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GENERALIZED M^*-SIMPLE GROUPS

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ABSTRACT. Let X be a compact bordered Klein surface of algebraic genus $p \geq 2$, and let $G = \Gamma^*/\Lambda$ be a group of automorphisms of X where Γ^* is an NEC group and Λ is a bordered surface group. If the order of G is $4q/(q-2)(p-1)$, for $q \geq 3$ a prime number, then the signature of Γ^* is $(0; +; [-]; \{(2, 2, 2, q)\})$. These groups of automorphisms are called generalized M^*-groups. In this paper, we define generalized M^*-simple groups and give some examples of them. Also, we classify solvable generalized M^*-simple groups.

1. Introduction. A compact bordered Klein surface X of algebraic genus $p \geq 2$ admits at most $12(p-1)$ automorphisms [**10**]. Groups isomorphic to the automorphism group of such a compact bordered Klein surface with this maximal number of automorphisms are called *M^*-groups*. These groups were first introduced in [**11**], and have been studied in several papers ([**3–5, 9**]). Also, the survey article in [**5**] contains a nice survey of known results about M^*-groups.

An important result about M^*-groups is that they must have a certain partial presentation. This is established by considering an M^*-group as an epimorphic image of a quadrilateral group $\Gamma^*[2, 2, 2, 3]$. A quadrilateral group Γ^* is a non-Euclidean crystallographic (NEC) group with signature

$$(0; +; [-]; \{(2, 2, 2, 3)\}).$$

Also Γ^* is isomorphic to the abstract group with the presentation

$$\langle c_0, c_1, c_2, c_3 \mid c_i^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^3 = I\rangle.$$

For some bordered surface group Λ the group $G = \Gamma^*/\Lambda$ satisfies $|G| = 12(p-1)$ and there is a bordered smooth epimorphism $\theta : \Gamma^* \to G$

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which maps $c_0 \to r_1$, $c_1 \to I$, $c_2 \to r_2$ and $c_3 \to r_3$. It is clear that $\mathrm{Ker}(\theta) = \Lambda$. Thus, r_1r_2 and r_1r_3 have orders 2 and 3, respectively, and each group G admits the following partial presentation:

$$\langle r_1,\ r_2,\ r_3 \mid r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^3 = \cdots = I\rangle.$$

In [**12**, Proposition 2, page 223], May extended this in the following proposition to an extended quadrilateral group $\Gamma^*[2,2,2,n]$, $n \geq 3$, integer.

Proposition 1.1. *Let G be a finite group, and let $\Gamma^* = \Gamma^*[2,2,2,n]$ be an extended quadrilateral group. If there is a homomorphism $\phi : \Gamma^* \to G$ onto G such that $K = \ker\phi$ is a bordered surface group, then G is generated by three distinct nontrivial elements r_1, r_2 and r_3 satisfying the relations*

$$r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^n = I. \tag{1.1}$$

It is clear that the order of G is $[4n/(n-2)](p-1)$, where $p \geq 2$ is an integer, and the signature of Γ^* is $(0; +; [-]; \{(2,2,2,n)\})$.

In [**12**], for $n \geq 3$ a prime number, Sahin et al. referred to these finite groups, which were obtained by May in [**12**, Proposition 2, page 223], as *generalized M^*-groups.* A generalized M^*-group associated to $n \geq 3$ a prime number, is a finite group G generated by three distinct nontrivial elements r_1, r_2 and r_3 which satisfy the relations (1.1). Notice that, if $n = 3$, then generalized M^*-groups are M^*-groups.

On the other hand, in [**16, 17**] the extended Hecke group $\overline{H}(\lambda_q)$ has been defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$, where $q \geq 3$ is an integer, and it has been extensively studied (for examples, see [**1, 8**] and [**6**, page 70]). The extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_1R_3)^q = I\rangle,$$

where $R_1 = 1/\overline{z}$, $R_2 = -\overline{z}$ and $R_3 = -\overline{z} - \lambda_q$, $\lambda_q = 2\cos(\pi/q)$ and $q \geq 3$ is an integer. The signature of the extended Hecke group $\overline{H}(\lambda_q)$ is $(0; +; [-]; \{(2, q, \infty)\})$. Since the extended Hecke group $\overline{H}(\lambda_q)$ contain a reflection, it is an NEC group. Thus, the quotient space $\mathcal{U}/\overline{H}(\lambda_q)$ is a Klein surface and $\mathcal{U}/H(\lambda_q)$ is the canonical double cover of $\mathcal{U}/\overline{H}(\lambda_q)$ where $\mathcal{U}$ is the upper half-plane. If a bordered surface group Γ is a normal subgroup of finite index in $\overline{H}(\lambda_q)$, then $\overline{H}(\lambda_q)/\Gamma$ is a group of automorphisms of the compact bordered Klein surface $X = \mathcal{U}/\overline{H}(\lambda_q)$. Also, the automorphism groups G of order $[4q/(q-2)](p-1)$ which act on compact bordered Klein surfaces X of genus $p \geq 2$, are finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_q)$, where $q \geq 3$ is an integer. For example, the groups of orders $|G| = 12(p-1)$, $|G| = 8(p-1)$, $|G| = (20/3)(p-1)$, respectively, are the finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_3)$ (the extended modular group $\mathrm{PGL}\,(2,\mathbf{Z})$), $\overline{H}(\lambda_4)$ or $\overline{H}(\lambda_5)$ [**20**]. Here the orders of these groups are the highest three among the automorphism groups of the compact Klein surfaces of genus $p \geq 2$ (see [**12**, page 221, Proposition 1]).

