulas

$$w(x) = \begin{cases} x - 4k, & x \in [4k-1, 4k+1),\ k \in \mathbf{Z}, \\ 2 + 4k - x, & x \in [4k+1, 4k+3),\ k \in \mathbf{Z}, \end{cases}$$

and

$$u(t) = w\left(\frac{t}{\sqrt{1-t^2}}\right).$$

Then u is "locally semialgebraic" (and not semialgebraic).

Example 12. Consider the semialgebraic set $S = (-1,1)^2 \cup \{(1,1)\}$ in $\mathbf{R}^2$. The fact of being a "locally semialgebraic function" (in the sense of [**16**, Definition 7.1 (b)]) on S (into $\mathbf{R}$) does not reduce to being a morphism of any locally (and even weakly) semialgebraic space that can be formed by redefining the notion of an admissible covering of the space S. In particular, each of the functions $F_n : S \to \mathbf{R}$ ($n = 1, 2, 3, \dots$), where

$$F_n(x,y) = \begin{cases} 0, & y \geq 1 - (1/n), \\ w[(1 - (1/n) - y)/(1-x)], & y < 1 - (1/n), \end{cases}$$

is "locally semialgebraic" (function w is defined as in the previous example).

In general, definable spaces and locally definable spaces do not behave well enough for being used in homotopy theory. The right choice of assumptions (as in the semialgebraic case of [**10**]) are: regularity and a new one called "paracompactness," which is only a rough analogue of the topological notion.

3. Regular paracompact locally definable spaces. One of the reasons why we pass to the locally definable spaces is the need of existence of covering mappings with infinite (for example, countable) fibers.

The following example is a generalization of an example from [**9**].

Example 13 (cf. [**9**, 5.14]). *The space* $\mathrm{Fin}(R)$. We look for the *universal covering* of the unit circle $S^1 \subseteq R^2$. We will soon see that (as in topology) $\pi_1(S^1) = \mathbf{Z}$, so the universal covering should have countable fibers. Let $\mathrm{Fin}(R)$ be the locally definable space introduced by the admissible covering by open intervals $\{(-n,n)\}_{n\in\mathbf{N}}$ in R. There is a surjective semialgebraic (so definable) morphism $e : [0,1] \to S^1$ that maps 0 and 1 to the distinguished point on S^1 and is injective elsewhere. Then the universal covering mapping $p : \mathrm{Fin}(R) \to S^1$ defined by $p(m + x) = e(x)$, where $m \in \mathbf{Z}$, $x \in [0,1]$, is a morphism of locally definable spaces.

A family of subsets of a locally definable space is *locally finite* if each open definable subset of the space meets only finitely many members of the family.

A locally definable space is called *paracompact* if there is a locally finite covering of the whole space by open definable subsets. (A locally finite covering must be admissible, since "admissible" means: when restricted to an open definable subspace, there is a finite subcovering. Shortly: "admissible" means exactly "locally *essentially* finite".)

Remark 14. The locally definable space $\mathrm{Fin}(R)$ given by the admissible covering $\{(-n,n) : n \in \mathbf{N}\}$ is paracompact for each R, since there exists a locally finite covering giving the same space. (Notice that if R contains $\mathbf{R}$, then

$$\bigcup_{r\in\mathbf{R}_+}^{a} (-r,r) = \bigcup_{n\in\mathbf{N}}^{a} (-n,n).)$$

In the language of nonstandard analysis, we can say that each galaxy may be considered to be a regular paracompact locally definable space.

Direct (i.e., Cartesian) products preserve regularity and paracompactness of locally definable spaces (cf. [**10**, I.4.2c) and I.4.4e)]). We will denote the category of regular paracompact locally definable spaces over R by RPLDS(R).

Example 15. The spaces from the proofs of Theorems 9 and 10 are objects of RPLDS(R).

Remark 16. A *connected* (in the sense of generalized topology: the space cannot be decomposed into two open disjoint nonempty subspaces) regular paracompact locally definable space has a countable admissible covering by definable open subsets (so-called *Lindelöf property* in [**10**]). If it has finite dimension k, then it can be embedded into the cartesian power $\mathrm{Fin}(R)^{2k+1}$. This holds by embedding into a partially complete space, triangulation (see Theorems 26, 27 below) and Theorem 3.2.9 from a book of Spanier [**30**] (see also [**10**, II.3.3]).

Topological Remark 17. The notion of paracompactness introduced above differs from the topological one. Each definable space is paracompact. There are Hausdorff definable (so paracompact) spaces which are not regular. With the regularity assumption, each paracompact space is normal and admits partition of unity. Paracompactness is inherited by all subspaces and Cartesian products. The Lindelöf property gives paracompactness only with the assumption that the closure of a definable set is definable.

Fiber products exist in the category of locally definable spaces over R (cf. [**10**, I.3.5]). A morphism $f : M \to N$ between locally definable spaces is called *proper* if it is universally closed in the sense of the generalized topology. This means that, for each morphism of locally definable spaces $g : N' \to N$, the induced morphism $f' : M \times_N N' \to N'$ in the pullback diagram is a closed mapping in the sense of the generalized topological spaces (it maps closed subspaces onto closed subspaces). If all restrictions of f to closed definable subspaces are proper, then we call f *partially proper*.

A Hausdorff locally definable space M is called *complete* if the morphism from M to the one point space is proper. Each paracompact complete space is affine definable (compare [**10**, I.5.10]). Moreover, M is called *locally complete* if each point has a complete neighborhood.

(Each locally complete locally definable space is regular, cf. [**10**, I.7, page 75]). It is *partially complete* if every closed definable subspace is complete. Every partially complete regular space is locally complete (cf. [**10**, I.7.1 a)]).

Topological Remark 18. This notion of properness is analogical to a notion from algebraic geometry. Partial completeness is the key notion.

Let M be a locally complete paracompact space. Take the family $\mathring{\gamma}_c(M)$ of all such open definable subsets U of M that $\overline{U}$ is complete. Introduce a new locally definable space M_{loc}, the *localization* or *partial completization* of M, on the same underlying set taking $\mathring{\gamma}_c(M)$ as an admissible covering by small open subspaces (cf. [**10**, I.2.6]). The new space is regular partially complete (not only locally complete) and the identity mapping from $M_{\rm loc}$ to M is a morphism, but $M_{\rm loc}$ may not be paracompact, see Warning-Example 24. Notice that localization leaves the strong topology unchanged.

Topological Remark 19. Localization is similar to the process of passing to k-spaces (they are exactly the compactly generated spaces if Hausdorffness is assumed) in homotopy theory. (Complete spaces play the role of compact spaces.) But notice that each topological locally compact space is a k-space.

Remark 20. Only one of the four locally definable spaces mentioned explicitly in the statement of Theorem 9 for each $\mathbf{R}_{\mathcal{S}}$ is partially complete, namely $(\mathbf{R}_{\mathcal{S}})_{\rm loc}$.

A *paracompact locally definable manifold* of dimension n over R is a Hausdorff locally definable space over R that has a locally finite covering by definable open subsets that are isomorphic to (open balls in) R^n. (Such a space is paracompact and locally complete, so regular, cf. [**10**, I.7, page 75].) If additionally the transition maps are (definable) C^k-diffeomorphisms ($k = 1, \dots, \infty$), then we get *paracompact locally definable C^k-manifolds.* Notice that the differential structure of such manifolds may be encoded by sheaves (in the sense of the strong topology) of C^k functions. We get the following original result:

Theorem 21. *Paracompact* (*in the topological sense*) *analytic manifolds of dimension n are in bijective correspondence with partially complete paracompact locally definable C^∞-manifolds over $\mathbf{R}_{\mathrm{an}}$ of the same dimension.*

Proof. A paracompact analytic manifold induces a paracompact locally definable C^∞-manifold over $\mathbf{R}_{\mathrm{an}}$. Each paracompact manifold (even a topological one) is regular. We may assume (by shrinking the covering of the manifold by chart domains if necessary) that the analytic structure of the manifold is given by a locally finite atlas consisting of charts whose domains and ranges are relatively compact subanalytic sets, and the charts extend analytically beyond the closures of chart domains. By taking a nice locally finite refinement, we additionally can get the chart domains and chart ranges (analytically and globally subanalytically) isomorphic to open balls in $\mathbf{R}^n$. Now the chart domains form a locally finite covering of the analytic manifold that defines a paracompact locally definable manifold over $\mathbf{R}_{\mathrm{an}}$. The transition maps (being analytic diffeomorphisms) are $\mathbf{R}_{\mathrm{an}}$-definable C^∞-diffeomorphisms of open, relatively compact, subanalytic subsets of some $\mathbf{R}^n$. Thus, we get a locally definable C^∞-manifold. (Notice that the relatively compact subanalytic sets are now the definable sets and the subanalytic sets are now the locally definable sets.)

