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SURVEY ARTICLE–GRAPHICAL REPRESENTATIONS OF FACTORIZATIONS IN COMMUTATIVE RINGS

M. AXTELL, N. BAETH AND J. STICKLES

ABSTRACT. This article surveys the recent and active area of irreducible divisor graphs of commutative rings. Notable algebraic and graphical results are given, and alternate constructions for irreducible divisor graphs and higher dimensional analogs are explored.

1. Introduction and motivation. One of the main themes in the study of abstract algebra is the study of how elements of a commutative ring factor. From the fundamental theorem of arithmetic to Galois theory, the study of how elements can be written as a product of irreducible elements has been a fruitful one. In more recent years, this area of study has taken an interesting turn by looking to graph theory in search of a better understanding of factorization theory. This paper will survey recent results along these lines.

The idea of transforming a ring-theoretic question into a graph-theoretic one is certainly not a novel concept. For example, the past decade or so has seen a great deal of research being done in the area of zero-divisor graphs with the goal of trying to better understand the role of zero-divisors in a commutative ring. Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$ (cf. [**3**]). The interested reader can find an overview of zero divisor graphs in [**2**]. With the success of zero-divisor graphs in mind, we turn our attention to factorizations, for the most part, in integral domains.

Throughout, D will denote an integral domain, D^* will denote the nonzero elements of D, and $U(D)$ will denote the units of D. Recall

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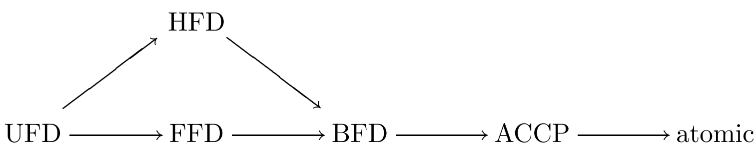
that $x \in D^*\backslash U(D)$ is said to be irreducible if, whenever $x = ab$, we have that $(x) = (a)$ or $(x) = (b)$; i.e., x is an associate of either a or b. Let $\mathrm{Irr}\,(D) = \{x \in D : x \text{ is irreducible}\}$. We can then define $\overline{\mathrm{Irr}}\,(D)$ to be a (pre-chosen) set of associate class representatives, one representative from each class of associates of a given *nonzero* irreducible. We call a domain *atomic* if each nonzero, nonunit element can be factored into a (finite) product of irreducible elements.

Atomic domains exhibit a variety of factorization behaviors. Most familiar is that of being a unique factorization domain. An atomic domain is a *unique factorization domain* (UFD) if every nonzero nonunit has a unique (up to associates) factorization into a product of irreducible elements. An atomic domain is a *finite factorization domain* (FFD) if every nonzero nonunit has only finitely many distinct nonassociate irreducible divisors. An atomic domain D is a *half-factorial domain* (HFD) provided the number of terms in every factorization of a given $x \in D^*\backslash U(D)$ into irreducibles is constant. An atomic domain D is a *bounded factorization domain* (BFD) if there is a bound on the length of factorizations into products of irreducible elements for each $x \in D^*\backslash U(D)$. Recall that a commutative ring satisfying the ascending chain condition on principal ideals is said to be ACCP. It is well known (cf. [**1**]) that the following relations hold and that none of the arrows are reversible.



In an effort to better understand how factoring nonzero nonunit elements of an atomic domain into irreducibles behave, Coykendall and Maney introduced the concept of an irreducible divisor graph of a nonzero nonunit in an atomic domain in [**12**]. This topic has been further studied in [**7, 16, 17**] and generalized in [**6, 8, 9, 13**]. We begin by formally defining the irreducible divisor graph.

Definition 1.1. Let D be an atomic domain, and let $x \in D^* \setminus U(D)$. The *irreducible divisor graph of x*, denoted $G_D(x)$, is given by (V, E), where the vertex set is $V = \{y \in \overline{\mathrm{Irr}}\,(D) : y \mid x\}$, and, given $y_1, y_2 \in V$, there is an edge $y_1 - y_2 \in E$ between vertices y_1 and y_2 if and only if $y_1 y_2 \mid x$.

We will write $G(x)$ instead of $G_D(x)$ if the domain D is clear from context. If multiple powers of an irreducible element divide x, then we place a numbered loop on the given irreducible divisor. If a^2 is the maximum power of a that divides x, the loop on a will have no number; however, if a^n is the maximal power of a dividing x with $n \geq 3$, then a's loop will be numbered with $n - 1$. When calculating the degree of a vertex v, we need to consider both the number of distinct neighbors of v as well as the number of edges, including loops, emanating from v. Thus, we define, for a vertex $v \in V(G(x))$, $\deg(v) = |\{w \in V : w \neq v,\ v - w \in E(G(x))\}|$ and $\mathrm{degl}\,(v) = \deg(v) + K(v) - 1$ where $K(v) = \sup\{n : v^n \mid x\}$.

In general, if $x_1, x_2, \ldots, x_n$ are vertices of a graph, then $x_1 - x_2 - \cdots - x_n$ denotes a *walk* from vertex x_1 to vertex x_n, where x_i is adjacent to x_{i+1} for $1 \leq i \leq n - 1$. If $x_1, x_2, \ldots, x_n$ are distinct from one another, then the walk is called a *path*. A graph is *complete* if any two distinct vertices are connected by an edge, while a graph is *connected* if a path exists between any two distinct vertices. A complete graph with no loops or multiple edges is also referred to as a *clique*. A complete graph having some number (possibly zero) of loops is referred to as a *pseudoclique*. For two distinct vertices a and b in a graph, the distance between a and b, denoted $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $d(a, b) = \infty$. The *diameter* of a graph G is $\mathrm{diam}\,(G) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of a graph G, denoted $\mathrm{g}(G)$, is the length of the shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{g}(G) = \infty$. Below are two illustrative examples of irreducible divisor graphs.

Example 1.2 [**12**, Example 2.1]. Let $D = \mathbf{Z}[\sqrt{-5}]$. The only factorizations of 18 into irreducibles are $18 = 2 \cdot 3^2$, $18 = 3(1+\sqrt{-5})(1-\sqrt{-5})$ and $18 = 2(2 + \sqrt{-5})(2 - \sqrt{-5})$. Therefore $G(18)$, shown in Figure 1, is a connected, non-complete graph on six vertices.