In [**11**], May showed that there is a relationship which says a finite group of order at least 12 is an M^*-group if and only if it is a finite homomorphic image of the extended modular group.

It is easy to see that there is a relationship between extended quadrilateral groups $\Gamma^*[2,2,2,n]$, $n \geq 3$ integer and extended Hecke groups $\overline{H}(\lambda_q)$, $q \geq 3$ integer. Of course, there is a relationship between extended Hecke groups $\overline{H}(\lambda_q)$, $q \geq 3$ prime, and generalized M^*-groups (see the following diagram).

$$\begin{aligned} \overline{H}(\lambda_3) = PGL(2,\mathbf{Z}) &\longleftrightarrow M^*\text{-groups} \\ \overline{H}(\lambda_q) &\longleftrightarrow \text{generalized } M^*\text{-groups.} \end{aligned}$$

In fact, many results can be obtained by using these relations. As a consequence, in [**18**], Sahin et al. show that a finite group of order at least $4q$ is a generalized M^*-group if and only if it is the homomorphic image of the extended Hecke group $\overline{H}(\lambda_q)$. By using known results about normal subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$ given in [**19**], they obtained an infinite family of generalized M^*-groups. Also, using known results about commutator subgroups of $\overline{H}(\lambda_p)$, the authors obtained that if G is a generalized M^*-group, then $|G : G'|$ divides 4 and $|G' : G''|$ divides q^2. Finally, they proved that, if $q \geq 3$ is a prime number and G is a generalized M^*-group associated to q, then G is supersolvable if and only if $|G| = 4 \cdot q^r$ for some positive integer r.

In this paper, we define generalized M^*-simple groups and give some examples of them. Also, we classify solvable generalized M^*-simple groups.

2. Generalized M^*-simple groups. In this section, we want to define generalized M^*-simple groups in a manner analogous to M^*-simple groups, O^*-simple groups and $LO1$-simple groups for the automorphism groups of Klein and Riemann surfaces given by May et al. in [**9, 13, 14**], respectively. To do this, we need the following theorem first.

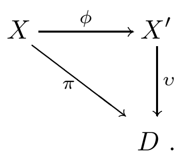
Theorem 2.1. *Let $q \geq 3$ be a prime number, and let G be a generalized M^*-group associated to an extended quadrilateral group* $\Gamma^*[2,2,2,q]$, *with genus action on the bordered Klein surface X of genus $g \geq 2$. Let N be a normal subgroup of G of index $r > 2q$. Set $G' = G/N$, $X' = X/N$, let $\phi : X \to X'$ be the quotient map, and let g' be the genus of X'. Then*:

(1) $g' \geq 2$;

(2) G' *is a generalized M^*-group*;

(3) ϕ *is a full covering.*

Proof. Firstly, we prove that $g' \geq 2$. The quotient space X/G is the disc D, and the quotient map $\pi : X \to D$ is ramified at four points, all of ∂D, with ramification indices $2, 2, 2, q$, where $q \geq 3$ is a prime number. From the induced action of $G' = G/N$ on $X' = X/N$, we have the following diagram of quotient maps.



Applying the Riemann-Hurwitz formula to the mapping ν yields

$$2g' - 2 = r\left[-2 + \sum\left(1 - \frac{1}{e_i}\right)\right],$$

where the e_i's are chosen from $\{2,\ 2,\ 2,\ q\}$. Then it is easy to see that there are no solutions for $g' = 0$ or 1, $r > 2q$. Thus, $g' \geq 2$.

Let G have generators r_1, r_2 and r_3 satisfying

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^2 = (r_1 r_3)^q = I,$$

and let $\mu : G \to G' = G/N$ be a natural quotient map since the index $r = [G : N] > 2q$. It is easy to see that r_1, r_2, r_3, $r_1 r_2$ and $r_1 r_3$ are not in N. Then G' is generated by $\mu(r_1)$, $\mu(r_2)$ and $\mu(r_3)$, and therefore G' is a generalized M^*-group.

Now, if we apply the Riemann-Hurwitz formula to the mapping ϕ, we find $2g' - 2 = |G'|(q-2)/8q$ since $|G'| = |G/N|$ is the index in the NEC-group Γ^* of the obvious smooth epimorphism from Γ^* to G'. Then $|G'| = [4q/(q-2)](g'-1)$. Therefore, ϕ is unramified. □

This theorem leads to the following notion.

Definition 2.1. A generalized M^*-group is called a generalized M^*-simple group if it has no non-trivial normal subgroups of index greater than $2q$, or equivalently, if it has no proper generalized M^*-quotient groups.