The so obtained locally definable space is partially complete.

Vice versa: A paracompact locally definable C^∞-manifold over $\mathbf{R}_{\mathrm{an}}$ induces a Hausdorff (analytic) manifold with analytic, globally subanalytic transition maps and globally subanalytic chart ranges. We may assume that the manifold is connected. Its locally finite atlas is countable (cf. [**10**, I.4.17]), so the manifold is a second countable topological space, and finally a paracompact analytic manifold. All locally definable subsets are now subanalytic (they are globally subanalytic in every chart).

One-to-one correspondence: If the paracompact locally definable manifold is partially complete, then the closure of a chart domain is a closed definable set (cf. [**10**, I.4.6]) and a complete definable set, which means it is a compact subanalytic set. Thus, chart domains are relatively compact. "Locally definable" in the sense of locally definable spaces means exactly "locally definable" in the topological sense. It follows that the definable subsets are exactly the relatively compact subanalytic subsets, and the locally definable subsets are exactly the subanalytic subsets of the paracompact analytic manifold obtained. Notice that the strong topology does not change when we pass from one type of a manifold to the other. So the structure of the partially complete locally definable space is uniquely determined (see Remark 7) by the analytic manifold. Both the structures of a C^∞ locally definable manifold over $\mathbf{R}_{\mathrm{an}}$ and the structure of an analytic manifold do not change during the above operations (only a convenient atlas was chosen). □

Remark 22. A real function on a (paracompact) analytic manifold M_{an} is analytic if and only if it is a C^∞ morphism from the corresponding partially complete paracompact locally definable C^∞-manifold (call it M_{ldm}) into $\mathbf{R}_{\mathrm{an}}$ as an affine definable space. (See [**12**, 5.3].)

Analogously, for each expansion $\mathbf{R}_{\mathcal{S}}$ of the field $\mathbf{R}$ that is a reduct of $\mathbf{R}_{\mathrm{an}}$, partially complete paracompact locally definable C^∞-manifolds over $\mathbf{R}_{\mathcal{S}}$ correspond uniquely to paracompact analytic manifolds of some special kinds. Then the locally definable subsets in the sense of a given locally definable manifold (as well as in the sense of its "expansions," see below) form nice "geometric categories." This in particular generalizes the *analytic-geometric categories* of van den Dries and Miller [**12**].

The above phenomenon may be explained in the following way: the analytic manifolds $\mathbf{R}^n$ ($n \geq 1$), which model all analytic manifolds, have a natural notion of smallness. A subset $S \subset \mathbf{R}^n$ is *topologically small* if it is *bounded* or, equivalently, *relatively compact.* In the corresponding partially complete paracompact locally definable C^∞-manifolds

$$\mathrm{Fin}(\mathbf{R}_{\mathrm{an}})^n = (\mathbf{R}_{\mathrm{an}})^n_{\mathrm{loc}} = \bigcup_{k \in \mathbf{N}}^{a} (-k,k)^n = \mathrm{Fin}((\mathbf{R}_{\mathrm{an}})^n)$$

over $\mathbf{R}_{\mathrm{an}}$ this means that S is a *small* subset in the sense of the generalized topology (if S is subanalytic, then this means *definable*). One could also use the notion of being *relatively complete* in this context. It is partial completeness that gives analogy between the usual topology and the generalized topology.

Remark 23. The generalized topology of the space M_{ldm} of Remark 22 is "the subanalytic site" considered by microlocal analysts (see [**19**]). More generally, the generalized topology of each paracompact locally definable manifold may be considered as a "locally definable site." It is also possible to consider all subanalytic subsets of a real analytic manifold as open sets of a generalized topological space, but then the strong topology becomes discrete.

Warning-Example 24 (cf. [**10**, I.2.6]). (*The space* R_{loc}). Structure R, as an affine definable space, is locally complete but not complete. For such a space R_{loc} is introduced by the admissible covering $\{(-r,r) : r \in R_+\}$. This is a locally (even regular partially) complete space which is not definable. If the cofinality of R is uncountable, then R_{loc} is not paracompact! Here the morphisms from R to R are "the continuous definable functions," and the morphisms from R_{loc} to R are "the continuous locally (in the sense of R_{loc}) definable functions." (The latter case includes some nontrivial periodic functions for an Archimedean R.)

A series of topological facts have counterparts for regular paracompact locally definable spaces.

Lemma 25 (cf. [**9**] and [**10**, Chapter I]). *Let M be an object of* RPLDS(R). *Then*:

a) Tautness. *The closure of a definable set is definable* (cf. [**10**, I.4.6]);

b) Shrinking of coverings lemma. *For each locally finite covering (U_λ) of M by open locally definable sets, there is a covering (V_λ) of M by open locally definable sets such that $\overline{V_\lambda} \subseteq U_\lambda$* (cf. [**10**, I.4.11]);

c) Partition of unity. *For every locally finite covering (U_λ) of M by open locally definable subsets there is a subordinate partition of unity, i.e., there is a family of morphisms $\phi_\lambda : M \to [0,1]$ such that* $\operatorname{supp}\phi_\lambda \subseteq U_\lambda$ *and* $\sum_\lambda \phi_\lambda = 1$ *on M* (cf. [**10**, I.4.12]);

d) Tietze's extension theorem. *If A is a closed subspace of M and $f : A \to K$ is a morphism into a convex definable subset K of R, then there exists a morphism $g : M \to K$ such that $g \mid A = f$* (cf. [**10**, I.4.13]);

e) Urysohn's lemma. *If A, B are disjoint closed locally definable subsets of M, then there is a morphism $f : M \to [0,1]$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$* (cf. [**10**, I.4.15]).

Each locally definable space M over R has a natural "base field extension" $M(S)$ over any elementary extension S of R (cf. [**10**, I.2.10]) and an "expansion" $M_{R'}$ to a locally definable space over any o-minimal expansion R' of R. Analogously, we may speak about a base field extension of a morphism.

The rules of conservation of the main properties under the base field extension are the same as for the locally semialgebraic case:

a) the base field extensions of the family of the connected components of a locally definable space M form the family of connected components of $M(S)$ (cf. [**10**, I.3.22 i)]);

b) if M is Hausdorff, then: the space M is definable if and only if $M(S)$ is definable, M is affine definable if and only if $M(S)$ is affine definable, M is paracompact if and only if $M(S)$ is paracompact, M is regular and paracompact if and only if $M(S)$ is regular and paracompact (cf. [**10**, B.1]);

c) if M is regular and paracompact, then: M is partially complete if and only if $M(S)$ is partially complete and M is complete if and only if $M(S)$ is complete (cf. [**10**, B.2]).

If we expand R to an o-minimal R', then:

a) any locally definable space M is regular over R if and only if $M_{R'}$ is a regular space over R', since they have the same strong topologies;

b) a locally definable space M is connected over R if and only if $M_{R'}$ is connected over R' (for an affine space: a clopen subset of a set definable over R is definable over R; generally: apply an admissible covering by affine subspaces "over R");

c) a locally definable space M is Lindelöf over R if and only if $M_{R'}$ is Lindelöf over R': if M is Lindelöf, then $M_{R'}$ is obviously Lindelöf; if $M_{R'}$ is Lindelöf, then each member of a countable admissible covering $\mathcal{V}$ of $M_{R'}$ by definable open subspaces is covered by a finite union of elements of the admissible covering $\mathcal{U}$ of M by definable open subspaces that allowed to construct $M_{R'}$. Then $\mathcal{U}$ has a countable subcovering $\mathcal{U}'$. (Up to this moment our proof goes like the proof of Proposition

2.9 iii) in [**4**], but they do not care about admissibility.) The family $\mathcal{U}''$ of finite unions of elements of $\mathcal{U}'$ is a countable coarsening of $\mathcal{V}$, hence is admissible in $M_{R'}$. Since "admissible" means "locally essentially finite," $\mathcal{U}''$ is, in particular, admissible in M;

d) if M is a Hausdorff locally definable space over R, then M is paracompact over R if and only if $M_{R'}$ is paracompact over R': if M is paracompact, then $M_{R'}$ is obviously paracompact; if $M_{R'}$ is paracompact, then we can assume that it is connected. Then $M_{R'}$ is Lindelöf (cf. [**10**, I.4.17]) and taut (i.e., the closure of a definable set is definable, cf. [**10**, I.4.6]). Now, by c), space M is Lindelöf, and it is taut by the construction of $M_{R'}$ and considerations of Fundamental Example 1, so M is paracompact (see [**10**, I.4.18] and [**4**, Proposition 2.9 iv)]).