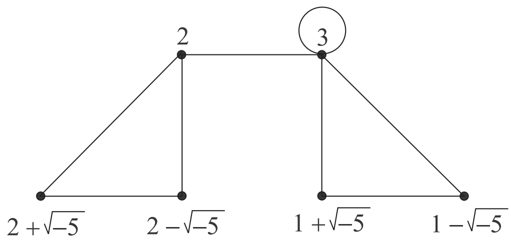


FIGURE 1. $G(18)$ in $D = \mathbf{Z}\sqrt{-5}$.

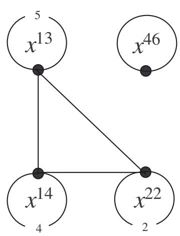


FIGURE 2. $G(x^{92})$ in $D = \mathbf{F}[x^{13}, x^{14}, x^{22}, x^{46}]$.

Example 1.3 [**7**, Example 1.3]. Let $D = \mathbf{F}[x^{13}, x^{14}, x^{22}, x^{46}]$ be the numerical semigroup ring over a field $\mathbf{F}$ with indeterminant x. The irreducible divisors of x^{92} are x^{13}, x^{14}, x^{22} and x^{46}. Moreover, x^{92} factors only as

$$(x^{13})^2(x^{22})^3 = (x^{22})(x^{14})^5 = (x^{13})^6(x^{14}) = (x^{46})^2.$$

Therefore, $G(x^{92})$, shown in Figure 2, has two disjoint connected components: one which is a complete graph on the vertices x^{13}, x^{14} and x^{22} and the one which consists of the single vertex x^{46}.

These two examples illustrate that irreducible divisor graphs need not be complete or even connected. This might indicate that there is insufficient structure in irreducible divisor graphs to exploit in investigating the ring-theoretic properties of factorizations. However, the following result by Coykendall and Maney showed otherwise, providing the impetus to further investigate these graphs. The proof presented is from [**7**, Theorem 2.1].

Theorem 1.4 [**12**, Theorem 5.1]. *Let D be an atomic domain. The following statements are equivalent.*

(1) *D is a* UFD.

(2) *$G(x)$ is complete for all $x \in D^*\backslash U(D)$.*

(3) *$G(x)$ is connected for all $x \in D^*\backslash U(D)$.*

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

To prove (3) implies (1), we show that the set $\mathcal{A}$ of all $x \in D^*\backslash U(D)$ that admit at least two distinct factorizations into irreducibles is empty.

Assume otherwise, and let $n = \min_{z\in\mathcal{A}}\{k : z = \pi_1\pi_2\cdots\pi_k$ with each π_i irreducible$\}$. Clearly, $n \geq 2$. Thus, there exists a $y \in D^*\backslash U(D)$ such that $y = \pi_1\pi_2\cdots\pi_n$. Since $y \in \mathcal{A}$, $y = \gamma_1\gamma_2\cdots\gamma_t$ with each γ_j irreducible and $t \geq n$. Since $G(y)$ is connected, without loss of generality, we may assume that there is an edge connecting π_1 and γ_1. If π_i is an associate to γ_j, then $y/\pi_i = u\pi_1\cdots\widehat{\pi}_i\cdots\pi_n = v\gamma_1\cdots\widehat{\gamma_j}\cdots\gamma_t$, with $u, v \in U(D)$, which gives two distinct factorizations of y/π_i into irreducibles, contradicting the minimality of n. Thus, π_i is not an associate of any γ_j. Since there is an edge connecting π_1 and γ_1, $\pi_1\gamma_1 \mid y$, hence $y = \pi_1\gamma_1\alpha_1\cdots\alpha_m = \pi_1\cdots\pi_n$ with α_i irreducible. Then $\gamma_1\alpha_1\cdots\alpha_m = \pi_2\cdots\pi_n$ are two distinct factorizations of y/π_1 into irreducibles, contradicting the minimality of n. Therefore, $\mathcal{A} = \varnothing$ and D is a UFD. □

In Section 2 we investigate other characterization results similar to Theorem 1.4 as well as some possible limitations to such results. Section 3 discusses a compressed version of the irreducible divisor graph, while Section 4 investigates the homology and simplicial complex structure of irreducible divisor graphs. In Section 5 we extend the study of irreducible divisor graphs to commutative rings with zero-divisors, and Section 6 investigates graph invariants of these graphs as well as realization results. Section 6 also provides a detailed discussion of such graphs in numerical monoids and numerical semigroup rings.

2. Characterization results. The process of characterizing several domain properties by various aspects of the irreducible divisor graphs of an atomic ring has proved very fruitful in many circumstances.

These results will be explored below, followed by an explanation of the likely impossibility of characterizing BFD's via their irreducible divisor graphs.

We begin with finite factorization domains and a characterization result below.

Theorem 2.1. [**7**, Theorem 3.1] and [**12**, Proposition 3.1]. *Let D be an atomic domain. The following statements are equivalent.*

(1) *D is a finite factorization domain.*

(2) *$G(x)$ is finite for every nonzero nonunit $x \in D$.*

(3) $\deg(\pi) < \infty$ *in $G(x)$ for every nonzero nonunit $x \in D$ and for every $\pi \in V(G(x))$.*

(4) $\mathrm{degl}(\pi) < \infty$ *in $G(x)$ for every nonzero nonunit $x \in D$ and for every $\pi \in V(G(x))$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear from the definition of an FFD and the construction of $G(x)$.

To prove (3) $\Rightarrow$ (1), assume D is not an FFD. Then there exists a $y \in D^*\backslash U(D)$ such that the set $\{\pi_\lambda\}_{\lambda\in\Lambda}$ of nonassociate irreducible divisors of y is infinite. Then, in $G(y^2)$, the vertices associated to π_{λ_i} and π_{λ_j} are connected by an edge for all pairs λ_i and λ_j. Therefore, $\deg(\pi_{\lambda_1}) = \infty$, and thus (3) fails to hold.

Thus, we have the equivalence of (1), (2) and (3). Clearly, (4) implies (3). Suppose that (4) fails. Then either (3) fails or some vertex π in $G(x)$ has infinitely many loops. In this case, D is not a BFD and hence not an FFD. Therefore, (1) implies (4), and thus all four conditions are equivalent. □

The following corollary is immediately obvious, although a proof of the result that does not involve Theorem 2.1 is provided as a representative argument involving irreducible divisors.

Corollary 2.2 [**7**, Proposition 3.2]. *Let D be an atomic domain such that, for every nonzero nonunit $x \in D$,* $\mathrm{degl}(\pi) < \infty$ *for all $\pi \in V(G(x))$. Then D satisfies the* ACCP.

