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SOME CRITERIA FOR $C_p(X)$ TO BE AN $L\Sigma(\leq \omega)$-SPACE

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ABSTRACT. Given a cardinal κ say that X is an $L\Sigma(< \kappa)$-space ($L\Sigma(\leq \kappa)$-space) if X has a countable network $\mathcal{F}$ with respect to a cover $\mathcal{C}$ of X by compact subspaces of weight strictly less than κ (less than or equal to κ, respectively), i.e., given any $C \in \mathcal{C}$, we have $w(C) < \kappa$ ($w(C) \leq \kappa$) and, for any $U \in \tau(X)$ with $C \subset U$, there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. These concepts were introduced and studied by Kubiś, Okunev and Szeptycki. We show that if $C_p(X)$ is a Lindelöf Σ-space and $|X| \leq \mathfrak{c}$, then $C_p(X)$ is an $L\Sigma(\leq \omega)$-space. This answers two questions of Kubiś, Okunev and Szeptycki. We also prove that if X is a space and $C_p(X)$ has the $L\Sigma(< \omega)$-property, then X is cosmic, i.e., $nw(X) \leq \omega$. This answers (in a stronger form) a question of Okunev published in Open Problems in Topology II.

0. Introduction. Lindelöf Σ-spaces constitute the smallest class which contains all compact spaces, all second countable spaces and is invariant under continuous images, closed subspaces and finite products. This explains why the Lindelöf Σ-property is so important in topology, functional analysis and descriptive set theory. One of a dozen equivalent definitions says that X is a Lindelöf Σ-space if and only if there exists a second countable space M and an upper semicontinuous compact-valued onto map $\varphi : M \to X$.

Given a class $\mathcal{K}$ of compact spaces, Kubiś, Okunev and Szeptycki introduced and studied in [**5**] the class $L\Sigma(\mathcal{K})$ of spaces X for which there exists a second countable space M and an upper semicontinuous onto map $\varphi : M \to X$ such that $\varphi(x)$ belongs to the class $\mathcal{K}$ for any $x \in M$. Let κ be a (not necessarily infinite) cardinal. If $\mathcal{K}$ consists of compact spaces of weight at most κ (or strictly less than κ, respectively)

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then the class $L\Sigma(\mathcal{K})$ is denoted in [**5**] by $L\Sigma(\leq \kappa)$ (or by $L\Sigma(< \kappa)$, respectively).

Compact spaces from the class $L\Sigma(\leq \omega)$ were studied (under a different name) by Tkachuk in [**11**] and Tkachenko in [**10**]; the paper [**5**] presents answers to some questions formulated in [**10, 11**]. Kubiś, Okunev and Szeptycki proved, in particular, that not every Corson compact space of weight ω_1 belongs to the class $L\Sigma(\leq \omega)$ and that all elements of an important subclass of the class of compact $L\Sigma(\leq \omega)$-spaces have a dense metrizable subspace.

In the paper [**9**] published in *Open problems in topology* II, Okunev outlined the current progress and some lines of research in the study of $L\Sigma(\leq \kappa)$-spaces and $L\Sigma(< \kappa)$-spaces. It is one of the open questions of the paper [**9**] whether a space X must be cosmic if $C_p(X)$ belongs to the class $L\Sigma(\leq n)$ for some $n \in \omega$. We prove that, if $C_p(X)$ is an $L\Sigma(< \omega)$-space, then X is cosmic giving thus a positive answer (in a stronger form) to Problem 5 from [**9**].

Therefore, the next natural step is to study for which X the space $C_p(X)$ has the $L\Sigma(\leq \omega)$-property. Molina Lara and Okunev proved in [**6**] that every Gul'ko compact space of cardinality at most $\mathfrak{c}$ is an $L\Sigma(\leq \omega)$-space and established that, for any Eberlein compact space X such that $|X| \leq \mathfrak{c}$, the space $C_p(X)$ has the $L\Sigma(\leq \omega)$-property. We prove that, if X is a Lindelöf Σ-space with the unique non-isolated point and $|X| \leq \mathfrak{c}$, then $C_p(X)$ is a Lindelöf Σ-space. This gives a positive answer to Problem 3.5 of [**6**]. We also show that, for any space X for which $C_p(X)$ has the Lindelöf Σ-property, $C_p(X)$ is a $L\Sigma(\leq \omega)$-space if and only if $|X| \leq \mathfrak{c}$. This result and its consequences give a positive answer to Problems 4.5 and 4.6 from [**6**].