4. Homotopies. Here basic definitions of homotopy theory are reintroduced. The unit interval $[0,1]$ of R will be considered to be an affine definable space over R.

Let M, N be objects of Space(R), and let f, g be morphisms from M to N. A *homotopy* from f to g is a morphism $H : M \times [0,1] \to N$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. If H exists, then f and g are called *homotopic*. If additionally $H(x,t)$ is independent of $t \in [0,1]$ for each x in a subspace A, then we say that f and g are *homotopic relative* to A. A subspace A of a space M is called a *retract* of M if there is a morphism $r : M \to A$ such that $r \mid A = \mathrm{id}_A$. Such r is called a *retraction*. A subspace A of M is called a *strong deformation retract* of M if there is a homotopy $H : M \times [0,1] \to M$ such that H_0 is the identity and H_1 is a retraction from M to A. Then H is called *strong deformation retraction*.

A *system of spaces* over R is any tuple $(M, A_1, \dots, A_k)$ where M is a space over R and $A_1, \dots, A_k$ are subspaces of M. A *closed pair* is a system (M, A) of a space with a closed subspace. A system $(A_0, A_1, \dots, A_k)$ is *decreasing* if A_{i+1} is a subspace of A_i for $i = 0, \dots, k-1$. A *morphism* of systems of spaces $f : (M, A_1, \dots, A_k) \to (N, B_1, \dots, B_k)$ is a morphism of spaces $f : M \to N$ such that $f(A_i) \subseteq B_i$ for each $i = 1, \dots, k$. A *homotopy* between two morphisms of systems of spaces f, g from $(M, A_1, \dots, A_k)$ to $(N, B_1, \dots, B_k)$ is a morphism

$$H : (M \times [0,1], A_1 \times [0,1], \dots, A_k \times [0,1]) \longrightarrow (N, B_1, \dots, B_k)$$

with $H_0 = f$ and $H_1 = g$. The *homotopy class* of such a morphism f will be denoted by $[f]$ and the *set of all homotopy classes* of morphisms from $(M, A_1, \dots, A_k)$ to $(N, B_1, \dots, B_k)$ by

$$[(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)].$$

If C is a closed subspace of M and $h : C \to N$ is a pregiven morphism such that $h(C \cap A_i) \subseteq B_i$, then we denote the sets of classes of homotopy relative to C of mappings extending h by

$$[(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h.$$

Let us adopt the notation: $I = [0,1]$, $\partial I^n = I^n \setminus (0,1)^n$, and $J^{n-1} = \overline{\partial I^n \setminus (I^{n-1} \times \{0\})}$. For every pointed space (M, x_0) over R and $n \in \mathbf{N}^*$ we define the (*absolute*) *homotopy groups* as sets

$$\pi_n(M, x_0) = [(I^n, \partial I^n), (M, x_0)]$$

where the multiplication $[f] \cdot [g]$, for $n \geq 1$, is the homotopy class of

$$(f * g)(t_1, t_2, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n), & 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n), & \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

For $n = 0$ we get (only) a set $\pi_0(M, x_0)$ of connected components of M with the base point the connected component of x_0. Also, as in topology, we define *relative homotopy groups*

$$\pi_n(M, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (M, A, x_0)].$$

A morphism $f : M \to N$ is a *homotopy equivalence* if there is a morphism $g : N \to M$ such that $g \circ f$ is homotopic to id_M and $f \circ g$ is homotopic to id_N. We call $f : M \to N$ a *weak homotopy equivalence* if f induces bijections in homotopy sets ($\pi_0(\cdot)$) and group isomorphisms in all homotopy groups ($\pi_n(\cdot)$, $n \geq 1$). Analogously, we define homotopy equivalences and weak homotopy equivalences for systems of spaces.

The following operations, known from the usual homotopy theory, may not be executable in the category of regular paracompact locally definable spaces over a given R: the *smash product* of two pointed spaces M, N, which is $M \wedge N = M \times N / M \vee N$, where $M \vee N$ denotes the wedge product of such spaces; the *reduced suspension* SM of M, which is $S^1 \wedge M$; the *mapping cylinder* $Z(f)$ of $f : M \to N$, which is the space obtained as the quotient of $(M \times [0,1]) \cup N$ by the equivalence relation that identifies each point of the form $(x, 1)$, $x \in M$, with $f(x)$; the *mapping cone* of f, which is the mapping cylinder of f divided by $M \times \{0\}$; the *cofiber* $C(f)$ of $f : M \to N$, which is the "switched" mapping cylinder $(([0,1] \times M) \cup_{1 \times M, f} N)/\{0\} \times M$.

5. Comparison theorems for locally definable spaces. In this section the two comparison theorems from [**10**] are extended, and the third is added. The first steps to do this are embedding in a partially complete space and triangulation.

Theorem 26 (Embedding into a partially complete space, cf. [**10**, II.2.1]). *Each regular paracompact locally definable space over R is isomorphic to a dense locally definable subset of a partially complete regular paracompact space over R.*

We restate the triangulation theorem, keeping the notation from [**10**] to avoid confusion.

Theorem 27 (Triangulation, cf. [**10**, II.4.4]). *Let M be a regular paracompact locally definable space over R. For a given locally finite family $\mathcal{A}$ of locally definable subsets of M, there is a simultaneous triangulation $\phi : X \to M$ of M and $\mathcal{A}$ (i.e., an isomorphism from the underlying set X, considered to be a locally definable space, of a strictly locally finite geometric simplicial complex $(X, \Sigma(X))$ to M such that all members of $\mathcal{A}$ are unions of images of open simplices from $\Sigma(X)$).*

In particular, each object of RPLDS(R) is locally (pathwise) connected and even locally contractible.

As an illustration of the methods available by triangulation, the following Bertini or Lefschetz type theorem (compare [**1**, (39.7)]) is proven. (See [**24**] for a topological version. Here the difficulty lies in the possibility that two different points are of an infinitesimal distance, and that a curve has an infinite velocity.)

A subspace Δ of a locally definable space Y *nowhere disconnects* Y if, for each connected open neighborhood W of any $y \in Y$, there is an open neighborhood $U \subseteq W$ of y such that $U \setminus \Delta$ is connected.

A morphism $p : E \to B$ in $\mathrm{LDS}(R)$ is a *branched covering* if there is a closed, nowhere dense *exceptional subspace* $\Delta \subseteq B$ such that $p|_{p^{-1}(B\setminus\Delta)} : p^{-1}(B \setminus \Delta) \to B \setminus \Delta$ is a *covering mapping* (this means that there is an admissible covering of $B \setminus \Delta$ by open subspaces, each of them well covered, analogically to the topological setting). If each of the *regular points* $b \in B \setminus \Delta$ of the branched covering $p : E \to B$ have the fiber of the same cardinality, then this cardinality is called the *degree* of a branched covering $p : E \to B$.

Theorem 28 (cf. [**24**, Theorem 1]). *Let Y be a simply connected* (*this assumes connected*) *object of* $\mathrm{RPLDS}(R)$*, Z a connected, paracompact locally definable manifold over R of dimension at least* 2 *and* $\pi : Y \times Z \to Y$ *the canonical projection.*

Assume that $V \subset Y \times Z$ is a closed subspace such that the restriction $\pi_V : V \to Y$ is a branched covering of finite degree and an exceptional set Δ of this branched covering nowhere disconnects Y. Put $X = (Y \times Z) \setminus V$, and $L = \{p\} \times Z$, for some $p \in Y \setminus \Delta$.