Proof. Assume that D does not satisfy the ACCP. Then there exists an ascending chain $(x_1) \subsetneq (x_2) \subsetneq (x_3) \subsetneq \cdots$ of proper principal ideals. Thus we have $x_1 = a_2x_2 = a_2a_3x_3 = \cdots$ for some $a_i \in D^*\backslash U(D)$. Since D is atomic, we may express these a_i's as a product of irreducibles, i.e.,

$$x_1 = \left(\prod_{i=1}^{n_2} a_{2,i}\right)x_2 = \left(\prod_{i=1}^{n_2} a_{2,i}\right)\left(\prod_{j=1}^{n_3} a_{3,j}\right)x_3 = \cdots,$$

where $a_{k,l} \in \mathrm{Irr}\,(D)$. If the set $\{a_{j,i}\}$ is infinite, then $a_{2,1}$ has infinite degree in $G(x_1)$. Otherwise, some $a_{j,i}$ appears infinitely often in the factorization of x, and thus $a_{2,i}$ is connected to $a_{j,i}$, which has an infinite number of loops in $G(x_1)$. Either of these conditions implies that $\deg l\,(\pi) = \infty$ for some irreducible divisor π of x. $\square$

The field of algebraic number theory gave rise to the concept of a half-factorial domain (HFD). This type of atomic domain was thought to have been nicely categorized via irreducible divisor graphs, although, as will be shown below, this does not appear to be the case. An excellent survey of HFD's is provided by [**11**]. The following definition first found in [**12**] is required:

Definition 2.3. Let D be an atomic domain, and let $x \in D^* \setminus U(D)$. If $A \subseteq V(G(x))$, then by $G_A(x)$, we mean the induced subgraph of $G(x)$ on $N(A) = \{u \in V(G(x)) : u - v \in E(G(x)) \text{ for some } v \in A\}$. If $A = \{\pi_1, \pi_2, \ldots, \pi_n\}$, then G_A will be denoted $G_{(\pi_1,\pi_2,\ldots,\pi_n)}(x)$ and if $A = \{\pi\}$, then $G_A(x)$ will be denoted by $G_\pi(x)$.

It was stated in ([**12**, Theorem 3.3]) that an atomic domain D was an HFD if and only if, for every nonzero nonunit $x \in D$, and for any irreducible factorization $\pi_1^{a_1} \cdots \pi_n^{a_n}$ of x with the π_i's pairwise nonassociate, the sum of the number of vertices and the number of loops in $G_{(\pi_1,\ldots,\pi_n)}(x)$ is constant [**12**]. However, this statement is incorrect, as can be demonstrated by Example 1.2. In this example we consider the factorizations of 18 into irreducibles: $18 = 2 \cdot 3^2$, $18 = 3(1+\sqrt{-5})(1-\sqrt{-5})$ and $18 = 2(2+\sqrt{-5})(2-\sqrt{-5})$. Setting $A_1 = \{2, 3\}$ and $A_2 = \{3, 1+\sqrt{-5}, 1-\sqrt{-5}\}$, the graph provided in Example 1.2 shows that the number of loops and vertices in $G_{A_1}(18)$ does not equal the number of loops and vertices in $G_{A_2}(18)$.

A recently studied type of HFD, called a BVD, also yields a nice classification result via irreducible divisor graphs. Let D be an HFD with quotient field K. If $D \neq K$, define a boundary map $\partial_D : K^* \to \mathbf{Z}$ by $\partial_D(\alpha) = t - s$ where $\alpha = (\pi_1\pi_2\cdots\pi_t)/(\delta_1\delta_2\cdots\delta_s)$ where $\pi_i, \delta_j \in \mathrm{Irr}\,(D)$ for each i and j. If $D = K$, then we declare $\partial_D(\alpha) = 0$ for all $\alpha \in K^*$. We say that D is a boundary valuation domain (BVD) if, given any $\alpha \in K^*$ with $\partial_D(\alpha) \neq 0$, either $\alpha \in D$ or $\alpha^{-1} \in D$. This boundary map was first introduced in [**10**] and BVDs first appeared in [**15**]. We now present, without proof, a classification of BVDs via irreducible divisor graphs.

Proposition 2.4 [**12**, Proposition 4.4]. *Let D be an atomic domain. The following are equivalent.*

(1) *D is a* BVD.

(2) *For every nonzero nonunit $x \in D$, the following hold*:

(a) *Either x is irreducible or $V(G(x)) = \overline{\mathrm{Irr}}\,(D)$.*

(b) *If $x = \alpha\beta$ with $\alpha, \beta \in \overline{\mathrm{Irr}}\,(D)$, then $G(x)$ is a disjoint union of graphs, each of which is isomorphic to K_2 or to a vertex with a single loop.*

(c) *If $x = \alpha_1 \cdots \alpha_n$ with $\alpha_i \in \overline{\mathrm{Irr}}\,(D)$ and $n \geq 3$, then $G(x)$ is a pseudoclique on $V(G(x))$.*

The irreducible divisor graph has been shown to provide strong graph equivalencies to the algebraic property of a domain being a UFD, an FFD and (to a lesser extent) some types of HFDs. However, this promise is not fully borne out in the realm of HFDs. In [**7**, Example 3.3], an HFD D that is not an FFD is constructed having the property that the vertices of $G(x)$ have infinite degree not counting loops for a given $x \in D$. In this same example, for a given $y \in D$, $G(y)$ consists of uncountably many disjoint complete graphs on 2 vertices. Thus, many pleasing graphical properties need not be present in the irreducible graph of an element of an HFD. This example and others in [**7**] indicate that irreducible divisor graphs may not be a useful tool in the study of HFDs or BFDs that are not FFDs.

2.1. Elasticity. While a characterization theorem of bounded factorization domains using irreducible divisor graphs seems unlikely, we can use these graphs to provide bounds on the elasticity of an element x in a BFD D by considering subgraphs of $G_D(x)$. These results of this section will be somewhat improved in Section 4.

The set of lengths of an element x in a bounded factorization domain D is $L(x) = \{t : x = a_1 a_2 \cdots a_t$ where each a_i is irreducible$\}$. It is then clear that D is an HFD if and only if each $L(x)$ is a singleton. The *elasticity* of x, denoted $\rho(x)$, gives a measure of how far the element x is from having unique factorization. We define the elasticity of x in D as $\rho(x) = \max(L(x))/\min(L(x))$. (Note that the standard definition of elasticity involves the ratio of the supremum to the infimum of $L(x)$, but that we may replace these with the maximum and the minimum since we are working in a BFD.)

