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L^2-INVERSE SPECTRAL PROBLEMS FOR DIFFUSIVE LOGISTIC EQUATIONS OF POPULATION DYNAMICS

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ABSTRACT. We consider the nonlinear eigenvalue problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad u(t) > 0,$$
$$t \in I := (0, 1), \quad u(0) = u(1) = 0,$$

where $\lambda > 0$ is an eigenvalue parameter. For a given $\alpha > 0$, there exists a unique solution pair $(\lambda(\alpha), u_\alpha)$ which satisfies $\|u_\alpha\|_2 = \alpha$ ($\|u_\alpha\|_2$: L^2-norm of u_α). $\lambda(\alpha)$ is continuous for $\alpha > 0$ and is called an L^2-bifurcation curve. We propose a new framework of inverse nonlinear eigenvalue problems from a viewpoint of the asymptotic expansion formula for $\lambda(\alpha)$ as $\alpha \to \infty$.

1. Introduction. We consider the following nonlinear eigenvalue problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0, 1), \tag{1.1}$$
$$u(t) > 0, \quad t \in I, \tag{1.2}$$
$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is an eigenvalue parameter. We assume that $f(u)$ satisfies the following conditions (A.1)–(A.3).

(A.1) $f(u)$ is a function of C^1 for $u \ge 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $g(u) := f(u)/u$ is strictly increasing for $u \ge 0$ ($g(0) := 0$).

(A.3) $g(u) \to \infty$ as $u \to \infty$.

Then, for each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\overline{I})$ with $\|u_\alpha\|_2 = \alpha$. Here, $\|u_\alpha\|_2$ is the L^2-norm of u_α. The set $\{(\lambda(\alpha), u_\alpha); \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and

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is an unbounded curve of class C^1 in $\mathbf{R}_+ \times L^2(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda(\alpha)$ is strictly increasing for $\alpha > 0$ and $\lambda(\alpha) \to \infty$ as $\alpha \to \infty$ (cf. [**1, 8**]).

The purpose of this paper is to propose a new framework of nonlinear inverse eigenvalue problems from a viewpoint of the asymptotic expansion formula for $\lambda(\alpha)$ as $\alpha \to \infty$.

To clarify our intention in detail, let us explain the background of our problem. Equation (1.1)–(1.3) is motivated by the logistic equation of population dynamics and vibration of string with self-interaction, and has been studied by many authors. We refer to [**1, 8, 10, 12**] for the works which treated the problems by bifurcation theory of L^∞-framework.

On the other hand, since (1.1)–(1.3) is regarded as an eigenvalue problem, it seems meaningful to study (1.1)–(1.3) in the L^2-framework. We refer to [**2–7**] for the works in this direction. One of the main interests in this field is to study the shape of the bifurcation curve $(\alpha, \lambda(\alpha))$ in $\mathbf{R}^2$. It should be emphasized that the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to 0$ has been studied intensively in [**3–7**].

As for the behavior of $\lambda(\alpha)$ and u_α as $\alpha \to \infty$, it is known from [**1**] that

$$\frac{u_\alpha(t)}{g^{-1}(\lambda(\alpha))} \longrightarrow 1 \tag{1.4}$$

locally uniformly on I as $\alpha \to \infty$. By this, it is easy to see that, for $\alpha \gg 1$,

$$\alpha = \|u_\alpha\|_2 = (1 + o(1))g^{-1}(\lambda(\alpha)).$$

This implies that, in many cases, as $\alpha \to \infty$,

$$\lambda(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.5}$$

In particular, we note that, if $f(u) = u^p$ $(p > 1)$, then $g(\alpha) = \alpha^{p-1}$. Motivated by (1.5), the following asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ has been given in [**12**].

Theorem 1.1 [**12**]. *Let $f(u) = u^p$ $(p > 1)$. Let an arbitrary $n \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$ be fixed. Then the following asymptotic formula* *holds as* $\alpha \to \infty$:

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_0^{k+2} \alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}), \tag{1.6}$$

where

$$C_0 = (p+3) \int_0^1 \sqrt{\frac{p-1}{p+1} - \xi^2 + \frac{2}{p+1}\xi^{p+1}}\, d\xi$$

and $a_k(p)$ *is the polynomial* $(\deg a_k(p) \le k+1)$ *which is determined by* $a_0 = 1, a_1, \ldots, a_{k-1}$ *inductively.*

On the other hand, the following *uniqueness result for the L^2-bifurcation* curve has been proved in [**13**] by using the variational approach.