If there is a morphism of locally definable spaces $h : Y \to Z$ over R with the graph contained in X, then the inclusion $i : L \setminus V \to X$ induces an epimorphism in the fundamental groups $i_* : \pi_1(L \setminus V) \to \pi_1(X)$.

Lemma 29 (Straightening property, cf. [**24**, Lemma 3]). *Every paracompact locally definable manifold M over R has the following straightening property*:

For each set $J \subset [0,1] \times M$ such that the natural projection $\beta : [0,1] \times M \to [0,1]$ restricted to J is a covering mapping of finite degree, there exists an isomorphism, called the straightening isomorphism*, $\tau : [0,1] \times M \to [0,1] \times M$ which satisfies the following three conditions*:

1) $\beta \circ \tau = \beta$,

2) $\tau \mid \{0\} \times M = \mathrm{id}$,

3) $\tau(J) = [0,1] \times (\alpha(J \cap (\{0\} \times M)))$, where $\alpha : [0,1] \times M \to M$ is the natural projection.

Proof. Special case. Assume M is a unit open ball in R^m. The set J is a finite union of graphs of definable continuous mappings $\gamma_i : [0,1]_R \to M$ $(i = 1, \dots, n)$. We apply induction on the number n of these graphs.

If $n = 1$, then obviously the straightening exists (compare [**24**, Lemma 2]), and the isomorphism may be chosen to extend continuously to the identity on the unit sphere.

If $n > 1$ and the lemma is true for $n-1$, then we can assume that the first $n-1$ graphs (of the functions $\gamma_1, \dots, \gamma_{n-1}$) are already straightened and that the distances between images of the corresponding mappings (points $p_1, \dots, p_{n-1}$) are not infinitesimals. Moreover, since the distance from the value $\gamma_n(t)$ of the last function γ_n to any of the distinguished points has a positive lower bound, we can assume $\gamma_n(t)$ is always outside some closed balls centered at p_i's with radius larger than some rational number. Now, we can cover the rest of the unit ball by finitely many regions that are each isomorphic to the open unit ball. Since the last function is definable, there are only finitely many transitions from one region to another when $t \in [0,1]_R$. We have the straightening inside each of the regions. By gluing such straightening as in the proof of Lemma 3 of [**24**], we get the straightening of the whole n-th mapping. Again, the straightening extends continuously to the identity on the unit sphere.

General case. Again J is a finite union of graphs of definable functions on $[0,1]_R$ (by arguments similar to those of the usual topological context). Since J is definable, it is contained in a finite union of open sets each isomorphic to the open unit ball in R^m. The thesis of the lemma extends by arguments similar to these of the special case. □

Proof of Theorem 28. Clearly, X is a connected and locally simply connected space. Let $j : L \setminus V \hookrightarrow X \setminus (\Delta \times Z)$ and $k : X \setminus (\Delta \times Z) \hookrightarrow X$ be the inclusions. Then the proof falls naturally into two parts.

Step 1. *The induced mapping $j_* : \pi_1(L \setminus V) \to \pi_1(X \setminus (\Delta \times Z))$ is an epimorphism.* This step is analogous to Part 1 of the proof of Theorem 1 in [**24**]. Here Lemma 29 is used.

Step 2. *The mapping $k_* : \pi_1(X \setminus (\Delta \times Z)) \to \pi_1(X)$ induced by k is an epimorphism.* Notice that $(\Delta \times Z) \cap X$ nowhere disconnects X. Let $u = (f, g)$ be a loop in X at $(p, h(p))$. The set $\mathrm{im}\,(u)$ has an affine open neighborhood W.

We use a (locally finite) triangulation of $Y \times Z$ (that is an isomorphism $\phi : K \to Y \times Z$ for some strictly locally finite, not necessarily closed, simplicial complex $(K, \Sigma(K))$, following the notation of [**10**]), compatible with $\mathrm{im}(u), \Delta \times Z, V, L, h, W$ "over $\mathbf{Q}$."

There is $\varepsilon \in \mathbf{Q}$ such that the "distance" from $\phi^{-1}(\mathrm{im}(u))$ to $\phi^{-1}(V \cap W)$ in some ambient affine space is at least ε. Moreover, the "velocity" of $\phi^{-1} \circ u$ (existing almost everywhere) is bounded from above by some rational number. Now since all the sets and functions considered (appearing in the context of K) are piecewise linear over $\mathbf{Q}$, the Lebesgue number argument is available. By the use of the "distance" function in the ambient affine space and the barycentric coordinates for the chosen triangulation, we find a loop $\widetilde{u} = (\widetilde{f}, \widetilde{g})$ homotopic to u rel$\{0,1\}$ with image in $X \setminus (\Delta \times Z)$. □

The following facts and theorems, whose proofs use the machinery of *good triangulations*, are straightforward generalizations of the corresponding semialgebraic versions from [**10**]:

Fact 30 (Canonical neighborhood retraction, cf. [**10**, III.1.1]). *Let M be an object of* RPLDS(R) *and A a closed subspace. There is an open neighborhood U (in particular a subspace) of A and a strong deformation retraction*

$$H : \overline{U} \times [0,1] \longrightarrow \overline{U}$$

from $\overline{U}$ to A such that the restriction $H|U \times [0,1]$ is a strong deformation retraction from U to A.

Fact 31 (Extension of morphisms, cf. [**10**, III.1.2]). *Let M be an object of* RPLDS(R)*, A a closed subspace and U a neighborhood of A from the previous theorem. Any morphism $f : A \to Z$ into a regular paracompact locally definable space extends to a morphism $\widetilde{f} : \overline{U} \to Z$. Moreover, if $\widetilde{f_1}$, $\widetilde{f_2}$ are extensions of f to $\overline{U}$, then they are homotopic in $\overline{U}$ relative to A.*

Fact 32 (Homotopy extension property, cf. [**10**, III.1.4]). *Let M be an object of* RPLDS(R)*. If A is a closed subspace of M, then $(A \times [0,1]) \cup (M \times \{0\})$ is a strong deformation retract of $M \times [0,1]$.*

In particular, the pair (M, A) has the following Homotopy extension property: for each morphism $g : M \to Z$ into a regular paracompact locally definable space Z and a homotopy $F : A \times [0,1] \to Z$ with $F_0 = g \mid A$, there exists a homotopy $G : M \times [0,1] \to Z$ with $G_0 = g$ and $G|A \times [0,1] = F$.

Since our spaces may be triangulated, the method of *simplicial approximations* ([**10**, III.2.5]) does a good job. In particular, the method of *well-cored systems* and *canonical retractions* from [**10**, III.2] gives the following.

Fact 33. *Each object of* RPLDS(R) *is homotopy equivalent to a partially complete one. A system $(M, A_1, \dots, A_k)$ of a regular paracompact locally definable space with closed subspaces is homotopy equivalent to an analogous system of partially complete spaces.*

The following two main theorems from [**10**] generalize, but the *mapping spaces* from III.3, which depend on the degrees of polynomials, should be replaced with similar mapping spaces depending on concrete formulas $\Psi(\overline{x}, \overline{y}, \overline{z})$, with parameters $\overline{z}$, of the language of the structure R (one "mapping space" per each formula Ψ).