We say that a collection of vertices $V = \{a_1, a_2, \ldots a_t\}$ and connecting edges is a complete subgraph of a graph G if (1) each element in V is a vertex of G, and (2) for all pairs $i \neq j$, $a_i - a_j$ is an edge in G.

Proposition 2.5 [**7**, Proposition 4.1]. *Let x be a non-irreducible nonzero nonunit of a* BFD D. *Then*

$$\rho(x) \leq \frac{1}{2} \max \left\{ t + l : \begin{array}{c} G(x) \text{ contains a complete subgraph with } t \text{ vertices} \\ \text{having a total of } l \text{ loops on its } t \text{ vertices} \end{array} \right\}.$$

Proof. Since x is not irreducible, $\min(L(x)) \geq 2$. Let $M = \max(L(x))$. Then we can write $x = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}$ for some irreducibles a_i with $a_i \neq a_j$ unless $i = j$ and $\sum_{i=1}^{t} n_i = M$. Then each a_i is a vertex in $G(x)$. Moreover, since $a_i a_j \mid x$ for each pair i, j, there is an edge in $G(x)$ connecting vertex a_i to vertex a_j, that is, $G(x)$ contains a copy of the complete graph K_t with vertices $a_1, a_2, \ldots a_t$. Also, for each i, $1 \leq i \leq t$, there are $n_i - 1$ loops drawn on vertex a_i. Thus, for any factorization of x of length M, we can find a complete subgraph of $G(x)$ that contains at least M vertices/loops. □

We note that, if x is irreducible, then $G(x)$ contains a single vertex with no loops, but yet $\rho(x) = 1$. It is often the case that many factorizations of x are jumbled together in $G(x)$, thus making the bound in Proposition 2.5 rather crude.

Example 2.6. Let $D = \mathbf{Z}[\sqrt{-5}]$. It is well known that D is an HFD, and thus $\rho(x) = 1$ for all $x \in D^*\backslash U(D)$. The bound given in Proposition 2.5 gives the exact elasticity for some elements of D, but is off by a factor of two for the element $18 \in D$. The irreducible divisor graph of 6 consists of two disjoint copies of a complete graph on two vertices and has no loops. Thus, according to Proposition 2.5, $\rho(6) \leq 2/2 = 1$, which is the exact elasticity of 6. However, for the element $18 \in D$ whose irreducible divisor graph is given in Figure 1, Proposition 2.5 gives $\rho(18) \leq 4/2 = 2$, which is true but off by a factor of two from the actual elasticity.

Example 2.7. Let $D = \mathbf{F}[x^{13}, x^{14}, x^{22}, x^{46}]$ be as in Example 1.3. Then, applying Proposition 2.5 to the irreducible divisor graph of x^{92} in D as shown in Figure 2, we have that $7/2 = \rho(x^{92}) \leq 14/2$, and we are again off by a factor of two.

This result does read easier if we know that $a^2 \nmid x$ for any irreducible a in D. Indeed, in this case we have the following corollary.

Corollary 2.8 [**7**, Corollary 4.4]. *Let x be a non-irreducible nonzero nonunit of a domain D such that $a^2 \nmid x$ for any irreducible a in D. Then*

$$\rho(x) \leq \frac{1}{2}\max\{t : G(x) \textit{ contains a complete subgraph with } t \textit{ vertices}\}.$$

The next result gives an alternate way of looking at elasticity in terms of irreducible divisor graphs. We say that a graph G contains disjoint copies of two complete subgraphs X and Y if (1) X and Y are complete graphs, (2) the vertex sets of X and Y are contained in the vertex set of G, (3) the edge sets of X and Y are contained in the edge set of G and (4) the intersection of vertex sets of X and Y is empty.

Proposition 2.9 [**7**, Proposition 4.5]. *Let x be a nonzero nonunit element of a* BFD *D with $\rho(x) = s/t$. Then $G(y)$ contains disjoint copies of complete subgraphs K_M with a total of m loops on its vertices and K_N having a total of n loops on its vertices for some divisor y of x such that $(M + m) - (N + n) = s - t$.*

Proof. Since $\rho(x) = s/t$, we can factor x into irreducibles as $x = a_1^{m_1} a_2^{m_2} \cdots a_M^{m_M} = b_1^{n_1} b_2^{n_2} \cdots b_N^{n_N}$ with $m_1 + m_2 + \cdots + m_M = s$ and $n_1 + n_2 + \cdots n_N = t$. If $a_i \neq b_j$ for all i, j, then set $y = x$. Each a_i and b_j is a vertex in $G(y)$. Moreover, $a_i a_k$ divides y for each pair i, k, and $b_j b_l$ divides y for each pair j, l. Hence, the vertex set $\{a_1, a_2, \ldots, a_M\}$ forms a copy of K_M with $m = s$ total loops on its vertices, while the vertex set $\{b_1, b_2, \ldots, b_N\}$ forms a copy of K_N with $n = t$ total loops on its vertices.

It is possible that $a_i = b_j$ for some pair i, j. If this is the case, cancel all like terms to arrive at two factorizations of some divisor y of x: $y = a_1^{m'_1} a_2^{m'_2} \cdots a_M^{m_M} = b_1^{n_1} b_2^{n'_2} \cdots b_N^{n'_N}$ where the smaller of m'_i and n'_j is zero whenever $a_i = b_j$. Let $m'_1 + m'_2 + \cdots + m'_M - M = m'$ and $n'_1 + n'_2 + \cdots n'_N - N = n'$. Since an equal number of terms were canceled from each side, we have that $m' - n' = s - t$. An argument similar to the one above gives the existence of two disjoint complete subgraphs of $G(y)$. □

Again, this result is cleaner if we insist that $a^2 \nmid x$ for any irreducible a in D.

Corollary 2.10 [**7**, Corollary 4.6]. *Let x be an element of a domain D with $\rho(x) = s/t$ such that $a^2 \nmid x$ for any non-unit $a \in D$. Then $G(y)$ contains disjoint copies of the complete graphs K_M and K_N for some divisor y of x and some positive integers M and N with $M - N = s - t$.*

3. Compressed irreducible divisor graphs. In [**7**], the authors propose an alternate definition in an attempt to simplify irreducible divisor graphs and their results. Consider two irreducible divisors π_1 and π_2 of an element x. If, for any factorization of x, it is always the case that both π_1 and π_2 appear in the factorization or neither π_1 nor π_2 appear in the factorization, then π_1 and π_2 will be connected to the same set of vertices and hence contribute the same information in $G(x)$. To eliminate this redundant information, we define an equivalence relation on the irreducible divisors of x as follows.