1. Notation and terminology. All spaces in this paper are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any $x \in X$; if $A \subset X$, then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. The unexplained notions can be found in [**1, 3**]; the definitions of cardinal invariants can be consulted in the survey of Hodel [**4**].

All ordinals are identified with the set of their predecessors and are assumed to carry the interval topology. As usual, $\mathbf{R}$ is the set of reals, $\mathbf{I} = [0, 1] \subset \mathbf{R}$ and $\mathbf{D}$ is the doubleton $\{0, 1\}$ with the discrete topology. For any infinite cardinal κ the space $A(\kappa)$ is the one-point compactification of the discrete space of cardinality κ. We use the symbol $\mathfrak{c}$ to denote the cardinality of the continuum, i.e., $\mathfrak{c} = 2^\omega$. The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic; we denote by υX the Hewitt realcompactification of the space X.

For any spaces X and Y the set $C(X, Y)$ consists of continuous functions from X to Y; if it has the topology induced from Y^X then the respective space is denoted by $C_p(X, Y)$. We write $C(X)$ instead of $C(X, \mathbf{R})$ and $C_p(X)$ instead of $C_p(X, \mathbf{R})$. Given a space X, let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$, i.e., $C_{p,n}(X)$ is the nth iterated function space of X.

If $\mathcal{A}$ is a family of sets of a space X, then $\wedge\mathcal{A}$ is the family of all finite intersections of the elements of $\mathcal{A}$ and $\mathcal{A} \mid Y = \{A \cap Y : A \in \mathcal{A}\}$ for any $Y \subset X$. Any map φ from a space X to the family $\exp(Y)$ of subsets of Y is called multivalued; we follow the usual practice writing $\varphi : X \to Y$ instead of $\varphi : X \to \exp(Y)$. A multi-valued map $\varphi : X \to Y$ is called compact-valued (finite-valued) if the set $\varphi(x)$ is compact (finite) for any $x \in X$. If $\varphi : X \to Y$ is a multi-valued map, then $\varphi(A) = \cup\{\varphi(x) : x \in A\}$ for any $A \subset X$; we say that *the map φ is onto* if $\varphi(X) = Y$. A multi-valued map $\varphi : X \to Y$ is called *upper semi-continuous* if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$ is open in X for any $U \in \tau(Y)$. We will often use the following characterization of Lindelöf Σ-spaces: a space X is Lindelöf Σ if and only if there exists a countable family $\mathcal{F}$ of subsets of X and a cover $\mathcal{C}$ of the space X such that every $C \in \mathcal{C}$ is compact and for any $U \in \tau(C, X)$ we can find a set $F \in \mathcal{F}$ with $C \subset F \subset U$. The family $\mathcal{F}$ is usually called a network with respect to the compact cover $\mathcal{C}$. If, in the characterization above, we require that all elements of the cover $\mathcal{C}$ belong to a given class $\mathcal{K}$, then we obtain an equivalent definition for the class $L\Sigma(\mathcal{K})$.

If X is a space then the expression $\dim X = 0$ says that every finite open cover of X has a disjoint open refinement. The spaces X with $\dim X = 0$ are also called *strongly zero-dimensional.* If X has a clopen base then it is called *zero-dimensional*; this is also denoted as $\operatorname{ind} X = 0$. A space X is called *simple* if it has at most one non-isolated point. A map $f : X \to Y$ is called *a condensation* if it is a continuous bijection; in this case we say that X *condenses onto* Y. If X condenses onto a subspace of Y, we say that X *condenses into* Y. If $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of X, then $\mathcal{A}$ is *a network with respect to* $\mathcal{B}$ if, for any $B \in \mathcal{B}$

and $U \in \tau(B, X)$, there exists an $A \in \mathcal{A}$ such that $B \subset A \subset U$. A family $\mathcal{N}$ is a network of X if every $U \in \tau(X)$ is the union of some subfamily of $\mathcal{N}$. The spaces with a countable network are called *cosmic*.

2. $L\Sigma(\mathcal{K})$**-properties in function spaces.** We will study the spaces X for which $C_p(X)$ has the $L\Sigma(\leq \omega)$-property; they are not necessarily cosmic. However, we will prove that, for any X, if $C_p(X)$ is an $L\Sigma(< \omega)$-space then X is cosmic.