Theorem 1.2 [**13**]. *Assume that $f_1(u)$ and $f_2(u)$ satisfy* (A.1)–(A.3). *Let $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ be the L^2-bifurcation curves of* (1.1)–(1.3) *associated with the nonlinear terms $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Further, assume that $\lambda_1(\alpha) = \lambda_2(\alpha)$ for any $\alpha > 0$. Then $f_1(u) \equiv f_2(u)$ if the connected components of the set $V := \{u \ge 0 : f_1(u) = f_2(u)\}$ are locally finite.*

These two theorems give us a new aspect for the nonlinear inverse eigenvalue problem. To explain the basic idea clearly, we consider the nonlinear terms f_1 and f_2 of the special form as follows. Let $p > 1$ be a constant, and

$$f_1(u) = u^p \quad (u \ge 0), \qquad f_2(u) = u^p(1 - h(u)).$$

We assume that f_1 and f_2 satisfy the conditions in Theorem 1.2 and $h(u)$ ($\not\equiv 0$) is *very small in some sense.* As in Theorem 1.2, let $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ be the L^2-bifurcation curves of (1.1)–(1.3) associated with the nonlinear terms $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Then we find that

(i) By Theorem 1.2, the bifurcation curve $\lambda_1(\alpha)$ for $f_1(u) = u^p$ does not coincide with $\lambda_2(\alpha)$ for $f_2(u) = u^p(1 - h(u))$ *identically.*

(ii) However, it is expected that the shapes of $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ are very close each other, since f_1 and f_2 are almost equal asymptotically.

Taking these considerations into account, we introduce the following inverse nonlinear eigenvalue problem (P), which seems to be reasonable from an asymptotical point of view.

(P) *If both asymptotic expansion formulas for* $\lambda_1(\alpha)$ *and* $\lambda_2(\alpha)$ *as* $\alpha \to \infty$ *satisfy* (1.6), *then can we say that* $f_1(u) = f_2(u)$ *in some sense?*

Now we state the result.

Theorem 1.3. *Let* $f_1(u) = u^2$ *and* $f_2(u) = u^2(1 - h(u))$, *where* $h(u) = (u-4)e^{-u}/2$. *Then* $\lambda_1(\alpha)$ *and* $\lambda_2(\alpha)$ *have the same asymptotic expression as* (1.6) *with* $p = 2$ *as* $\alpha \to \infty$.

Theorem 1.3 gives us not only evidence of the affirmative answer to (P), but also the further direction of this study. Namely, assume that $f_1(u)$ and $f_2(u)$ satisfy (A.1)–(A.3). Suppose that both asymptotic expansion formulas for $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ as $\alpha \to \infty$ satisfy (1.6). Then the expected conclusion of (P) is that $f_1(u) = f_2(u) + \zeta(u)$, where $\zeta(u)$ decays exponentially to 0 as $u \to \infty$. In other words, $f_1(u)$ *should be asymptotically equal to* $f_2(u)$ *in algebraic sense.*

Remark 1.4. (1) We see that $h(u) = (ue^{-u})^{(4)}/2$, and this property enable us to obtain the precise remainder estimate of $\|u_\alpha\|_2$ as $\alpha \to \infty$. Moreover, $f(u) = u^2(1 - h(u))$ must satisfy (A.1)–(A.3). Since it is quite difficult to find such a small perturbation term as $h(u)$ above generally, we are not able to improve Theorem 1.3 for more general $f(u)$ easily.

(2) It is clear that $f(u) = u^2(1 - h(u))$ satisfies (A.1) and (A.3). Furthermore, since $g(u) = f(u)/u = u(1 - h(u))$, for $u \geq 0$, we obtain

$$
\begin{aligned}
g'(u) &= e^{-u}\left(e^u + \frac{1}{2}u^2 - 3u + 2\right) \\
&\geq e^{-u}\left(1 + u + \frac{1}{2}u^2 + \frac{1}{2}u^2 - 3u + 2\right)
\end{aligned}
$$

$$= e^{-u}(u^2 - 2u + 3) > 0.$$

Therefore, $f(u) = u^2(1 - h(u))$ satisfies (A.2).

Our arguments here to prove Theorem 1.3 are quite straightforward and are totally different from those of Theorem 1.1. To prove Theorem 1.1, the special relationship between λ and the critical value of the corresponding solution, which holds only for the case $f(u) = u^p$ $(p > 1)$, was used.