Definition 3.1 [**7**, Section 5]. Let D be an atomic domain, and let $x \in D^* \backslash U(D)$. Then we say that $a, b \in \overline{\text{Irr}}\,(D)$ are x-equivalent, denoted $a \sim_x b$, if, whenever a appears as a factor in a factorization of x into irreducibles, then b does as well, and vice versa (up to associates).

It is easy to see that $\sim_x$ is an equivalence relation on the set of irreducible divisors of x. We now use the equivalence classes of these irreducible divisors of x, denoted $[y_i]_x$, as vertices of the compressed zero-divisor graph.

Definition 3.2 [**7**, Definition 5.1]. Let D be an atomic domain, and let $x \in D^* \backslash U(D)$. The *compressed irreducible divisor graph of* x, denoted $G_c(x)$, is given by (V, E), where $V = \{[y]_x \mid y \in \overline{\operatorname{Irr}}(R)$ and $y \mid x\}$, and given distinct $[y_1]_x, [y_2]_x \in V$, $[y_1]_x - [y_2]_x \in E$ if and only if $y_1 y_2 \mid x$.

We note that, while the edges of this graph are well defined, loops are not. Therefore, loops are not included in $G_c(x)$, and when comparing $G_c(x)$ with $G(x)$, we will ignore any loops in $G(x)$. We also note that $G(x)$ and $G_c(x)$ may or may not be identical.

Example 3.3. Recall the irreducible divisor graph $G(18)$ in the ring $D = \mathbf{Z}[\sqrt{-5}]$ as given in Figure 1. As we saw in Example 1.2, we can factor $18 = 2 \cdot 3^2 = 3(1+\sqrt{-5})(1-\sqrt{-5}) = 2(2+\sqrt{-5})(2-\sqrt{-5})$. Since $1+\sqrt{-5}$ and $1-\sqrt{-5}$ always appear together in any factorization of 18 into irreducibles, $[1+\sqrt{-5}]_{18} = [1-\sqrt{-5}]_{18}$. Similarly, $[2+\sqrt{-5}]_{18} = [2-\sqrt{-5}]_{18}$. Therefore, the compressed irreducible divisor graph $G_c(18)$ of 18 in D is as shown in Figure 3. We see that each of the two cliques of order three in Figure 1 have been collapsed to an edge between two vertices in Figure 3.

Example 3.4. Recall the irreducible divisor graph $G(x^{92})$ in the ring $D = \mathbf{F}[x^{13}, x^{14}, x^{22}, x^{46}]$ as in Example 1.3. Since the only factorizations of x^{92} in D are $(x^{13})^2(x^{22})^3 = (x^{22})(x^{14})^5 = (x^{13})^6(x^{14}) = (x^{46})^2$, we see that no pair of irreducible divisors of x^{92} always appear together in any factorization of x^{92}. Therefore, $G_c(x^{92})$ is the graph $G(x^{92})$ as shown in Figure 2 without the loops.

Other examples given in [**7**] further illustrate the similarities and differences between $G(x)$ and $G_c(x)$. While necessary and sufficient

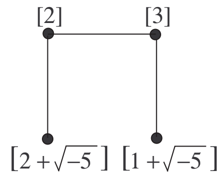


FIGURE 3. $G_c(18)$ in $D = \mathbf{Z}[\sqrt{-5}]$.

conditions guaranteeing the equality of $G(x)$ and $G_c(x)$ are unknown at this time, the following proposition does provide a sufficient condition.

Proposition 3.5 [**7**, Proposition 5.6]. *Let D be an atomic domain, let $x \in D$, suppose that $G(x) \ncong K_2$ and that $G(x)$ contains no subgraph isomorphic to K_n for $n \geq 3$. If $G(x)$ is connected, then $G(x) = G_c(x)$.*

Further investigation into the similarities between $G(x)$ and $G_c(x)$ leads to the following series of results, which includes a distinct improvement on Theorem 1.4.

Theorem 3.6 [**7**, Theorem 5.7]. *Let D be an atomic domain. Let $x \in D^* \backslash U(D)$. Then:*

(1) *$G(x)$ is connected if and only if $G_c(x)$ is connected.*

(2) *$G_c(x)$ is complete if and only if $G(x)$ is complete.*

(3) *D is a UFD if and only if $G_c(x) \cong K_1$ for all nonunits $x \in D$.*

Proof. Let D be an atomic domain and $x \in D^* \backslash U(D)$.

(1) ($\Rightarrow$). Consider $[a]_x \neq [b]_x$ in $V(G_c(x))$. If $G(x)$ is connected, then there is a path $a - c_1 - \cdots - c_n - b$ in $G(x)$. Then $[a]_x - [c_1]_x - \cdots - [c_n]_x - [b]_x$ is a walk in $G_c(x)$, and hence $G_c(x)$ is connected.

($\Leftarrow$). Assume $G_c(x)$ is connected, and let a, b be irreducible divisors of x. If $[a]_x = [b]_x$, then a and b must appear together in any factorization of x, so $a - b$ is in an edge in $G(x)$. Assume now that $[a]_x \neq [b]_x$ and that $[a]_x$ and $[b]_x$ are connected in $G_c(x)$ via a path

$[a]_x - [d_1]_x - \cdots - [d_n]_x - [b]_x$. Then $a - d_1 - \cdots - d_n - b$ is a path in $G(x)$ for some set of d_i, each an arbitrary representative from the equivalence class $[d_i]$, and hence $G(x)$ is connected.

(2) ($\Rightarrow$). Let a and b be distinct elements in $V(G(x))$. If $[a]_x = [b]_x$, then $ab \mid x$, and hence $a - b$ is an edge in $G(x)$. If $[a]_x \neq [b]_x$, then $[a]_x - [b]_x$ is an element of $E(G_c(x))$, and thus $ab \mid x$. Hence, $G(x)$ is complete.

($\Leftarrow$). If $[a]_x$ and $[b]_x$ are distinct elements in $V(G_c(x))$, then a and b are distinct elements in $V(G(x))$. Since $G(x)$ is complete, $ab|x$ and thus $[a]_x - [b]_x$. Therefore, $G_c(x)$ is complete.

(3) ($\Rightarrow$). If D is a UFD, then each element $x \in D$ has a unique factorization $x = a_1^{n_1} \cdots a_t^{n_t}$ with the a_i distinct and irreducible. Thus, $G_c(x)$ consists of the single vertex $[a_1]_x$.