2.1. Proposition. *Given a space X and a class $\mathcal{K}$ of compact spaces, $C_p(X)$ belongs to $L\Sigma(\mathcal{K})$ if and only if $C_p(\upsilon X)$ belongs to $L\Sigma(\mathcal{K})$.*

Proof. Let $\pi : C_p(\upsilon X) \to C_p(X)$ be the restriction map. The class $L\Sigma(\mathcal{K})$ is preserved by condensations; since π condenses $C_p(\upsilon X)$ onto $C_p(X)$, if $C_p(\upsilon X)$ is an $L\Sigma(\mathcal{K})$-space then so is $C_p(X)$.

Now assume that $C_p(X)$ is an $L\Sigma(\mathcal{K})$-space and fix a cover $\mathcal{C} \subset \mathcal{K}$ of the space $C_p(X)$ together with a countable network $\mathcal{N}$ with respect to the family $\mathcal{C}$; we can consider that $\mathcal{N}$ is closed under finite intersections. The space υX being Lindelöf Σ by [**8**, Theorem 3.5], we can apply [**14**, Theorem 2.6] to convince ourselves that $\pi^{-1}(K)$ is compact for any $K \in \mathcal{C}$ so $\pi|\pi^{-1}(K) : \pi^{-1}(K) \to K$ is a homeomorphism and hence $\mathcal{C}' = \{\pi^{-1}(K) : K \in \mathcal{C}\}$ is a compact cover of $C_p(\upsilon X)$ with $\mathcal{C}' \subset \mathcal{K}$.

Let $\mathcal{N}' = \{\pi^{-1}(N) : N \in \mathcal{N}\}$; it suffices to show that $\mathcal{N}'$ is a network with respect to $\mathcal{C}'$. Suppose not; then we can find a set $K \in \mathcal{C}$ and $U \in \tau(\pi^{-1}(K), C_p(\upsilon X))$ such that $N' \backslash U \neq \varepsilon$ for any $N' \in \mathcal{N}'$ with $K' = \pi^{-1}(K) \subset N'$. Consider the family $\mathcal{F} = \{N \in \mathcal{N} : K \subset N\}$ and choose an enumeration $\{F_n : n \in \omega\}$ of $\mathcal{F}$. By our choice of K there exists a point $z_n \in \pi^{-1}(F_n)\backslash U$ for every $n \in \omega$. Therefore $y_n = \pi(z_n) \in F_n$ for each $n \in \omega$; the family $\mathcal{F}$ being an outer network for K, for the set $Q = \{y_n : n \in \omega\}$ we can find a point $a \in \overline{Q} \cap K$; let $b = \pi^{-1}(a)$. If $P = \{z_n : n \in \omega\}$, then $P \subset C_p(\upsilon X)\backslash U$ and hence $\overline{P} \cap K' = \varnothing$ and, in particular, $b \notin \overline{P}$. However, the map $\pi \mid (P \cup \{b\}) : P \cup \{b\} \to Q \cup \{a\}$ is a homeomorphism by [**7**, Theorem 1] so we obtained a contradiction which shows that $\mathcal{N}'$ is a countable network with respect to the cover $\mathcal{C}' \subset \mathcal{K}$. $\square$

The following theorem gives a positive answer to Problem 5 from [**9**].

2.2. Theorem. *If $C_p(X)$ is an $L\Sigma(<\omega)$-space then X is cosmic.*

Proof. Say that a space X is *dubious* if it is not cosmic and $C_p(X)$ has the $L\Sigma(<\omega)$-property; striving for a contradiction, assume that there exists a dubious space X. Then $C_p(\upsilon X)$ is also an $L\Sigma(<\omega)$-space by Proposition 2.1 so it follows from the inequalities $nw(\upsilon X) \geq nw(X) > \omega$ that the space υX is also dubious. Therefore, we can assume, without loss of generality, that $X = \upsilon X$ and hence X is a Lindelöf Σ-space.