2. Proof of Theorem 1.3. In this section, C denotes various positive constants independent of $\lambda \gg 1$. We begin with the fundamental tools which play important roles in what follows. We know from [**1**] that, for a given $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\overline{I})$ of (1.1)–(1.3). We use this notation in what follows. Therefore, $\alpha = \|u_\lambda\|_2$. It is well known that

$$u_\lambda(t) = u_\lambda(1-t), \quad t \in I, \tag{2.1}$$

$$u_\lambda'(t) > 0, \quad 0 \le t < \frac{1}{2}, \tag{2.2}$$

$$\|u_\lambda\|_\infty = u_\lambda\left(\frac{1}{2}\right). \tag{2.3}$$

We know by [**1**] that

$$\lambda = \|u_\lambda\|_\infty(1 - h(\|u_\lambda\|_\infty)) + \lambda_1, \tag{2.4}$$

where $\lambda_1 > 0$ is the remainder term of λ with respect to $\|u_\lambda\|_\infty$ and depends upon λ. For $\lambda \gg 1$, we have

$$C^{-1}\lambda e^{-\sqrt{\lambda}/2} \le \lambda_1 \le C\lambda e^{-\sqrt{(1-\kappa)\lambda}/2}. \tag{2.5}$$

Here $0 < \kappa \ll 1$ is an arbitrary small constant, which is fixed in what follows. For completeness, we give the proof of (2.5) in the Appendix. Now multiply (1.1) by $u_\lambda'(t)$. Then

$$\left(u_\lambda''(t) + \lambda u_\lambda(t) - u_\lambda(t)^2 + u_\lambda^2(t)h(u_\lambda(t))\right) u_\lambda'(t) = 0.$$

This along with (2.3) implies that

$$\begin{aligned}(2.6)\quad \frac{1}{2}u_\lambda'(t)^2 + \frac{1}{2}\lambda u_\lambda(t)^2 - \frac{1}{3}u_\lambda(t)^3 + \int_0^{u_\lambda(t)} \xi^2 h(\xi)\,d\xi &\equiv \text{constant}\\ &= \frac{1}{2}\lambda\|u_\lambda\|_\infty^2 - \frac{1}{3}\|u_\lambda\|_\infty^3\\ &\quad + \int_0^{\|u_\lambda\|_\infty} \xi^2 h(\xi)\,d\xi \quad (\text{put } t = 1/2).\end{aligned}$$

Let

$$\begin{aligned}L_\lambda(\theta) &= \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{3}(\|u_\lambda\|_\infty^3 - \theta^3)\\ &\quad + 2\int_\theta^{\|u_\lambda\|_\infty} \xi^2 h(\xi)\,d\xi.\end{aligned} \tag{2.7}$$

This along with (2.2) and (2.6) implies that, for $0 \le t \le 1/2$,

$$u_\lambda'(t) = \sqrt{L_\lambda(u_\lambda(t))}. \tag{2.8}$$

By this and (2.1), we obtain
(2.9)

$$\begin{aligned}\|u_\lambda\|_\infty^2 - \alpha^2 &= 2\int_0^{1/2} \frac{(\|u_\lambda\|_\infty^2 - u_\lambda^2(t))u_\lambda'(t)}{\sqrt{L_\lambda(u_\lambda(t))}}\,dt\\ &= 2\int_0^{\|u_\lambda\|_\infty} \frac{(\|u_\lambda\|_\infty^2 - \theta^2)}{\sqrt{L_\lambda(\theta)}}\,d\theta\\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}}\int_0^1 \frac{1-s^2}{\sqrt{B_\lambda(s)}}\,ds\\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}}\left\{\int_0^1 \frac{1-s^2}{\sqrt{A(s)}}\,ds + \int_0^1\left(\frac{1-s^2}{\sqrt{B_\lambda(s)}} - \frac{1-s^2}{\sqrt{A(s)}}\right)ds\right\}\\ &= \frac{2\|u_\lambda\|_\infty^2}{\sqrt{\lambda}}(C_1 + M_\lambda),\end{aligned}$$

where

$$A(s) := 1 - s^2 - \frac{2}{3}(1 - s^3), \tag{2.10}$$

$$B_\lambda(s) := 1 - s^2 - \frac{2}{3}\frac{\|u_\lambda\|_\infty}{\lambda}(1 - s^3) + Q_\lambda(s), \tag{2.11}$$

$$Q_\lambda(s) := \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_{\|u_\lambda\|_\infty s}^{\|u_\lambda\|_\infty} \xi^2 h(\xi)\, d\xi, \tag{2.12}$$

$$C_1 := \int_0^1 \frac{1 - s^2}{\sqrt{A(s)}}\, ds, \tag{2.13}$$

$$M_\lambda := \int_0^1 \left(\frac{1 - s^2}{\sqrt{B_\lambda(s)}} - \frac{1 - s^2}{\sqrt{A(s)}}\right) ds. \tag{2.14}$$

By (2.9), we prove Theorem 1.3. To do this, the estimate for M_λ as $\lambda \to \infty$ plays an important role.