($\Leftarrow$). Assume now that $G_c(x)$ consists of a single vertex for each $x \in D$. If $x \in D$ is a nonzero nonunit, then every factorization of x has the form $a_1^{m_1} a_2^{m_2} \cdots a_t^{m_t}$ for some fixed set of irreducible divisors a_i of x and a collection of positive integers m_i. If $a_1^{m_1} a_2^{m_2} \cdots a_t^{m_t} = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}$ are distinct factorizations, then there must be some i for which $m_i \neq n_i$. Without loss of generality, assume that $m_1 < n_1$. Then $a_2^{m_2} a_3^{m_3} \cdots a_t^{m_t} = a_1^{n_1 - m_1} a_2^{n_2} \cdots a_t^{n_t} =: y$. But then $G_c(y)$ contains at least two distinct vertices, contradicting our hypotheses. Thus, $m_i = n_i$ for all i, and hence x has a unique factorization. □

The proof of Theorem 2.6 also proves the following diameter result.

Corollary 3.7 [**7**, Corollary 5.9]. *Let D be an atomic domain. Let $x \in D^* \backslash U(D)$. Then* $\operatorname{diam}(G_c(x)) \leq \operatorname{diam}(G(x))$.

In addition, Theorems 1.4 and 3.6 provide a classification result for UFDs.

Corollary 3.8 [**7**, Corollary 5.8]. *Let D be an atomic domain. The following statements are equivalent.*

(1) *D is a* UFD.

(2) *$G_c(x)$ is a single vertex for every $x \in D^* \backslash U(D)$.*

(3) *$G_c(x)$ is complete for every $x \in D^* \backslash U(D)$.*

(4) *$G_c(x)$ is connected for every $x \in D^* \backslash U(D)$.*

As discussed in the previous section, irreducible divisor graphs appear to be more useful to study within the realm of FFDs. To that end, compressed irreducible divisor graphs yield a characterization theorem similar to Theorem 2.1.

Theorem 3.9 [**7**, Theorem 5.10]. *Let D be an atomic domain. Then D is an* FFD *if and only if $G_c(x)$ is finite for every $x \in D^*\backslash U(D)$.*

Proof. ($\Rightarrow$). Clear.

($\Leftarrow$). Assume $G_c(x)$ is finite for every $x \in D^*\backslash U(D)$, and suppose that there exists some $x \in D^*\backslash U(D)$ with infinitely many nonassociate irreducible divisors $\{a_1, a_2, \dots\}$. Since $G_c(x)$ is finite, without loss of generality, assume that $|[a_1]_x| = \infty$, i.e., assume $[a_1]_x = \{b_j\}_{j\in\Lambda}$ for some infinite indexing set Λ. Thus, factorizations of x into irreducibles in which one factor is a_1 must also involve b_j for all $j \in \Lambda$, a contradiction, since infinitely long factorizations are not allowed. □

Corollary 3.10 [**7**, Corollary 5.11]. *Let D be an atomic domain. The following statements are equivalent.*

(1) *D is a finite factorization domain.*

(2) *For all $x \in D^*\backslash U(D)$, $G_c(x)$ is finite.*

(3) *For all $x \in D^*\backslash U(D)$, $\deg([\pi]_x) < \infty$ for all $[\pi]_x \in V(G_c(x))$.*

4. Homology and simplicial complexes. In [**16**], Maney considered homology (calculated over the field $\mathbf{Z}_2$) in order to better understand the structure of irreducible divisor graphs. A careful and thorough introduction to these ideas, terminology and notation can be found in his paper. In addition, a characterization of UFDs is given by considering homology groups. The following two results consider the zeroth and first homologies, as defined in [**16**] and give conditions for when an irreducible divisor graph $G(x)$ is connected and when D is a UFD. In light of Theorem 1.4, these results are perhaps not surprising.

Theorem 4.1 [**16**, Theorem 3.3]. *Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be the set of distinct components of $G(x)$. Then $H_0 \cong \oplus_{\alpha\in\Lambda}\mathbf{Z}_2$. In particular, $H_0 \cong \mathbf{Z}_2$ if and only if $G(x)$ is connected.*

Proposition 4.2 [**16**, Corollary 4.5]. *Let D be a* UFD. *Then, for each nonzero nonunit $x \in D$, $H_1(x; D) = 0$.*

The main result from [**16**] is given below, where necessary and sufficient conditions are given for an atomic domain D to be a UFD.

Theorem 4.3 [**16**, Theorem 5.4]. *Let D be an atomic domain. The following are equivalent.*

(1) *D is a* UFD.

(2) *For each nonzero nonunit $x \in D$ and each $n \in \mathbf{N}$, $H_0(x; D) \cong \mathbf{Z}_2$ and $H_n(x; D) = 0$.*

(3) *For each nonzero nonunit $x \in D$, $H_0(x; D) \cong \mathbf{Z}_2$.*

Note that, in this result, higher homologies are considered as well as H_0 and H_1. In fact, even though they are not explicitly discussed in [**16**], it is not just irreducible divisor graphs that are being considered, but irreducible divisor *simplicial complexes* in which faces of dimension larger than one occur. This more generalized approach to studying graphical representations of factorizations was studied in [**9**]. We now discuss the major results from this paper, including a result that shows that the *irreducible divisor simplicial complex* of every element contains all possible subsimplicies if and only if D is a UFD, thus putting the characterization in Theorem 4.3 into a more familiar context.

Definition 4.4 [**9**]. If D is an atomic domain and x is a nonzero nonunit of D, then the *irreducible divisor simplicial complex* of x in D is $S_D(x)$ given by (V, F), where $V = \{y \in \overline{\text{Irr}}\,(D) : y \mid x\}$ and $\{y_1, y_2, \dots, y_n\} \in F$ if and only if $y_1 y_2 \cdots y_n \mid x$. In addition, we place $n-1$ loops on vertex y if $y^n \mid x$ but $y^{n+1} \nmid x$.

Note that, if we ignore loops, this definition matches the standard definition of a simplicial complex, which we provide in Definition 4.6 along with other relevant definitions.

Example 4.5. As in Example 1.2, let $D = \mathbf{Z}[\sqrt{-5}]$ and consider the factorizations of 18 into irreducibles: $18 = 2 \cdot 3^2$, $18 = 3(1+\sqrt{-5})(1-\sqrt{-5})$ and $18 = 2(2+\sqrt{-5})(2-\sqrt{-5})$. If $S(18) = (V, E)$, then

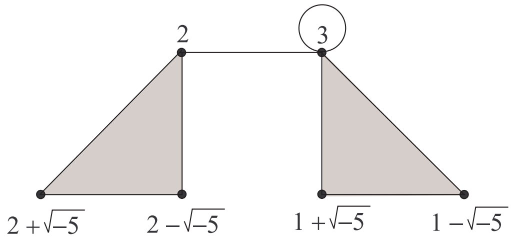


FIGURE 4. $S(18)$ in $D = \mathbf{Z}[\sqrt{-5}]$.