Lindelöf Σ-property of $C_p(X)$ and $s(X) \leq \omega$ imply that the space X is cosmic (see [**13**, Theorem 3.6]) so we can find a discrete subspace $D \subset X$ with $|D| = \omega_1$. If $Y = \overline{D}$, then normality of X implies that $C_p(Y)$ is a continuous image of $C_p(X)$, and hence $C_p(Y)$ is also an $L\Sigma(<\omega)$-space, i.e., Y is also dubious. Let Z be the space obtained from Y by collapsing the set $Y \backslash D$ of non-isolated points of Y to a point. If $q : Y \to Z$ is the respective quotient map, then the dual map $q^* : C_p(Z) \to C_p(Y)$ is an embedding of $C_p(Z)$ onto a closed subspace of $C_p(X)$ (recall that $q^*(f) = f \circ q$ for any $f \in C_p(Z)$). As a consequence, $C_p(Z)$ must also be an $L\Sigma(<\omega)$-space which shows that Z is a dubious space as well.

Let a be the unique non-isolated point of Z. Consider the topology ν on the set Z such that $Z' = (Z, \nu)$ is the one-point compactification of the discrete space D and a is the unique non-isolated point of the space Z'. It is clear that the identity map $j : Z \to Z'$ is continuous and hence $C_p(Z')$ embeds in $C_p(Z)$. Observe that $C_p(Z')$ is homeomorphic to the space $\Sigma_*(\omega_1) = \{x \in \mathbf{R}^{\omega_1} :$ for any $\varepsilon > 0$ the set $\{\alpha < \omega_1 : |x(\alpha)| > \varepsilon\}$ is finite$\}$ which is universal for all Eberlein compact spaces of weight $\leq \omega_1$.

In particular, $A(\omega_1)^\omega$ embeds in $C_p(Z')$ and hence in $C_p(Z)$; thus, $A(\omega_1)^\omega$ must be an $L\Sigma(<\omega)$-space. However, Theorem 4.11 of [**5**] says that, if E is a space and E^ω has the $L\Sigma(<\omega)$-property , then X^ω is hereditarily separable. Since $A(\omega_1)^\omega$ is not even separable, we obtained a contradiction. $\square$

2.3. Theorem. *Suppose that X is a space with $|X| \leq \mathfrak{c}$ and $a \in X$ is the unique non-isolated point of X. Then the following conditions are equivalent*:

(i) *X is a Lindelöf Σ-space*;

(ii) *X is an $L\Sigma(\leq 2)$-space*;

(iii) *there exists a separable metric topology μ on the set $D = X\backslash\{a\}$ such that $U \cap D \in \mu$ for any $U \in \tau(a, X)$.*

Proof. Assume that X is a Lindelöf Σ-space, and fix a compact cover $\mathcal{C}$ of the space X for which there exists a closed countable network $\mathcal{N}$ with respect to $\mathcal{C}$; let $\mathcal{M} = \{N \cap D : N \in \mathcal{N}\}$. Take a base $\mathcal{B}$ for a separable metric topology on D and consider the topology μ on the set D for which the family $\wedge(\mathcal{M} \cup \mathcal{B})$ is a clopen base. It is clear that μ is second countable; take any set $U \in \tau(a, X)$. If $x \in U \cap D$, then there exists a set $C \in \mathcal{C}$ with $x \in C$. The set $H = C\backslash U$ is finite being compact and discrete. Observe that $U \cup H$ is an open neighborhood of C, and hence we can find a set $N \in \mathcal{N}$ such that $C \subset N \subset U \cup H$. The set $N \cap D$ is open in (D, μ), and hence so is the set $M = N\backslash H$. Since $x \in M \subset U \cap D$, we proved that every point $x \in U \cap D$ has a μ-neighbourhood contained in $U \cap D$. Thus, $U \cap D \in \mu$ for any $U \in \tau(a, X)$ and hence we proved that (i) $\Rightarrow$ (iii).

The implication (ii) $\Rightarrow$ (i) being evident, assume that (iii) holds and take a separable metric topology μ on D as in (iii). Denote by Z the space (D, μ) and let $\varphi(z) = \{z, a\}$ for any $z \in Z$. Then $\varphi : Z \to X$ is a two-valued onto map. Suppose that $z \in Z$ and we are given $U \in \tau(X)$ with $\varphi(z) \subset U$. Then $V = U \cap D \in \tau(Z)$ and $z \in V$. We have $\varphi(V) = V \cup \{a\} \subset U$, and hence φ is upper semi-continuous so X is an $L\Sigma(\leq 2)$-space. □

2.4. *Remark.* The statement (i) $\Rightarrow$ (ii) was proved (by a different method) in a paper of Molina Lara and Okunev (see [**6**, Corollary 2.4]).