If a generalized M^*-group G has a quotient group of order $2q$ or less (the possibilities being the trivial group, C_2, $C_2 \times C_2$, C_q and $C_2 \times C_q$), then these quotient groups are not generalized M^*-groups.

On the other hand, a simple generalized M^*-group is a generalized M^*-simple group. Thus, if G is a simple generalized M^*-group, then G acts only on non-orientable surfaces since otherwise the orientation-preserving maps would be subgroup of index 2 in G [**10**, page 206].

Now we give some examples related to generalized M^*-simple groups.

Example 2.1. Let q be an odd prime. The group $C_2 \times D_q \cong D_{2q}$ is the smallest generalized M^*-simple group. Let $D_{2q} = \langle A, B \mid A^2 = B^2 = (AB)^{2q} = I \rangle$. If we choose $r_1 = (AB)^s A$, $r_2 = B(AB)^s$ and $r_3 = B(AB)^t$, where $s = (q-1)/2$ and $t = (q-3)/2$, then D_{2q} has the following relations:

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^2 = (r_1 r_3)^q = (r_2 r_3)^{2q} = (r_1 r_2 r_3)^2 = I.$$

Thus it is easy to check that the group D_{2q} is a generalized M^*-simple group.

Example 2.2. Let q be an odd prime. Let $G^{q,n,r}$ be the group with generators A, B and C and defining relations:

$$A^q = B^n = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = I.$$

If we set $r_1 = BC$, $r_2 = CA$ and $r_3 = BCA$, then we obtain the presentation:

$$r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^q = (r_2r_3)^n = (r_1r_2r_3)^r = I.$$

Thus, G is a quotient of $\Gamma^*[2,2,2,q]$ by a bordered surface group if and only if G is a quotient of the group $G^{q,n,r}$ for some n and r. If $q \geq 3$ prime and the group is finite, then we obtain a generalized M^*-group with index n. Some values of n and r which make the group finite are given in [**7**]. It is clear that, if (q', n', r') is any permutation of (q, n, r), then $G^{q',n',r'}$ is isomorphic to $G^{q,n,r}$. For $q = 7$, the groups $G^{7,9,3}$ and $G^{7,12,3}$ are generalized M^*-simple groups.

Example 2.3. Let q be an odd prime. The group PSL$(2, q)$ when $q \equiv 1 \pmod 4$ and the group PGL$(2, q)$ when $q \equiv 3 \pmod 4$ have the following presentation:

$$A^3 = B^{n(q)} = C^q = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = I,$$

where $n(q)$ is the ordinal of the first Fibonacci number that is divisible by q ([**3**, page 54] and [**9**, page 277]). Thus, for the above values of q, the groups PSL$(2, q)$ and PGL$(2, q)$ are generalized M^*-simple groups.

It is hard to classify all generalized M^*-simple groups, or even simple generalized M^*-groups. In fact, classifying which simple groups are generalized M^*-groups is equivalent to classifying which simple groups are quotients of the extended Hecke groups $\overline{H}(\lambda_q)$. But to study these groups is very difficult. Here we can only classify solvable generalized M^*-simple groups.

Theorem 2.2. *Let $q \geq 3$ be a prime number, and let G be a solvable generalized M^*-simple group associated to q.*

i) *If $q = 3$, then $G \cong C_2 \times S_3$ or $G \cong S_4$;*

ii) *If $q > 3$ then $G \cong C_2 \times D_q$.*

Proof. i) Please see the proof in [**9**, Theorem 15, page 278].

ii) Let $q > 3$ be a prime number. If we check the groups of order $4qs$ where $s = 1, 2, \dots, q$, then we obtain that the two smallest solvable generalized M^*-groups are $C_2 \times D_q$ and $D_q \times D_q$ of orders $4q$ and $4q^2$, respectively. But the group $D_q \times D_q$ is not a generalized M^*-simple group. Then we assume that G is a solvable generalized M^*-group with $o(G) > 4q^2$. If we show that G has a generalized M^*-quotient group, then the proof will be complete.

We know that G is solvable. Then $G \neq G'$. Therefore, using Corollary 11 in [**18**, page 1215], we find $[G : G''] \geq 2q$. Also, since $o(G) > 4q^2$, we have $G'' \neq 1$. Now we consider two cases. If $[G : G''] > 2q$, then using Theorem 2.1, we obtain that the quotient group G/G'' is a generalized M^*-group. Now we consider the case $[G : G''] = 2q$, that is, $G/G'' \cong D_q$. If G'' were a minimal normal subgroup of G, then, from [**15**, Theorem 5.24, page 105], G'' would be an elementary Abelian 2-group. Also, G'' has no elements with order larger than $2q$. Thus, G'' is a quotient group of a group $G^{q,m,n}$, where $m \leq n \leq 2q$. But if we check the table in [**7**, pages 138–139] for each $q > 3$, then we see that there is no such generalized M^*-group G such that $[G : G''] = 2q$ and $o(G) > 4q^2$. Hence, G'' contains a minimal normal subgroup N of G with $[G : N] > [G : G''] = 2q$. Therefore, the quotient group G/N is a generalized M^*-group. $\square$

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