$V = \{2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}, 2+\sqrt{-5}, 2-\sqrt{-5}\}$ and $F = F_1 \cup F_2 \cup F_3$ where $F_1 = \{\{2\}, \{3\}, \{1+\sqrt{-5}\}, \{1-\sqrt{-5}\}, \{2+\sqrt{-5}\}, \{2-\sqrt{-5}\}\}$, $F_2 = \{\{2, 3\}, \{3, 1+\sqrt{-5}\}, \{3, 1-\sqrt{-5}\}, \{1+\sqrt{-5}, 1-\sqrt{-5}\}, \{2, 2+\sqrt{-5}\}, \{2, 2-\sqrt{-5}\}, \{2+\sqrt{-5}, 2-\sqrt{-5}\}\}$ and $F_3 = \{\{3, 1+\sqrt{-5}, 1-\sqrt{-5}\}, \{2, 2+\sqrt{-5}, 2-\sqrt{-5}\}\}$. Graphically, $S(18)$ is shown in Figure 4 and looks exactly like $G(18)$ but with the two 2-dimensional faces shaded in to indicate that the products $3(1+\sqrt{-5})(1-\sqrt{-5})$ and $2(2+\sqrt{-5})(2-\sqrt{-5})$ divide 18 in $\mathbf{Z}[\sqrt{-5}]$.

As is evident by looking at Example 4.5, if we consider only 0- and 1-dimensional faces of $S_D(x)$, faces of cardinality 1 and 2, we obtain $G_D(x)$. We now make this more precise. For consistency, we use the notation $\{u, v\}$ to denote an edge in $G_D(x)$, since this now matches the notation for the 1-dimensional face $\{u, v\}$ in $S_D(x)$.

Definition 4.6. A *simplicial complex* is a pair $S = (V, F)$ where V is a finite set of vertices and F is a collection of finite non-empty subsets of V (called *faces*) satisfying:

(1) $\{v\} \in F$ for all $v \in V$, and

(2) if $\sigma \in F$, and $\varnothing \neq \tau \subseteq \sigma$, then $\tau \in F$.

(3) A face $\sigma \in F$ maximal with respect to set containment is called a *facet* of S.

(4) A face $\sigma = \{a_1, a_2, \ldots, a_{d+1}\}$ is said to have *dimension* d.

(5) For a nonnegative integer k, the *k-skeleton* of S, denoted S_k, is the subcomplex of S consisting of all faces of S whose dimension is at most k.

Proposition 4.7 [**9**]. *Let D be an atomic domain, and let x be a nonzero nonunit of D. The 1-skeleton of $S_D(x)$ is precisely $G_D(x)$.*

Proof. Let $G_D(x) = (V, E)$ denote the irreducible divisor graph of x, and let $S_D(x) = (V', F)$ denote the irreducible divisor simplicial complex of x. Clearly, $V = V'$ since both sets are precisely the elements in $\overline{\text{Irr}}\,(D)$ that divide x. By the definition of $G_D(x)$, if $a, b \in V$ with $\{a, b\} \in E$, then $ab \mid x$, whence $\{a, b\} \in F$. Moreover, if vertices a and b are adjacent in $S_D(x)$, then $\{a, b\} \in F$, and hence $ab \mid x$. Therefore $\{a, b\} \in E$. ◻

Therefore, we see that the concept of the irreducible divisor simplicial complex really is a higher-dimensional analog of the irreducible divisor graph. Because this structure generally contains components of dimension 2 and higher, $S_D(x)$ carries more information than $G_D(x)$ about the factorizations of the element x in domain D. For example, although occasionally a factorization of x is visible in $G_D(x)$ by looking at complete subgraphs of $G_D(x)$ (see Example 1.2), this is not always the case as illustrated by Example 1.3: the three vertices x^{13}, x^{14} and x^{22} form a clique, yet these three elements do not appear together in any one factorization of x^{92}. However, the next few results will demonstrate that the irreducible divisor simplicial complex offers far more hope. The following two results show that it is easier to find factorizations of x by looking at $S_D(x)$.

Theorem 4.8 [**9**]. *If $A = \{a_1, a_2, \ldots, a_n\}$ is a facet of $S(x)$, then x has a factorization of the form $x = a_1^{m_1} \cdots a_n^{m_n}$ where $m_i \geq 1$ for all i.*

Proof. Since A is a face of $S(x)$, it is clear from the definition of $S(x)$ that $a_1 a_2 \cdots a_n \mid x$. Now, if $x = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} b_1 \cdots b_t$ for some nonnegative integers $m_1, \ldots, m_n$ and some irreducible elements $b_1, \ldots, b_t \notin \{a_1, a_2, \ldots, a_n\}$, then $\{a_1, a_2, \ldots, a_n, b_1\}$ is a face of $S(x)$ strictly larger than A, contradicting that A is a facet of $S(x)$. Therefore, x has a factorization $x = a_1^{m_1} \cdots a_n^{m_n}$ where each m_i is a positive integer. ◻

As is illustrated in the following example, the converse does not always hold, since $\{2, 3\}$ is a face of $S(108)$ that does not correspond to a facet.

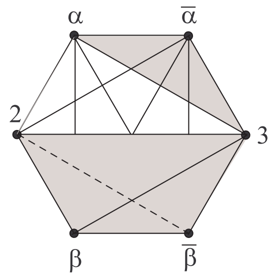


FIGURE 5. $S(108)$ in $D = \mathbf{Z}[\sqrt{-5}]$.

Example 4.9. As in Examples 1.2 and 4.5, let $D = \mathbf{Z}[\sqrt{-5}]$. Now let $x = 180$, let $\alpha = 1 + \sqrt{-5}$, and let $\beta = 2 + \sqrt{-5}$. By using norms, we see that 108 factors into irreducibles only as

$$108 = 2^3 \cdot 3^3 = 2 \cdot 3^2 \alpha \cdot \overline{\alpha} = 2^2 \cdot 3 \cdot \beta \cdot \overline{\beta} = 3 \cdot \alpha^2 \cdot \overline{\alpha}^2 = 2 \cdot \alpha \cdot \overline{\alpha} \cdot \beta \cdot \overline{\beta}.$$

in D. Graphically, a portion of $S(108)$ is shown in Figure 5, where the facets $\{3, \alpha, \overline{\alpha}\}$ and $\{2, 3, \beta, \overline{\beta}\}$ are shaded. Note that there are other facets of dimensions 3 and 4 that are not represented in this illustration. However, the converse does hold if x is square free.