2.5. Proposition. *If X is a space with a unique non-isolated point and $C_p(X)$ is a Lindelöf Σ-space, then X is also a Lindelöf Σ-space.*

Proof. The space X being normal, we have $\operatorname{ext}(X) \leq \omega$; since X embeds in $C_p(C_p(X))$, Baturov's theorem [**2**] is applicable so the space X is Lindelöf and hence realcompact. Now, apply [**8**, Theorem 3.5] to see that $X = \upsilon X$ a Lindelöf Σ-space. □

The following result on a universal covering of $\mathbf{I}$ is well known for compact spaces (see, e.g., [**1**, Lemma IV.3.7]); however, it turns out that compactness can be omitted if we assume that X is strongly zero-dimensional.

2.6. Theorem. *If X is a space such that* $\dim X = 0$, *then* $C_p(X, \mathbf{I})$ *is a continuous image of the space* $C_p(X, \mathbf{D}^\omega)$.

Proof. It follows from $\dim X = 0$ that βX is a zero-dimensional compact space so we can apply [**1**, Lemma IV.3.6] to find a continuous map $r : \mathbf{D}^\omega \to \mathbf{I}$ such that, for any $g \in C(\beta X, \mathbf{I})$ there exists a continuous function $u_g : \beta X \to \mathbf{D}^\omega$ for which we have $g = r \circ u_g$.

For any function $h \in C_p(X, \mathbf{D}^\omega)$, let $\varphi(h) = r \circ h$; it is standard to prove that the map $\varphi : C_p(X, \mathbf{D}^\omega) \to C_p(X, \mathbf{I})$ is continuous so it suffices to show that we have the equality $\varphi(C_p(X, \mathbf{D}^\omega)) = C_p(X, \mathbf{I})$. Take any $f \in C_p(X, \mathbf{I})$; there exists $g \in C_p(\beta X, \mathbf{I})$ with $g|X = f$. Then $h = u_g|X \in C_p(X, \mathbf{D}^\omega)$; given any point $x \in X$, we have $f(x) = g(x) = r(u_g(x)) = r(h(x))$ which shows that $f = r \circ h$, i.e., $f = \varphi(h)$ so the map φ is surjective. □

The following theorem answers positively Problem 3.5 from the paper [**6**]. The equivalence of (i) and (ii) is established in [**6**, Corollary 3.4] by a different method.

2.7. Theorem. *Let X be a space with a unique non-isolated point such that $|X| \leq \mathfrak{c}$. Then the following conditions are equivalent*:

(i) $C_p(X)$ *is a* $L\Sigma(\leq \omega)$*-space*;

(ii) $C_p(X)$ *is a Lindelöf* Σ*-space*;

(iii) X *is a Lindelöf* Σ*-space.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) is a consequence of Proposition 2.5 so assume that X is a Lindelöf Σ-space and denote by a the unique non-isolated point of X. Our first step is to establish that the space $C_p(X, \mathbf{D})$ has the $L\Sigma(\leq \omega)$-property. To that end, consider the sets $Q = \{f \in C_p(X, \mathbf{D}) : f(a) = 0\}$ and $D = X \backslash \{a\}$; the space $E = \{f \in \mathbf{D}^X : f(a) = 0\}$ is easily seen to be a compactification of Q.

By Theorem 2.3, there exists a separable metrizable topology μ on the set D such that $U \cap D \in \mu$ for any $U \in \tau(a, X)$; fix a base $\mathcal{B} = \{B_n : n \in \omega\}$ of the space (D, μ). The set $K_n = \{f \in \mathbf{D}^X : f(B_n) \subset \{0\}\}$ is compact being homeomorphic to $\mathbf{D}^{X \backslash B_n}$ for any $n \in \omega$.

Take any functions $f \in Q$ and $g \in E\backslash Q$; since g is discontinuous at a, there exists a point $x \in D \cap f^{-1}(0) \cap g^{-1}(1)$. The set $D \cap f^{-1}(0)$ being open in (D, μ), we can find $n \in \omega$ such that $x \in B_n \subset D \cap f^{-1}(0)$. It is immediate that $f \in K_n$ and $g \notin K_n$ and hence the family $\mathcal{K} = \{K_n : n \in \omega\}$ separates the points of Q from the points of $E\backslash Q$.