Let $(M, A_1, \dots, A_r)$ and $(N, B_1, \dots, B_r)$ be systems of regular paracompact locally definable spaces over R, where each A_i $(i = 1, \dots, r)$ is closed in M. Let $h : C \to N$ be a given morphism from a closed subspace C of M such that $h(C \cap A_i) \subseteq B_i$ for each $i = 1, \dots, r$. Then we have:

Theorem 34 (First comparison theorem, cf. [**10**, III.4.2]). *Let $R \prec S$ be an elementary extension. Then the "base field extension" functor from R to S induces a bijection between the homotopy sets*:

$$\kappa : [(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h \longrightarrow [(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h(S).$$

Theorem 35 (Second comparison theorem, cf. [**10**, III.5.1]). *Let R be an* o*-minimal expansion of* $\mathbf{R}$. *Then the "forgetful" functor*

RPLDS(R) $\to$ Top *to the topological category induces a bijection between the homotopy sets*

$$\lambda : [(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h \longrightarrow [(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h_{\mathrm{top}}.$$

Moreover, a version of the proof of the First comparison theorem gives:

Theorem 36 (Third comparison theorem). *If R' is an* o*-minimal expansion of R, then the "expansion" functor induces a bijection between the homotopy sets*

$$\mu : [(M, A_1, \dots, A_k), (N, B_1, \dots, B_k)]^h_R \longrightarrow [(M, A_1, \dots, A_k)_{R'}, (N, B_1, \dots, B_k)_{R'}]^h_{R'}.$$

Sketch of proof. Baro and Otero [**3**] have written a detailed proof of this theorem in the case of systems of definable sets. They use a natural tool of "normal triangulations" from [**2**] to get an applicable version of II.4.3 from [**10**]. The theorem extends to the general case as in [**10**]. ◻

Because of the locally finite character of the regular paracompact locally definable spaces, by inspection of the proof of the Triangulation theorem ([**10**, II.4.4]), each such space has an isomorphic copy that is built from sets definable without parameters glued together along sets that are definable without parameters. It is possible to triangulate even "over the field of real algebraic numbers $\overline{\mathbf{Q}}$ " or "over the field of rational numbers $\mathbf{Q}$". Moreover, if two 0-definable subsets of R^n are isomorphic as definable spaces (i.e., definably homeomorphic), then there is a 0-definable isomorphism between them (we may change arbitrary parameters into 0-definable parameters in the defining formula of an isomorphism).

By the (*noncompact*) o*-minimal version of Hauptvermutung* for structure R, we understand the following statement, which is a version of [**5**, Question 1.3]:

Given two semialgebraic (*definable in the field structure of* R) *sets in some* R^n, *if they are definably homeomorphic, then they are semialgebraically homeomorphic.*

In other words: *if two affine semialgebraic spaces are isomorphic as definable spaces, then they are isomorphic as semialgebraic spaces.*

It follows from [**29**, Theorem 2.5] that this statement is true for every R. Thus, the category of regular paracompact locally semialgebraic spaces RPLSS(R) over (the underlying field of) R may be viewed as a subcategory of RPLDS(R), but not as a full subcategory. Moreover, by triangulation with vertices having coordinates in the field of real algebraic numbers $\overline{\mathbf{Q}}$, we have the following fact:

Fact 37. *Each regular paracompact locally definable space over* R *is* isomorphic *to a regular paracompact locally semialgebraic space over* (*the underlying field of*) R.

Thus, by the Third comparison theorem, the homotopy categories HRPLSS (R) and HRPLDS(R) are equivalent. Analogously, we get

Corollary 38. *The homotopy categories of systems* $(M, A_1, \dots, A_k)$ *of regular paracompact locally definable spaces with finitely many closed subspaces and systems* $(M, A_1 \dots, A_k)$ *of regular paracompact locally semialgebraic spaces with finitely many closed subspaces* (*over the "same"* R) *are equivalent.*

Proof. By the triangulation theorem (Theorem 27), every object of the former category is isomorphic to an object of the later category. Thus, the "expansion" functor is essentially surjective. By the Comparison theorem (Theorem 36), it is also full and faithful. This implies that this functor is an equivalence of categories. □

It follows that the homotopy theory for regular paracompact locally definable spaces can, to a large extent, be transferred from the semialgebraic homotopy theory and, eventually, from the topological homotopy theory, as in [**10**].

Other important facts about regular paracompact locally definable spaces will be developed in a more general setting of definable CW-complexes and weakly definable spaces.

6. Weakly definable spaces. In homotopy theory one needs to use quotient spaces (e.g., mapping cylinders, mapping cones, cofibers, smash products, reduced suspensions, CW-complexes), and this operation is not always executable in the category of locally definable spaces (as in the semialgebraic case). That is why weakly definable spaces, which are analogues of arbitrary Hausdorff topological spaces, need to be introduced. We start here to re-develop the theory of Knebusch from [**20**].

Let (M, O_M) be a space over R, and let K be a small subset of M. We can induce a space on K in the following way:

i) *open* sets in K are the intersections of open sets on M with K,

ii) *admissible* coverings in K are such open coverings that some finite subcovering already covers the union,

iii) a function $h : V \to R$ is a *section* of $O_K(V)$ if it is a finite open union of restrictions to K of sections of the sheaf O_M. We call (K, O_K) a *small subspace* of (M, O_M).

A subset K of M is called *closed definable* in M if K is closed, small and the space (K, O_K) is a definable space. The collection of closed definable subsets of M is denoted by $\overline{\gamma}(M)$. The set K is called a *polytope* if it is a closed definable complete space. We denote the collection of polytopes of M by $\gamma_c(M)$.

A *weakly definable space* (over R) is a space M (over R) having a family, indexed by a partially ordered set A, of regular closed definable subsets $(M_\alpha)_{\alpha \in A}$ such that the following conditions hold:

WD1) M is the union of all M_α,

WD2) if $\alpha \leq \beta$ then M_α is a (closed) subspace of M_β,

WD3) for each α there is only a *finite* number of β such that $\beta \leq \alpha$,

WD4) the family (M_α) is strongly inverse directed, i.e., for each α, β there is some γ such that $\gamma \leq \alpha$, $\gamma \leq \beta$ and $M_\gamma = M_\alpha \cap M_\beta$,

WD5) the set of indices is directed: for each α, β there is γ with $\gamma \geq \alpha$, $\gamma \geq \beta$,

WD6) the space M is the *inductive limit* of the spaces (M_α), which means the following:

a) a subset U of M is *open* if and only if each $U \cap M_\alpha$ is open in M_α,

b) an open family (U_λ) is *admissible* if and only if for each α the restricted family $(M_\alpha \cap U_\lambda)$ is admissible in M_α,

c) a function $h : U \to R$ on some open U is a *section* of O_M if and only if all the restrictions $h|U \cap M_\alpha$ are sections of respective sheaves O_{M_α}.

Such a family (M_α) is called an *exhaustion* of M.

A space M is called a *weak polytope* if M has an exhaustion composed of polytopes. *Morphisms* and *isomorphisms* of weakly definable spaces are their morphisms and isomorphisms as spaces (we get the full subcategory WDS(R) of Space(R)).

A *weakly definable subset* is such a subset $X \subseteq M$ that has definable intersections with all members of some exhaustion (M_α), and is considered with the exhaustion $(X \cap M_\alpha)$; hence, it may be considered as a *subspace* of M (cf. [**20**, IV.3]).

A subset X of M is *definable* if it is weakly definable and the space (X, O_X) is definable. A subset X of M is definable if and only if it is weakly definable and is contained in a member of an exhaustion M_α (cf. [**20**, IV.3.4]).

The *strong topology* on M is the topology that makes the topological space M the respective inductive limit of the topological spaces M_α. The unpleasant fact about the weakly definable spaces (in comparison with the locally definable spaces) is that points may not have small neighborhoods (see Example 41). Moreover, open sets from the generalized topology may not form a basis of the strong topology (cf. [**20**, Appendix C]).

The closure of a definable subset of M is always definable (cf. [**20**, IV.3.6]), so the topological closure operator restricted to the class $\gamma(M)$ of definable subsets of M may be treated as the closure operator of the generalized topology. The weakly definable subsets are "piecewise constructible" from the generalized topology (compare [**26**]).

All weakly definable spaces are Hausdorff, actually even "normal," see [**20**, IV.3.12]. We can consider "expansions" and "base field extensions" of weakly definable spaces (compare considerations in [**20**, IV.2]) or morphisms (in the case of a base field extension) similar to the operations defined for locally definable spaces. They do not depend on the chosen exhaustion and preserve connectedness (cf. [**20**, IV.2, IV.3]).

Remark 39. Assume that a weakly definable space M is also locally definable. Then $\overline{\gamma}(M)$ is the family of all closed small subsets (as in [**10**, page 57]), since closed small subsets are definable as subspaces. We can speak about complete subspaces of M. It is easy to see that complete subspaces are always closed. Thus, the family $\gamma_c(M)$ contains exactly the definable complete subspaces (as in [**10**, page 81]).