Proposition 2.1. *For* $\lambda \gg 1$,

$$|M_\lambda| \le Ce^{-\sqrt{(1-2\kappa)\lambda}/2}. \tag{2.15}$$

We accept Proposition 2.1 tentatively and prove Theorem 1.3.

Proof of Theorem 1.3. By (2.4), (2.9) and Proposition 2.1, for $\lambda \gg 1$,

$$\|u_\lambda\|_\infty^2 - \alpha^2 = \frac{2}{\sqrt{\lambda}} C_1 \|u_\lambda\|_\infty^2 + O(\lambda^{3/2} e^{-\sqrt{(1-2\kappa)\lambda}/2}). \tag{2.16}$$

This implies that

$$\|u_\lambda\|_\infty^2 \left(1 - \frac{2}{\sqrt{\lambda}} C_1\right) = \alpha^2 + O\left(\lambda^{3/2} e^{-\sqrt{(1-2\kappa)\lambda}/2}\right). \tag{2.17}$$

By (2.4) and (2.5), for $\lambda \gg 1$,

$$\lambda = \|u_\lambda\|_\infty + O(\lambda e^{-\sqrt{(1-\kappa)\lambda}/2}). \tag{2.18}$$

By (1.5), we see that $\lambda = \alpha(1 + o(1))$ for $\lambda \gg 1$. By this, (2.17) and (2.18), we obtain

$$\lambda^2 \left(1 - \frac{2}{\sqrt{\lambda}} C_1\right) = \alpha^2 + O(\alpha^2 e^{-\sqrt{(1-\kappa)(1+o(1))\alpha}/2}). \tag{2.19}$$

We substitute $\lambda = \alpha + \alpha_2$ into (2.19), where $\alpha_2 = o(\alpha)$ for $\alpha \gg 1$. Then

$$
\begin{aligned}
\lambda^2 &= (\alpha+\alpha_2)^2 = \alpha^2 + 2\alpha\alpha_2 + \alpha_2^2 \\
&= 2\lambda^{3/2}C_1 + \alpha^2 + O\left(\alpha^2 e^{-\sqrt{(1-\kappa)(1+o(1))\alpha/2}}\right) \\
&= \alpha^2 + 2(\alpha+\alpha_2)^{3/2}C_1 \\
&\quad + O\left(\alpha^2 e^{-\sqrt{(1-\kappa)(1+o(1))\alpha/2}}\right).
\end{aligned}
\tag{2.20}
$$

By this, for $\alpha \gg 1$, we obtain

$$
2\alpha\alpha_2 = 2\alpha^{3/2}C_1(1+o(1)).
\tag{2.21}
$$

By this and the fact that $C_0 = C_1$, which can be obtained by direct calculation, for $\alpha \gg 1$, we obtain that $\alpha_2 = C_1\alpha^{1/2} = C_0\alpha^{1/2}$. Thus, we obtain the second term of (1.6). By the same argument as this, we obtain the third term α_3, fourth term α_4 and nth term α_n of $\lambda = \lambda(\alpha)$ inductively. By this procedure, we obtain the same asymptotic expansion formula as (1.6) for $p = 2$, since it is clear that formula (2.16) without the remainder term is valid for the case $f(u) = u^2$, and (1.6) is also obtained by (2.16) by using the fact that $C_0 = C_1$. Thus, the proof is complete. □

3. Proof of Proposition 2.1. In this section, we prove Proposition 2.1. Let an arbitrary $0 < \varepsilon \ll 1$ be fixed. We have

$$
\begin{aligned}
M_\lambda &= \int_0^1 \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}\,ds \\
&= \int_0^{1-\varepsilon} \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})},ds \\
&\quad + \int_{1-\varepsilon}^1 \frac{(1-s^2)(A(s)-B_\lambda(s))}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)}+\sqrt{B_\lambda(s)})}\,ds \\
&:= M_{1,\lambda} + M_{2,\lambda}.
\end{aligned}
\tag{3.1}
$$

To estimate $M_{1,\lambda}$ and $M_{2,\lambda}$, for $0 \le s \le 1$, we put

(3.2)
$$
\begin{aligned}
K_\lambda(s) &:= A(s) - B_\lambda(s) \\
&= \frac{2}{3}\left(\frac{\|u_\lambda\|_\infty}{\lambda} - 1\right)(1 - s^3) - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_{\|u_\lambda\|_\infty s}^{\|u_\lambda\|_\infty} \xi^2 h(\xi)\, d\xi \\
&= \frac{2}{3}\left(\frac{\|u_\lambda\|_\infty}{\lambda} - 1\right)(1 - s^3) \\
&\quad + \frac{1}{\lambda\|u_\lambda\|_\infty^2} e^{-\|u_\lambda\|_\infty}(\|u_\lambda\|_\infty^3 - \|u_\lambda\|_\infty^2 - 2\|u_\lambda\|_\infty - 2) \\
&\quad + \frac{1}{\lambda\|u_\lambda\|_\infty^2} e^{-\|u_\lambda\|_\infty s}(-\|u_\lambda\|_\infty^3 s^3 + \|u_\lambda\|_\infty^2 s^2 + 2\|u_\lambda\|_\infty s + 2).
\end{aligned}
$$