Given an arbitrary function $f \in Q$ the set $K_f = \cap\{K_n : f \in K_n\} \subset Q$ is compact. If $g \in K_f$ and $B_n \subset U = f^{-1}(0)$ then $g(B_n) \subset \{0\}$. The family $\mathcal{B}$ being a base of (D, μ), it follows from $U \in \mu$ that $U = \cup\{B_n : B_n \subset U\}$; as an immediate consequence, we have $g(U) \subset \{0\}$. This shows that we have the inclusions $K_f \subset \{g \in Q : g(U) \subset \{0\}\} \subset \{h\} \times \mathbf{D}^{X\backslash U}$ where $h \in \mathbf{D}^U$ is the function which is identically zero on U. Since $X\backslash U$ is countable, the set $\{h\} \times \mathbf{D}^{X\backslash U} \simeq \mathbf{D}^{X\backslash U}$ is second countable and hence so is K_f. It is standard that the family $\wedge\mathcal{K}$ is a network with respect to K_f for any $f \in Q$. Therefore the countable family $\mathcal{N} = (\wedge\mathcal{K}) \mid Q$ is a network with respect to the compact cover $\{K_f : f \in Q\}$ of the space Q. We already saw that every K_f is second countable; this proves that Q is an $L\Sigma(\leq \omega)$-space.

Since $C_p(X, \mathbf{D})$ is a union of two subspaces homeomorphic to Q, the space $C_p(X, \mathbf{D})$ also has the $L\Sigma(\leq \omega)$-property. Recall that the space X is Lindelöf Σ; being zero-dimensional, it embeds in $C_p(C_p(X, \mathbf{D}))$ which, together with Okunev's theorem [**8**, Corollary 2.11] implies that $C_p(X)$ is a Lindelöf Σ-space.

It is noted in [**6**] (and is easy to prove) that $L\Sigma(\leq \omega)$-property is preserved by countable products so $C_p(X, \mathbf{D})^\omega \simeq C_p(X, \mathbf{D}^\omega)$ is an $L\Sigma(\leq \omega)$-space. Any space with a unique non-isolated point is strongly zero-dimensional and hence $\dim X = 0$. Applying Theorem 2.6, we conclude that $C_p(X, \mathbf{I})$ is a continuous image of $C_p(X, \mathbf{D}^\omega)$ so $C_p(X, \mathbf{I})$ is an $L\Sigma(\leq \omega)$-space. Finally, observe that $C_p(X)$ embeds in $C_p(X, \mathbf{I})$ so we can apply a result of Molina Lara and Okunev (see [**6**, Lemma 2.3]) to see that $C_p(X)$ is an $L\Sigma(\leq \omega)$-space. $\square$

2.8. Proposition. *If X is a space such that $|X| \leq \mathfrak{c}$ and $C_p(X)$ is a Lindelöf Σ-space, then $|C_p(X)| \leq \mathfrak{c}$, $|C_p(\upsilon X)| \leq \mathfrak{c}$ and $|\upsilon X| \leq \mathfrak{c}$.*

Proof. Let $\pi : C_p(\upsilon X) \to C_p(X)$ be the restriction map; observe that both υX and $C_p(\upsilon X)$ are Lindelöf Σ-spaces by [**8**, Theorem 3.5] and [**12**, Theorem 2.3]. For any compact set $K \subset C_p(X)$, we have $w(K) \leq w(C_p(X)) \leq \mathfrak{c}$; since K is Gul'ko compact and hence Fréchet-Urysohn, we obtain the inequality $|K| \leq \mathfrak{c}$. Therefore, every compact subset of the space $C_p(X)$ has cardinality not exceeding $\mathfrak{c}$. Now if $K' \subset C_p(\upsilon X)$ is compact then $K = \pi(K')$ is a compact subset of $C_p(X)$ so $|K'| = |K| \leq \mathfrak{c}$; this shows that all compact subsets of $C_p(\upsilon X)$ also have cardinalities not exceeding $\mathfrak{c}$. Every Lindelöf Σ-space is the union of at most $\mathfrak{c}$-many compact subspaces so $|C_p(\upsilon X)| \leq \mathfrak{c}$ and $|C_p(X)| \leq \mathfrak{c}$. Finally, if K is a compact subset of υX, then $nw(K) \leq nw(\upsilon X) = nw(C_p(\upsilon X)) \leq \mathfrak{c}$. The space K is Fréchet-Urysohn being Gul'ko compact so $|K| \leq \mathfrak{c}$. Thus, all compact subspaces of υX have cardinality at most $\mathfrak{c}$ so $|\upsilon X| \leq \mathfrak{c}$ because υX is a Lindelöf Σ-space. $\square$