Fiber products exist in WDS(R) (cf. [**20**, IV.3.20]). So we (analogously to the case of locally definable spaces) define *proper* and *partially proper* mappings between weakly definable spaces as well as *complete* and *partially complete* spaces. It appears that the complete spaces are the polytopes, and the partially complete spaces are the weak polytopes (cf. [**20**, IV.5]).

The following examples from [**20**] remain relevant in the case of an o-minimal expansion of a real closed field.

Example 40 (cf. [**20**, IV.1.5]). The category RPLDS(R) is a full subcategory of WDS(R). An exhaustion of an object M of RPLDS(R) is given by all finite subcomplexes Y in X that are closed in X for some triangulation $\phi : X \to M$.

Example 41 (cf. [**20**, IV.1.8 and IV.4.7-8]). An infinite wedge of circles is a weak polytope but not a locally definable space. A "countable comb" or "uncountable comb" is a weak polytope which is not a locally definable space.

Warning-Example 42 (cf. [**20**, IV.4.7]). Consider the "countable comb" from [**20**, IV.4.7]. This example shows that the topological closure of a weakly definable subset may not be weakly definable. Moreover, the naive "Arc sellecting lemma for weakly definable spaces" does not hold.

On the other hand, the following examples did not appear explicitly in [**20**].

Example 43. Consider an uncountable proper subfield F of $\mathbf{R}$. Let X be a subset of the unit square $[0, 1]^2$ consisting of points that have at least one coordinate in F. This set has a natural exhaustion making X into a weak polytope over $\mathbf{R}$. This weak polytope is not locally simply connected.

Example 44. An open interval of R is a definable space but not a weak polytope, an infinite comb with such a "hand" is a weakly definable space but not a weak polytope.

Gluing weakly definable spaces is possible: for a closed pair (M, A) and a partially proper morphism $f : A \to N$ the quotient space of $M \sqcup N$ by an equivalence relation identifying each $a \in A$ with $f(a)$ is a weakly definable space $M \cup_f N$ called the *space obtained by gluing M to N along A by f*. Then the projection $\pi : M \sqcup N \to M \cup_f N$ is partially proper and strongly surjective, cf. [**20**, IV.8.6]. (A morphism $f : M \to N$ is *strongly surjective* if each definable subset of N is covered by the image of a definable subset of M.)

A family $\mathcal{A}$ of subsets of a weakly definable space M will be called *piecewise finite* if, for each $D \in \gamma(M)$, the set D meets only finitely many members of $\mathcal{A}$. (Such families are called "partially finite" in [**20**].)

A *definable partition* of a weakly definable space M is a piecewise finite partition of M into a subset Σ of the family $\gamma(M)$ of definable subsets of M. An element τ of Σ is an *immediate face* of σ if $\tau \cap (\overline{\sigma} \setminus \sigma) \neq \varnothing$. Then we write $\tau \prec \sigma$. A *face* of σ is an element of some finite chain of immediate faces finishing with σ. (Each σ has only finitely many immediate faces, even a finite number of faces, cf. [**20**, V.1.7]).

A *patch decomposition* of M is a definable partition Σ of M such that for each $\sigma \in \Sigma$ there is a number $n \in \mathbf{N}$ such that any chain $\tau_r \prec \tau_{r-1} \prec \cdots \prec \tau_0 = \sigma$ in Σ has length $r \leq n$. The smallest such n is called the *height* of σ and denoted by $h(\sigma)$. A *patch complex* is a pair $(M, \Sigma(M))$ consisting of a space M and a patch decomposition $\Sigma(M)$ of M. Elements of the patch decomposition are called *patches.*

Example 45 (cf. [**20**, V.1.4]). Each exhaustion gives a patch decomposition of M.

Instead of triangulations for RPLDS(R), we have available for WDS(R) so called special patch decompositions. A *special patch decomposition* is such a patch decomposition that for each $\sigma \in \Sigma$, the pair $(\overline{\sigma}, \sigma)$ is isomorphic to the pair with the second element being a standard open simplex, and the first element this standard open simplex with some added open proper faces.

Fact 46 (cf. [**20**, V.1.12]). *Let M be an object of* WDS(R)*, and let $\mathcal{A}$ be a piecewise finite family of subspaces. Then there is a simultaneous special patch decomposition of M and the family $\mathcal{A}$.*

A *relative patch decomposition* of a closed pair (M, A) is a patch decomposition Σ of the space $M \setminus A$. Then we denote by $\Sigma(n)$ the union of all patches of height n, by M_n the union of A and all $\Sigma(m)$ with $m \leq n$, $M(n)$ the "direct (generalized) topological sum" of all closures $\overline{\sigma}$ where $\sigma \in \Sigma(n)$, and $\partial M(n)$ the direct sum of all frontiers $\partial\sigma = \overline{\sigma} \setminus \sigma$ of $\sigma \in \Sigma(n)$.

By $\psi_n : M(n) \to M_n$, we denote the union of all inclusions $\overline{\sigma} \to M_n$ with $\sigma \in \Sigma(n)$, and by $\phi_n : \partial M(n) \to M_{n-1}$ the restriction of ψ_n, which is called the *attaching map*. Then, since ϕ_n is partially proper (cf. [**20**, VI.2]), we can express M_n as $M(n) \cup_{\phi_n} M_{n-1}$. The space M_n is called *n-chunk* and $M(n)$ is called *n-belt*. So each weakly definable space is built up by gluing direct (generalized) topological sums of definable spaces to the earlier constructed spaces in countably many steps. In particular, definable versions of CW-complexes are among weakly definable spaces (see below).

A family $(X_\lambda)_{\lambda \in \Lambda}$ from $\mathcal{T}(M)$, the class of weakly definable subsets of M, is called *admissible* if each definable subspace B of M is contained in the union of finitely many elements of the family. (One could call such families "piecewise essentially finite" or "partially essentially finite.") Thus definable partitions are exactly the admissible partitions into definable subsets.

An *admissible filtration* of a space X is an admissible increasing sequence of closed subspaces $(X_n)_{n \in \mathbf{N}}$ covering X. For example: the sequence $(M_n)_{n \in \mathbf{N}}$ of chunks of M (for a given patch decomposition) is an admissible filtration of M (cf. [**20**, VI.2]).

The next fact is very important in homotopy-theoretic considerations.

Fact 47 (Composition of homotopies, cf. [**20**, V.5.1]). *Let* $(C_n)_{n\in\mathbf{N}}$ *be an admissible filtration of space* M. *Assume* $(G_n : M \times [0,1] \to N)_{n\in\mathbf{N}}$ *is a family of homotopies such that* $G_{n+1}(\cdot, 0) = G_n(\cdot, 1)$ *and* G_n *is constant on* C_n. *For any given strictly increasing sequence* $0 = s_0 < s_1 < s_2 < \cdots$ *with all* s_m *less than 1, there is a homotopy* $F : M \times [0,1] \to N$ *such that*

$$F(x,t) = G_{k+1}\left(x, \frac{t - s_k}{s_{k+1} - s_k}\right), \quad \textit{for } (x,t) \in C_n \times [s_k, s_{k+1}], \\ 0 \le k \le n-2,$$

and $F(x,t) = G_n(x,0)$ *for* $(x,t) \in C_n \times [s_{n-1}, 1]$.

7. Comparison theorems for weakly definable spaces. Now, with *patch decompositions* playing the role of triangulations, we get the Comparison theorems for weakly definable spaces as in [**20**].

Fact 48 (Homotopy extension property, cf. [**20**, V.2.9]). *Let* (M, A) *be a closed pair of weakly definable spaces over* R. *Then* $(A \times [0,1]) \cup (M \times \{0\})$ *is a strong deformation retract of* $M \times [0,1]$. *In particular, the pair* (M, A) *has the following Homotopy extension property*:

For each morphism $g : M \to Z$ *into a weakly definable space* Z *and a homotopy* $F : A \times [0,1] \to Z$ *with* $F_0 = g \mid A$, *there exists a homotopy* $G : M \times [0,1] \to Z$ *with* $G_0 = g$ *and* $G \mid A \times [0,1] = F$.