Lemma 3.1. *For $\lambda \gg 1$,*

$$
|M_{1,\lambda}| \le Ce^{-\sqrt{(1-\kappa)\lambda}/2}. \tag{3.3}
$$

Proof. By (3.1) and (3.2),

(3.4)
$$
\begin{aligned}
M_{1,\lambda} &= \frac{2}{3}\int_0^{1-\varepsilon} \frac{(1-s^2)(1-s^3)(\|u_\lambda\|_\infty/\lambda - 1)}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)} + \sqrt{B_\lambda(s)})}\, ds \\
&\quad + \frac{1}{\lambda\|u_\lambda\|_\infty^2} e^{-\|u_\lambda\|_\infty}(\|u_\lambda\|_\infty^3 - \|u_\lambda\|_\infty^2 - 2\|u_\lambda\|_\infty - 2) \\
&\quad \times \int_0^{1-\varepsilon} \frac{(1-s^2)}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)} + \sqrt{B_\lambda(s)})}\, ds + \frac{1}{\lambda\|u_\lambda\|_\infty^2} \\
&\quad \times \int_0^{1-\varepsilon} \frac{(-\|u_\lambda\|_\infty^3 s^3 + \|u_\lambda\|_\infty^2 s^2 + 2\|u_\lambda\|_\infty s + 2)e^{-\|u_\lambda\|_\infty s}}{\sqrt{A(s)}\sqrt{B_\lambda(s)}(\sqrt{A(s)} + \sqrt{B_\lambda(s)})}\, ds \\
&\quad := M_{1,0,\lambda} + M_{1,1,\lambda} + M_{1,2,\lambda}.
\end{aligned}
$$

We first estimate $M_{1,2,\lambda}$. It is clear that, for $0 \le s \le 1 - \varepsilon$ and $\lambda \gg 1$,

$$
C^{-1} \le A(s) \le C, \qquad C^{-1} \le B_\lambda(s) \le C. \tag{3.5}
$$

By (3.4) and (3.5), for $\lambda \gg 1$, we obtain

(3.6)
$$\begin{aligned}
&\frac{C^{-1}}{\lambda\|u_\lambda\|_\infty^2}\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}(-\|u_\lambda\|_\infty^3 s^3 + \|u_\lambda\|_\infty^2 s^2 + 2\|u_\lambda\|_\infty s + 2) \\
&\quad \le M_{1,2,\lambda} \\
&\quad \le \frac{C}{\lambda\|u_\lambda\|_\infty^2}\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}(-\|u_\lambda\|_\infty^3 s^3 + \|u_\lambda\|_\infty^2 s^2 + 2\|u_\lambda\|_\infty s + 2)\, ds.
\end{aligned}$$

We put

$$\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}(-\|u_\lambda\|_\infty^3 s^3 + \|u_\lambda\|_\infty^2 s^2 + 2\|u_\lambda\|_\infty s + 2)\, ds := T_1 + T_2 + T_3 + T_4, \tag{3.7}$$

where

$$T_1 = -\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}\|u_\lambda\|_\infty^3 s^3 ds, \tag{3.8}$$

$$T_2 = \int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}\|u_\lambda\|_\infty^2 s^2 ds, \tag{3.9}$$

$$T_3 = 2\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s}\|u_\lambda\|_\infty s\, ds, \tag{3.10}$$

$$T_4 = 2\int_0^{1-\varepsilon} e^{-\|u_\lambda\|_\infty s} ds. \tag{3.11}$$

Then, by direct calculation,

(3.12)
$$\begin{aligned}
T_1 &= -\frac{1}{\|u_\lambda\|_\infty}\int_0^{(1-\varepsilon)\|u_\lambda\|_\infty} x^3 e^{-x} dx \\
&= \frac{1}{\|u_\lambda\|_\infty}\left[e^{-x}(x^3 + 3x^2 + 6x + 6)\right]_0^{(1-\varepsilon)\|u_\lambda\|_\infty} \\
&:= R_1(\|u_\lambda\|_\infty) - \frac{6}{\|u_\lambda\|_\infty},
\end{aligned}$$