2.9. Proposition. *Suppose that Y is a Lindelöf Σ-space and $|Y| \leq \mathfrak{c}$. If $C_p(Y)$ is a Lindelöf Σ-space, then Y can be condensed into $C_p(T)$ for some simple Lindelöf Σ-space T with $|T| \leq \mathfrak{c}$.*

Proof. Tkachuk proved in [**15**, Theorem 4.11] that there exists a closed simple subspace $T \subset C_p(Y)$ which separates the points of Y. Since T is closed in $C_p(Y)$, it has the Lindelöf Σ-property. Let $\varphi(x)(f) = f(x)$ for any $x \in Y$ and $f \in T$. Then $\varphi(x)$ is a continuous function on T and $\varphi : Y \to C_p(T)$ is a condensation. Finally observe that $|T| \leq |C_p(Y)| \leq \mathfrak{c}$ by Proposition 2.8. $\square$

The following result gives a positive answer (in a stronger form) to Problem 4.5 from [**6**].

2.10. Theorem. *Suppose that X is an arbitrary space and $C_p(X)$ has the Lindelöf Σ-property. Then $C_p(X)$ is an $L\Sigma(\leq \omega)$-space if and only if $|X| \leq \mathfrak{c}$.*

Proof. If $C_p(X)$ is an $L\Sigma(\leq\omega)$-space, then it is easy to see that $|C_p(X)| \leq \mathfrak{c}$. The restriction map condenses the space $C_p(\upsilon X)$ onto $C_p(X)$ so $|C_p(\upsilon X)| \leq \mathfrak{c}$, and hence $nw(\upsilon X) = nw(C_p(\upsilon X)) \leq |C_p(\upsilon X)| \leq \mathfrak{c}$. Given a compact subspace $K \subset \upsilon X$ we have $nw(K) \leq \mathfrak{c}$; since K is a Gul'ko compact and hence Fréchet-Urysohn, we conclude that $|K| \leq \mathfrak{c}$. Every Lindelöf Σ-space is the union of at most $\mathfrak{c}$-many compact subspaces so $|\upsilon X| \leq \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. Consequently, $|X| \leq |\upsilon X| \leq \mathfrak{c}$, so we proved necessity.

Now assume that $|X| \leq \mathfrak{c}$ and $C_p(X)$ is a Lindelöf Σ-space; we will need the restriction map $\pi : C_p(\upsilon X) \to C_p(X)$. Since $L\Sigma(\leq \omega)$-property is preserved by continuous maps, it suffices to show that $C_p(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space. Observe first that both υX and $C_p(\upsilon X)$ are Lindelöf Σ-spaces (see [**8**, Theorem 3.5] and [**2**, Theorem 2.3]).

It follows from Proposition 2.8 that $|\upsilon X| \leq \mathfrak{c}$ and $|C_p(\upsilon X)| \leq \mathfrak{c}$. Apply Okunev's result [**8**, Corollary 2.11] to see that $C_p(C_p(\upsilon X))$ is a Lindelöf Σ-space. It follows from Proposition 2.9 (applied for $Y = C_p(\upsilon X)$) that there exists a simple Lindelöf Σ-space Z and a condensation of $C_p(\upsilon X)$ into $C_p(Z)$. If Z is countable, then $nw(C_p(\upsilon X)) \leq \omega$, and hence $C_p(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space being a continuous image of a second countable space. If Z is uncountable then it follows from the Lindelöf property of Z that it must have a non-isolated point so Theorem 2.7 is applicable to see that $C_p(Z)$ is an $L\Sigma(\leq \omega)$-space; now a result of Molina Lara and Okunev (see [**6**, Lemma 2.3]) implies that $C_p(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space. □

2.11. Corollary. *If $|X| \leq \mathfrak{c}$ and $C_p(X)$ is a Lindelöf Σ-space, then:*

(i) *$|C_{p,n}(X)| \leq \mathfrak{c}$ and $|C_{p,n}(\upsilon X)| \leq \mathfrak{c}$ for every $n \in \omega$;*

(ii) *$C_{p,n}(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space for each $n \in \omega$;*