Let $(M, A_1, \ldots, A_r)$ and $(N, B_1, \ldots, B_r)$ be systems of weakly definable spaces over R where each A_i is closed in M. Let $h : C \to N$ be a given morphism from a closed subspace C of M such that $h(C \cap A_i) \subseteq B_i$ for each $i = 1, \ldots, r$. Then we have:

Theorem 49 (First comparison theorem, cf. [**20**, V.5.2 i)]). *For an elementary extension* $R \prec S$ *the following map, induced by the "base field extension" functor, is a bijection*

$$\kappa : [(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]^h \\ \longrightarrow [(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]^h(S).$$

Theorem 50 (Second comparison theorem, cf. [**20**, V.5.2 ii)]). *If* $R = \mathbf{R}$ *as fields, then the following map to the topological homotopy* *sets, induced by the "forgetful" functor, is a bijection*

$$\lambda : [(M, A_1, \dots , A_r), (N, B_1, \dots , B_r)]^h \longrightarrow [(M, A_1, \dots , A_r), (N, B_1, \dots , B_r)]^h_{\mathrm{top}}.$$

Again, a version of the proof of the First comparison theorem (so also of a version of the proof of [**20**, V.5.2 i)]; we present the proof for the convenience of the reader) gives:

Theorem 51 (Third comparison theorem). *If R' is an* o*-minimal expansion of R, then the following map, induced by the "expansion" functor, is a bijection*

$$\mu : [(M, A_1, \dots , A_k), (N, B_1, \dots , B_k)]^h_R \longrightarrow [(M, A_1, \dots , A_k)_{R'}, (N, B_1, \dots , B_k)_{R'}]^h_{R'}.$$

Proof. It suffices to prove the surjectivity, and only the case $k = 0$. We have a map $f : M \to N$ (over R') extending $h : C \to N$ (over R), and we seek for a mapping $g : M \to N$ (over R) such that g is homotopic to f relative to C (the homotopies appearing in this proof are allowed to be over R').

We choose a relative patch decomposition (over R) of (M, C), and will construct maps $h_n : M_n \to N$ (over R), $f_n : M \to N$ (over R') for $n \geq -1$, and a homotopy $H_n : M \times [0, 1] \to N$ relative to M_{n-1} such that: $h_{-1} = h$, $h_n|_{M_{n-1}} = h_{n-1}$, $f_{-1} = f$, $f_n|_{M_n} = h_n$, $H_n(\cdot, 0) = f_{n-1}$, $H_n(\cdot, 1) = f_n$. If we do this, we are done: we have a map $g : M \to N$ with $g|_{M_n} = h_n$ for each n. Composing, by Fact 47, the homotopies $(H_n)_{n\geq 0}$ along a sequence $s_n \in [0, 1)$ with $s_{-1} = 0$, we obtain a homotopy $G : M \times [0, 1] \to N$ relative to C from f to g as desired.

We start with $h_{-1} = h$ and $f_{-1} = f$. Assume that h_i, f_i and H_i are given for $i < n$. Then we get a pushout diagram over R (see [**20**, page 149]) and we define:

$$k_n = h_{n-1} \circ \phi_n : \partial M(n) \longrightarrow N \text{ (over } R),$$
$$u_n = (f_{n-1}|_{M_n}) \circ \psi_n : M(n) \longrightarrow N \text{ (over } R').$$

Notice that u_n extends k_n. By the Third comparison theorem for locally definable spaces (Theorem 36) there is a map $v_n : M(n) \to N$ over R extending k_n and a homotopy $F_n : M(n) \times [0,1] \to N$ relative to $\partial M(n)$ from u_n to v_n. The maps v_n and h_{n-1} combine to a map $h_n : M_n \to N$, with $h_n \circ \psi_n = v_n$ and $h|_{M_{n-1}} = h_{n-1}$. The maps F_n and $M_{n-1} \times [0,1] \ni (x,t) \mapsto h_{n-1}(x) \in N$ combine (cf. [**20**, IV.8.7.ii)]) to the homotopy $\widetilde{H}_n : M_n \times [0,1] \to N$ relative to M_{n-1} from $f_{n-1}|_{M_n}$ to h_n. It can be extended (by Fact 48) to the homotopy $H_n : M \times [0,1] \to N$ with $H_n(\cdot, 0) = f_{n-1}$. Put $f_n = H_n(\cdot, 1)$. This finishes the induction step and the proof of the theorem. $\square$

Again, the category of weakly semialgebraic spaces over (the underlying field of) R may be considered a (not full in general) subcategory of WDS(R). But see the following important new example:

Warning-Example 52. *Let Q be the square $[0,1]^2_R$. Now form $\widetilde{Q}$ in the following way: for each definable subset A of Q, glue $A \times S^1$ to Q by identifying $A \times \{1\}$ with A. If there are definable non-semialgebraic sets in R^2, then $\widetilde{Q}$ as a weakly definable space is not isomorphic to* (*an expansion of*) *a weakly semialgebraic space over R.*

8. Definable CW-complexes. A *relative definable* CW-*complex* (M, A) over R is a relative patch complex (M, A) satisfying the conditions:

(CW1) immediate faces of patches have smaller dimensions than the original patches in the patch decomposition of $M \setminus A$,

(CW2) for each patch $\sigma \in \Sigma(M, A)$ there is a morphism $\chi_\sigma : E_n \to \overline{\sigma}$ (E_n denotes the unit closed ball of dimension n) that maps the open ball isomorphically onto σ and the sphere onto $\partial\sigma$.

For $A = \varnothing$, we have an *absolute definable* CW-*complex* over R. All definable CW-complexes are weak polytopes (absolute or relative, see [**20**, V.7, page 165]). A *system of definable* CW-*complexes* is a system of spaces $(M, A_1, \dots, A_k)$ such that each A_i is a closed subcomplex of the definable CW-complex M (cf. [**20**, V.7, page 178]). Such a system is *decreasing* if A_i is a (closed) subcomplex of A_{i-1} for $i = 1, \dots, k$, where $A_0 = M$. As in the semialgebraic case, we have the following.

Example 53. Each partially complete object of RPLDS(R) admits a definable CW-complex structure over R, since it is isomorphic to a closed (geometric) locally finite simplicial complex. (Compare considerations of [**10**, II.4] and [**20**, V.7.1 ii)].)

Fact 33 and Example 53 give

Fact 54. *Each object of* RPLDS(R) *is homotopy equivalent to a definable* CW-*complex over R. Each system* $(M, A_1, \ldots, A_k)$ *of a regular paracompact locally definable space with closed subspaces is homotopy equivalent to a system of definable* CW-*complexes.*

The following version of the Whitehead theorem for definable CW-complexes may be proved like its topological analogue (see [**23**, Theorem 7.5.4]).

Theorem 55. *Each weak homotopy equivalence between definable* CW-*complexes is a homotopy equivalence. Similar facts hold for any decreasing systems of definable* CW-*complexes.*

Proof. The proof is analogous to the proofs of 7.5.2, 7.5.3 and 7.5.4 in [**23**]. The argument from the long exact homotopy sequence may be proved as in [**18**] (compare [**10**, III.6.1] and [**20**, V.6.6]). The second part of the thesis follows from the definable analogue of [**20**, V.2.13]. □

Using the above instead of Theorem V.6.10 of [**20**], we can both pass to a reduct and eliminate parameters.

Theorem 56 (cf. [**20**, V.7.10]). *Each definable* CW-*complex is homotopy equivalent to an expansion of a base field extension of a semialgebraic* CW-*complex over* $\overline{\mathbf{Q}}$. *Analogous facts hold for decreasing systems of definable* CW-*complexes.*

Proof. This follows from reasoning with relative CW-complexes analogous to the proof of V.7.10 in [**20**] (instead of the case of an elementary extension of real closed fields, we have the case of an o-minimal expansion of a real closed field). The construction of the desired relative CW-complex "skeleton by skeleton" is similar. Since we are dealing only with decreasing systems of definable CW-complexes, the use of V.6.10 of [**20**] (whose role is the transition from finite unions to any unions) may be replaced with the use of Theorem 55. □

Moreover, combining the above with the Comparison theorems gives an extension of [**20**, Remarks VI.1.3].