(3.13)
$$T_2 = \frac{1}{\|u_\lambda\|_\infty}\int_0^{(1-\varepsilon)\|u_\lambda\|_\infty} x^2 e^{-x} dx$$

$$
\begin{aligned}
&= \frac{1}{\|u_\lambda\|_\infty}\left[-e^{-x}(x^2+2x+2)\right]_0^{(1-\varepsilon)\|u_\lambda\|_\infty}\\
&:= R_2(\|u_\lambda\|_\infty) + \frac{2}{\|u_\lambda\|_\infty},
\end{aligned}
$$

(3.14)

$$
\begin{aligned}
T_3 &= \frac{2}{\|u_\lambda\|_\infty}\int_0^{(1-\varepsilon)\|u_\lambda\|_\infty} xe^{-x}dx\\
&= \frac{2}{\|u_\lambda\|_\infty}\left[-e^{-x}(x+1)\right]_0^{(1-\varepsilon)\|u_\lambda\|_\infty}\\
&:= R_3(\|u_\lambda\|_\infty) + \frac{2}{\|u_\lambda\|_\infty},
\end{aligned}
$$

(3.15)

$$
\begin{aligned}
T_4 &= \frac{2}{\|u_\lambda\|_\infty}\int_0^{(1-\varepsilon)\|u_\lambda\|_\infty} e^{-x}dx\\
&= \frac{2}{\|u_\lambda\|_\infty}\left[-e^{-x}\right]_0^{(1-\varepsilon)\|u_\lambda\|_\infty}\\
&:= R_4(\|u_\lambda\|_\infty) + \frac{2}{\|u_\lambda\|_\infty}.
\end{aligned}
$$

Here, by (2.4) and (3.12)–(3.15),

$$
|R_j(\|u_\lambda\|_\infty)| \le C\lambda^{3-j}e^{-\lambda(1-2\varepsilon)} \quad (j=1,2,3,4).
$$

By this and (3.12)–(3.15),

$$
\begin{aligned}
|T_1+T_2+T_3+T_4| &= |R_1(s)+R_2(s)+R_3(s)+R_4(s)|\\
&\le |R_1(s)|+|R_2(s)|+|R_3(s)|+|R_4(s)|\\
&\le C\lambda^2 e^{-\lambda(1-2\varepsilon)}.
\end{aligned}
$$

By this, (2.4), (3.6) and (3.7), we obtain

$$
|M_{1,2,\lambda}| \le C\lambda^{-1}e^{-\lambda(1-2\varepsilon)}. \tag{3.16}
$$

By (2.4), (2.5), (3.4) and (3.5), for $\lambda \gg 1$,

$$
\begin{aligned}
|M_{1,0,\lambda}| &\le \frac{C}{\lambda}(\|u_\lambda\|_\infty h(\|u_\lambda\|_\infty)+\lambda_1)\\
&\le C\lambda e^{-\lambda(1-\varepsilon)} + Ce^{-\sqrt{(1-\kappa)\lambda}/2}.
\end{aligned} \tag{3.17}
$$

Finally, by (3.4) and (3.5),

$$|M_{1,1,\lambda}| \le Ce^{-\lambda(1-\varepsilon)}. \tag{3.18}$$

By this, (3.4), (3.16) and (3.17), we obtain (3.3). Thus, the proof is complete. □

Lemma 3.2. *For* $1-\varepsilon \le s \le 1$ *and* $\lambda \gg 1$

$$K_\lambda(s) = \frac{2\lambda_1}{\lambda}(s-1) + J_\lambda(s)(s-1)^2, \tag{3.19}$$

where

$$|J_\lambda(s)| \le C\frac{\lambda_1}{\lambda}. \tag{3.20}$$

Proof. We have $K_\lambda(1) = 0$. Further,

$$K'_\lambda(s) = \frac{2}{\lambda}s^2(\lambda - \|u_\lambda\|_\infty) + \frac{2s^2\|u_\lambda\|_\infty}{\lambda}h(\|u_\lambda\|_\infty s). \tag{3.21}$$

Therefore, by this and (2.4),

$$\begin{aligned} K'_\lambda(1) &= \frac{2}{\lambda}(\|u_\lambda\|_\infty(1 - h(\|u_\lambda\|_\infty)) + \lambda_1 - \|u_\lambda\|_\infty) \\ &\quad + \frac{2}{\lambda}\|u_\lambda\|_\infty h(\|u_\lambda\|_\infty) \\ &= \frac{2\lambda_1}{\lambda}. \end{aligned} \tag{3.22}$$