(iii) *$C_{p,2n+1}(X)$ is an $L\Sigma(\leq \omega)$-space for each $n \in \omega$.*

Proof. Apply Proposition 2.8 to see that $|C_p(\upsilon X)| \leq \mathfrak{c}$ and $|\upsilon X| \leq \mathfrak{c}$. A theorem of Okunev [**8**, Theorem 2.12] guarantees that $C_{p,n}(\upsilon X)$ is a Lindelöf Σ-space for any $n \in \omega$ so we can apply inductively Proposition 2.8 to convince ourselves that $|C_{p,n}(\upsilon X)| \leq \mathfrak{c}$ for any $n \in \omega$; Theorem 2.10 implies that $C_{p,n}(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space for any $n \in \omega$. It was proved in Tkachuk [**12**, Theorem 2.5] that the space $C_{p,2n+1}(X)$ is a continuous image of $C_{p,2n+1}(\upsilon X)$ so $C_{p,2n+1}(X)$ is an $L\Sigma(\leq \omega)$-space, and hence $|C_{p,2n+1}(X)| \leq \mathfrak{c}$ for each $n \in \omega$. Finally apply [**12**, Corollary 2.2] to see that $C_{p,2n+2}(\upsilon X)$ is homeomorphic to $\upsilon(C_{p,2n+2}(X))$, and therefore $|C_{p,2n+2}(X)| \leq |\upsilon(C_{p,2n+2}(X))| = |C_{p,2n+2}(\upsilon X)| \leq \mathfrak{c}$ for every $n \in \omega$. □

2.12. Corollary. *Suppose that* $|X| \leq \mathfrak{c}$ *and* $C_p(C_p(X))$ *is a Lindelöf* Σ*-space. Then*:

(i) $|C_{p,n}(X)| \leq \mathfrak{c}$ *and* $|C_{p,n}(\upsilon X)| \leq \mathfrak{c}$ *for every* $n \in \omega$;

(ii) $C_{p,n}(\upsilon X)$ *is an* $L\Sigma(\leq \omega)$*-space for each* $n \in \omega$;

(iii) $C_{p,2n}(X)$ *is an* $L\Sigma(\leq \omega)$*-space for each* $n \in \omega$.

Proof. Since X is homeomorphic to a closed subset of $C_p(C_p(X))$, the space X has to be Lindelöf Σ. It follows from [**8**, Theorem 3.5] that $\upsilon(C_p(X))$ is a Lindelöf Σ-space. Applying Corollary to Theorem 2 of [**7**], we can find a space Z which condenses onto X and $C_p(Z) \simeq \upsilon(C_p(X))$. Therefore, $|Z| \leq \mathfrak{c}$ and $C_p(Z)$ is a Lindelöf Σ-space which shows that Proposition 2.8 is applicable to see that $|C_p(Z)| = |\upsilon(C_p(X))| \leq \mathfrak{c}$. As a consequence, $|C_p(X)| \leq |\upsilon(C_p(X))| \leq \mathfrak{c}$ so we can apply Corollary 2.11 for the spaces $Y = C_p(X)$ and $C_p(Y) = C_p(C_p(X))$. □

The following corollary provides a positive answer to Problem 4.6 from [**6**].

2.13. Corollary. *If* X *and* $C_p(X)$ *are Lindelöf* Σ*-spaces and, additionally,* $|X| \leq \mathfrak{c}$*, then* $|C_{p,n}(X)| \leq \mathfrak{c}$ *and* $C_{p,n}(X)$ *is an* $L\Sigma(\leq \omega)$*-space for any* $n \in \omega$.

Proof. Observe that $C_p(C_p(X))$ is a Lindelöf Σ-space by [**8**, Theorem 2.12]; Corollary 2.11 and Corollary 2.12 do the rest. □

Let us conclude this paper with some open questions.

2.14. Question. Suppose that X is a Lindelöf Σ-space with a unique non-isolated point. Must $C_p(X)$ be a Lindelöf Σ-space?

Observe that it follows from Theorem 2.7 that the answer is positive if $|X| \leq \mathfrak{c}$.

2.15. Question. Is it true that $C_p(X, \mathbf{I})$ is a continuous image of $C_p(X, \mathbf{D}^\omega)$ for any space X such that $\operatorname{ind} X = 0$?

This question might be of interest because the answer is "yes" if $\dim X = 0$ (see Theorem 2.6).

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