Corollary 57. *The homotopy categories of topological* CW*-complexes, semialgebraic* CW*-complexes over* (*the underlying field of*) R*, and definable* CW*-complexes over* R *are equivalent. Similar facts hold for decreasing systems of* CW*-complexes.*

9. The case of bounded o-minimal theories. Let T be an o-minimal complete theory extending RCF. We may assume that the theory is already Skolemized, so every 0-definable function is in the language and T has quantifier elimination. We can build models of T using the *definable closure* operation in some huge model (or, equivalently, using the notion of a *generated substructure* of a huge model for the chosen rich language). Taking a "primitive extension" generated by a single element t over a model R gives a new model $R\langle t\rangle$ of T determined up to isomorphism by the type this single element realizes over the former model R.

Such a T will be called *bounded* if the model $P\langle t\rangle$ has countable cofinality, where P is the prime model of T and t realizes $+\infty$ over P. This condition can be expressed in the following way: there is a (countable) sequence of 0-definable unary functions that is cofinal in the set of all 0-definable unary functions at $+\infty$ (this property does not depend on a model of T: notice that $P\langle t\rangle$ is cofinal in $R\langle t\rangle$, for any model R of T, if t realizes $+\infty$ over R). In particular, polynomially bounded theories are bounded.

Each bounded theory T has the following property: each model R has an elementary extension S such that both S and its "primitive extension" $S\langle t\rangle$, with t realizing $+\infty$ over S, have countable cofinality. (Take $S = R\langle t_1\rangle$, with t_1 realizing $+\infty$ over R). This allows, by the First comparison theorem, to extend many facts about weakly definable spaces over "nice" models to spaces over any model of T.

The following example may be extracted from the proof of Theorem IV.9.2 in [**20**]. It shows the importance of the boundedness assumption. (The role of the boundedness assumption may also be seen by considering Example IV.9.12 in [**20**].)

Example 58. Consider the standard closed m-dimensional simplex with one open proper face removed ($m \geq 2$), call this set A, as a definable subset of R^{m+1}. We want to introduce a partially complete space on the same set A. If R and $R\langle t\rangle$ have countable cofinality, then we can find a sequence of internal points tending to the barycenter of the removed face, and we can use a "cofinal at 0_+" sequence of unary functions tending (uniformly and monotonically) to the zero function to produce an increasing sequence $(P_n)_{n\in\mathbf{N}}$ of polytopes covering our set A and such that any polytope contained in A is contained in some P_n. Then $(P_n)_{n\in\mathbf{N}}$ is an exhaustion of a weak polytope with the underlying set A. The old space and the new space on A have the same polytopes. (Compare the proof of [**20**, Theorem IV.9.2.]) A similar construction can be made if several open proper faces are removed.

By reasoning similar to that of [**20**, V.7.8], we get

Theorem 59 (CW-approximation, cf. [**20**, V.7.14]). *If T is bounded, then each decreasing system of weakly definable spaces $(M_0,\ldots,M_r)$ over R has a* CW*-approximation (that is, a morphism $\phi : (P_0,\ldots,P_r) \to (M_0,\ldots,M_r)$ from a decreasing system of definable* CW*-complexes over R that is a homotopy equivalence of systems of spaces).*

The methods to obtain this theorem include the use (as in [**20**, IV.9-10]) of a so-called *partially complete core $P(M)$ of a weakly definable space M*, which is an analogue and generalization of the localization M_{loc} for locally complete paracompact locally definable spaces M, and a *partially proper core p_f of a morphism $f : M \to N$* of weakly definable spaces. (Note that it is sensible to ask for a partially complete core only if R has countable cofinality.) In particular, the Strong Whitehead theorem (cf. [**20**, V.6.10]), proved by methods of [**20**, IV.9-10 and V.4.7, V.4.13], guarantees the extension of relevant results to weakly definable spaces. Thus, the homotopy category of decreasing systems of weakly definable spaces over R is equivalent to its full (homotopy) subcategory of decreasing systems of definable CW-complexes over R (one uses an analogue of Theorem V.2.13 in [**20**]).

The following corollary is an extension of Corollary 57 in the bounded case.

Corollary 60. *If T is bounded, then the homotopy categories of weakly definable spaces (over any model R of T) and of topological, semialgebraic and definable* CW*-complexes are all equivalent. Similar facts hold for decreasing systems of spaces.*

Still the homotopy category of $\mathrm{WDS}(R)$ may possibly be richer in the non-bounded case.

10. Generalized homology and cohomology theories. Now we have the operation of taking the (reduced) suspension $SM = S^1 \wedge M$ on the category of pointed weak polytopes $\mathcal{P}^*(R)$ over R, and on its homotopy category $H\mathcal{P}^*(R)$ (cf. [**20**, VI.1]). This allows us to define analogues of so-called *complete generalized homology and cohomology theories*, known from the usual homotopy theory, just as in [**20**, VI.2 and VI.3]. (Such theories do not necessarily satisfy the *dimension axiom.*) Denote the category of Abelian groups by Ab. For a pair (M, A) of pointed weak polytopes, M/A will denote the quotient space of M by a closed space A, with the distinguished point being that obtained from A.

A *reduced cohomology theory* k^* over R is a sequence $(k^n)_{n\in\mathbf{Z}}$ of contravariant functors $k^n : H\mathcal{P}^*(R) \to Ab$ together with natural equivalences $s^n : k^{n+1} \circ S \leftrightsquigarrow k^n$ such that the following hold:

Exactness axiom. For each $n \in \mathbf{Z}$ and each pair of pointed weak polytopes (M, A), the sequence

$$k^n(M/A) \xrightarrow{p^*} k^n(M) \xrightarrow{i^*} k^n(A)$$

is exact.

Wedge axiom. For each $n \in \mathbf{Z}$ and each family $(M_\lambda)_{\lambda\in\Lambda}$ of pointed weak polytopes, the mapping

$$(i_\lambda)^* : k^n\Big(\bigvee_\lambda M_\lambda\Big) \longrightarrow \prod_\lambda k^n(M_\lambda)$$

is an isomorphism.

A *reduced homology theory* h_* over R is a sequence $(h_n)_{n\in\mathbf{Z}}$ of covariant functors $h_n : H\mathcal{P}^*(R) \to Ab$ together with natural equivalences $s_n : h_n \leftrightsquigarrow h_{n+1} \circ S$ such that the following hold:

Exactness axiom. For each $n \in \mathbf{Z}$ and each pair of pointed weak polytopes (M, A), the sequence

$$h_n(A) \xrightarrow{i_*} h_n(M) \xrightarrow{p_*} h_n(M/A)$$

is exact.

Wedge axiom. For each $n \in \mathbf{Z}$ and each family $(M_\lambda)_{\lambda \in \Lambda}$ of pointed weak polytopes, the mapping

$$(i_\lambda)_* : \bigoplus_\lambda h_n(M_\lambda) \longrightarrow h_n\Big(\bigvee_\lambda M_\lambda\Big)$$

is an isomorphism.

If T is bounded, then these theories correspond uniquely (up to an isomorphism) to topological theories (cf. [**20**, VI.2.12 and VI.3]). All of these generalized homology and cohomology functors can be built by using spectra for homology theories, or Ω-spectra for cohomology theories as in [**20**, VI.8].

Similarly, *unreduced* generalized *homology* and *cohomology* theories may be considered on the category $H\mathcal{P}(2, R)$ of pairs of weak polytopes. If T is bounded, then these theories are equivalent to respective reduced theories, cf. [**20**, VI.4]; homology theories are extendable to $H\mathrm{WDS}(2, R)$, and some difficulties appear for cohomology theories, cf. [**20**, VI.5-6]. We get the following extension of Corollaries 57 and 60.

Corollary 61. *If T is bounded, then, by the equivalence of respective homotopy categories of topological pointed* CW*-complexes* (*with continuous mappings*) *and of pointed weak polytopes, we get "the same" generalized homology and cohomology theories as the classical ones, known from the usual topological homotopy theory.*

11. Open problems. The following problems are still open:

1) Can the assumption of boundedness of T in Theorem 59 and later be omitted? Is there a way of proving the Strong Whitehead theorem (the analogue of [**20**, V.6.10]) without methods of [**20**, IV.9-10]?

2) Do the above considerations lead to a "(closed) model category" (see [**17**, page 109], for the definition)? Such categories are desired in (abstract) homotopy theory.

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