Furthermore,

$$\begin{aligned} K''_\lambda(s) &= \frac{4}{\lambda}s(\lambda - \|u_\lambda\|_\infty) \\ &\quad + \frac{4\|u_\lambda\|_\infty}{\lambda}sh(\|u_\lambda\|_\infty s) \\ &\quad + \frac{2\|u_\lambda\|_\infty^2}{\lambda}s^2h'(\|u_\lambda\|_\infty s). \end{aligned} \tag{3.23}$$

Since

$$h'(u) = \frac{1}{2}(5-u)e^{-u}, \tag{3.24}$$

by (2.4) and (2.5), for $1-\varepsilon \le s \le 1$ and $\lambda \gg 1$,

$$|K''_\lambda(s)| \le C\left(\frac{\lambda_1}{\lambda} + \lambda^2 e^{-\|u_\lambda\|_\infty(1-\varepsilon)}\right) \le C\frac{\lambda_1}{\lambda}. \tag{3.25}$$

By this, (3.22) and the mean value theorem, for $1-\varepsilon \le s \le 1$ and $\lambda \gg 1$,

$$K_\lambda(s) = K'_\lambda(1)(s-1) + \frac{1}{2}K''_\lambda(s_1)(s-1)^2, \tag{3.26}$$

where s_1 satisfies $s \le s_1 \le 1$ and depends upon s. Then, we put $J_\lambda(s) = K''_\lambda(s_1)$. By (3.22), (3.25) and (3.26), we obtain (3.19). Thus, the proof is complete. □

Lemma 3.3. *For $1-\varepsilon \le s \le 1$ and $\lambda \gg 1$,*

$$B_\lambda(s) = \frac{2\lambda_1}{\lambda}(1-s) + K_1(\lambda,s)(1-s)^2, \tag{3.27}$$

where $1-\kappa \le K_1(\lambda,s) \le 1$.

Proof. We have $B_\lambda(1) = 0$. Furthermore,

$$B'_\lambda(s) = -2s + 2s^2\frac{\|u_\lambda\|_\infty}{\lambda} - \frac{2\|u_\lambda\|_\infty}{\lambda}s^2 h(\|u_\lambda\|_\infty s). \tag{3.28}$$

By this and (2.4),

$$B'_\lambda(1) = -2 + 2\frac{\|u_\lambda\|_\infty}{\lambda} - \frac{2\|u_\lambda\|_\infty}{\lambda}h(\|u_\lambda\|_\infty) = -\frac{2\lambda_1}{\lambda}. \tag{3.29}$$

Furthermore,

$$\begin{aligned} B''_\lambda(s) = {} & -2 + 4\frac{\|u_\lambda\|_\infty}{\lambda}s - 4\frac{\|u_\lambda\|_\infty}{\lambda}sh(\|u_\lambda\|_\infty s) \\ & - 2\frac{\|u_\lambda\|_\infty^2}{\lambda}s^2 h'(\|u_\lambda\|_\infty s). \end{aligned} \tag{3.30}$$

Therefore, by (3.29), (3.30) and the Taylor expansion, we obtain

$$\begin{aligned} B_\lambda(s) &= B_\lambda(1) + B_\lambda'(s)(s-1) + \frac{1}{2}B_\lambda''(s_2)(s-1)^2 \\ &= \frac{2\lambda_1}{\lambda}(1-s) + K_1(\lambda,s)(1-s)^2, \end{aligned} \tag{3.31}$$

where $1-\varepsilon < s < s_2 < 1$. Then by (2.4), (3.24) and (3.30), we see that $1-\kappa \le K_1(\lambda,s) \le 1$ for $\lambda \gg 1$. This implies (3.27). Thus, the proof is complete. □

Lemma 3.4. *For* $1-\varepsilon \le s \le 1$ *and* $\lambda \gg 1$,

$$|M_{2,\lambda}| \le Ce^{-\sqrt{(1-2\kappa)\lambda}/2}. \tag{3.32}$$

Proof. We know by Taylor expansion that, for $1-\varepsilon \le s \le 1$ and $\lambda \gg 1$,

$$C^{-1}(1-s)^2 \le A(s) \le C(1-s)^2. \tag{3.33}$$

By this and (3.1),

$$|M_{2,\lambda}| \le \int_{1-\varepsilon}^1 \frac{(1-s^2)|K_\lambda(s)|}{\sqrt{A(s)}B_\lambda(s)}\,ds \le C\int_{1-\varepsilon}^1 \frac{|K_\lambda(s)|}{B_\lambda(s)}\,ds. \tag{3.34}$$

By Hölder's inequality, for $a, b \ge 0$ and $q_1, q_2 > 1$ with $1/q_1 + 1/q_2 = 1$,

$$ab \le \frac{1}{q_1}a^{q_1} + \frac{1}{q_2}b^{q_2} \le C(a^{q_1} + b^{q_2}). \tag{3.35}$$

For $1-\varepsilon \le s \le 1$, we put

$$a^{q_1} = \frac{2\lambda_1}{\lambda}(1-s), \quad b^{q_2} = K_1(\lambda,s)(1-s)^2.$$

By this, Lemma 3.3, (3.31) and (3.35), for $1-\varepsilon \le s \le 1$ and $\lambda \gg 1$,

$$\begin{aligned} B_\lambda(s) &= \frac{2\lambda_1}{\lambda}(1-s) + K_1(\lambda,s)(1-s)^2 \\ &\ge C^{-1}\left(\frac{2\lambda_1}{\lambda}(1-s)\right)^{1/q_1}\left(K_2(\lambda,s)(1-s)^2\right)^{1/q_2} \\ &\ge C^{-1}\left(\frac{2\lambda_1}{\lambda}\right)^{1/q_1}(1-s)^{1+1/q_2}. \end{aligned} \tag{3.36}$$

By this, (2.5) and Lemma 3.2,

$$
\begin{aligned}
\frac{|K_\lambda(s)|}{B_\lambda(s)} &\le C\frac{(2\lambda_1/\lambda)\{(1-s)+(1-s)^2\}}{(2\lambda_1/\lambda)^{1/q_1}(1-s)^{1+1/q_2}} \\
&\le C\left(\frac{2\lambda_1}{\lambda}\right)^{1/q_2}\left\{(1-s)^{-1/q_2}+(1-s)^{1-1/q_1}\right\} \\
&\le Ce^{-\sqrt{(1-\kappa)\lambda}/(2q_2)}\{(1-s)^{-1/q_2}+(1-s)^{1/q_2}\}.
\end{aligned}
\tag{3.37}
$$

We choose $q_2 > 1$ near 1. By this and (3.34), we obtain (3.32). Thus, the proof is complete. □

Now Proposition 2.1 follows from (3.1) and Lemmas 3.1 and 3.4. Thus, the proof is complete. □

APPENDIX

4. We prove (2.5). We consider (1.1)–(1.3) with $f(u) = u^2(1-h(u))$. By (2.8), we obtain

$$
\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} \frac{u_\lambda'(t)}{\sqrt{L(u_\lambda(t))}}\,dt \\
&= \int_0^{\|u_\lambda\|_\infty} \frac{1}{\sqrt{L_\lambda(\theta)}}\,d\theta \\
&= \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{B_\lambda(s)}}\,ds.
\end{aligned}
\tag{4.1}
$$

Here, $B_\lambda(s)$ is defined by (2.11). Let an arbitrary $0 < \varepsilon \ll 1$ be fixed. Let $\xi := \lambda_1/\lambda > 0$. By (3.5), (3.31) and (4.1),

$$
\begin{aligned}
\frac{\sqrt{\lambda}}{2} &= \int_0^{1-\varepsilon} \frac{1}{\sqrt{B_\lambda(s)}}\,ds + \int_{1-\varepsilon}^1 \frac{1}{\sqrt{B_\lambda(s)}}\,ds \\
&\le C + \int_{1-\varepsilon}^1 \frac{\varepsilon}{\sqrt{2\xi(1-s)+(1-\kappa)(1-s)^2}}\,ds \\
&= C + \int_0^{\varepsilon} \frac{1}{\sqrt{2\xi v+(1-\kappa)v^2}}\,dv \\
&= (1-\kappa)^{-1/2}(\log C_\varepsilon - \log 2\xi).
\end{aligned}
\tag{4.2}
$$

By this, we obtain $\lambda_1 \le (C/2)\lambda e^{-\sqrt{(1-\kappa)\lambda}/2}$. By (3.31), for $1-\varepsilon \le s \le 1$ and $u \gg 1$, we have

$$B_\lambda(s) \le 2\xi(1-s) + (1-s)^2.$$

By this and (4.1), we obtain

$$\begin{aligned}
\frac{\sqrt{\lambda}}{2} &= \int_0^1 \frac{1}{\sqrt{B_\lambda(s)}}\,ds \\
&\ge \int_{1-\varepsilon}^1 \frac{1}{\sqrt{2\xi(1-s)+(1-s)^2}}\,ds \\
&= \int_0^\varepsilon \frac{1}{\sqrt{2\xi v + v^2}}\,dv \\
&= \Big[\log|2v + 2\xi + 2\sqrt{(v^2+2\xi v)}|\Big]_0^\varepsilon \\
&= \log C_\varepsilon - \log 2\xi.
\end{aligned}$$

By this, we obtain $\lambda_1 \ge (C/2)\lambda e^{-\sqrt{\lambda}/2}$. Thus, the proof of (2.5) is complete. □

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