Proposition 4.10 [**9**]. *If a subsimplex of $S(x)$ is loop free, then facets correspond exactly to factorizations.*

Proof. If $x = a_1 a_2 \cdots a_n$ with none of the vertices a_i looped in $S(x)$, then $\{a_1, a_2, \ldots, a_n\}$ is a facet of $S(x)$. Indeed, if $\{a_1, a_2, \ldots, a_n\} \subsetneq \{a_1, a_2, \ldots, a_n, b_1, \ldots, b_t\}$, then we can factor x both as $x = a_1 a_2 \cdots a_n$ and as $x = a_1 a_2 \cdots a_n b_1 y$ for some nonzero nonunit y. Setting the two factorizations equal and canceling, we arrive at a clear contradiction: $1 = b_1 y$. □

The following theorem extends the results of Theorem 1.4 and gives further characterizations of unique factorization domains. First, we provide a couple of definitions. Recall that, for a set X, the set $\mathcal{P}(X)$ denotes the *power set* of X and consists of all subsets of X. If $S = (V, F)$ and $T = (V', F')$ are two simplicial complexes, then their *join* is the simplicial complex $S * T = (V \cup V', F'')$ where $F'' = \{A \cup B : A \in F, B \in F'\}$. Note that, as simplicial complexes, $\mathcal{P}(X) * \mathcal{P}(Y) = \mathcal{P}(X \cup Y)$. We first require a simple lemma.

Lemma 4.11 [**9**]. *Let a, b be nonzero nonuits of a* UFD. *Then* $V(S(ab)) = V(S(a)) \cup V(S(b))$.

Proof. Let $x \in V(S(ab))$. Then $x \mid ab$. Note that x is irreducible and hence prime. If $x \mid a$, then $x \in V(S(a))$. If $x \nmid a$, then $x \mid b$ and hence $x \in V(S(b))$. Therefore, $x \in V(S(a)) \cup V(S(b))$. Conversely, suppose that $x \in V(S(a)) \cup V(S(b))$. Then, either $x \in V(S(a))$ or $x \in V(S(b))$, which implies that either $x \mid a$ or $x \mid b$. In either case, $x \mid ab$ and so $x \in V(S(ab))$. □

Note that the inclusion $V(S(ab)) \supseteq V(S(a)) \cup V(S(b))$ does not require D to be a UFD.

Theorem 4.12 [**9**]. *Let D be an atomic domain. The following are equivalent.*

(1) *D is a* UFD.

(2) $V(S(x)) = \mathcal{P}(\overline{Irr}(x))$ *for each nonzero nonunit $x \in D$.*

(3) $S(xy) = S(x) * S(y)$ *for all nonzero nonunits $x, y \in D$.*

Proof. Suppose that D is a UFD. If x is a nonzero nonunit in D, then factor x uniquely as $a_1^{m_1} \cdots a_n^{m_n}$, where $\overline{\mathrm{Irr}}\,(x) = \{a_1, \ldots, a_n\}$, and where each m_i is a positive integer. Since this is the only factorization of x in D, it is clear that $V(S_D(x)) = \mathcal{P}(\overline{\mathrm{Irr}}\,(x))$.

Conversely, if $V(S(x)) = \mathcal{P}(\overline{\mathrm{Irr}}\,(x))$ for each nonzero nonunit $x \in D$, then by Proposition 4.7, $G(x)$ is complete for each nonzero nonunit $x \in D$. Then the characterization in Theorem 1.4 guarantees that D is a UFD.

Assume that D is a UFD, and let x and y be nonzero nonunits of D. By the equivalence of (1) and (2), $V(S(a)) = \mathcal{P}(\overline{\mathrm{Irr}}\,(a)) = \mathcal{P}(V(S(a)))$

for each nonzero nonunit a of D. By Lemma 4.11, we have that

$$\begin{aligned} S(xy) &= \mathcal{P}(V(S(xy))) = \mathcal{P}(V(S(x)) \cup V(S(y))) \\ &= \mathcal{P}(V(S(x))) * \mathcal{P}(V(S(y))) = S(x) * S(y). \end{aligned}$$

Conversely, if D (which is atomic) is not a UFD, then there exists an irreducible $z \in D$ that is not prime. That is, there exist $a, b \in D$ with $z \mid ab$, but yet $z \nmid a$ and $z \nmid b$. Set $x = ab$ and factor $a = a_1^{m_1} \cdots a_n^{m_n}$ and $b = b_1^{t_1} \cdots b_s^{t_s}$ with $a_i, b_j \in \overline{\text{Irr}}\,(x)$ for each i and j. Since $z \mid x$, $\{z\}$ is a face of $S(x)$. Now consider $S(a)*S(b)$. Since $z \nmid a$ and $z \nmid b$, $z \notin V(S(a))$ and $z \notin V(S(b))$, and hence $z \notin V(S(a)) \cup V(S(b)) \subseteq V(S(x))$. Therefore, $S(a) * S(b) \neq S(x)$. □

4.1. Elasticity. We now give some improvements on the elasticity results given in subsection 2.1.

Proposition 4.13. [**9**]. *Let x be a non-irreducible nonzero nonunit of a* BFD *D. Then*

$$\rho(x) \leq \frac{1}{2} \max \left\{ t + l : \begin{array}{c} S(x) \textit{ contains a facet with } t \textit{ vertices} \\ \textit{having a total of } l \textit{ loops on its } t \textit{ vertices} \end{array} \right\}.$$

Proof. Let x be a non-irreducible nonzero nonunit of D. If $x = a_1^{n_1} \cdots a_t^{n_t}$ is a factorization of x in D, then $a_1 \cdots a_t \mid x$, and hence $\{a_1, a_2, \ldots, a_t\}$ is a face of $S(x)$. Moreover, $\{a_1, \ldots, a_t\}$ is contained in some facet B with cardinality at least t, and the total number of loops on the vertices of this facet is at least $n_1 + \cdots + n_t$. This proves the result, since this shows that $\max\{L(x)\}$ is at most

$$\max \left\{ t + l : \begin{array}{c} S(x) \text{ contains a facet with } t \text{ vertices} \\ \text{having a total of } l \text{ loops on its } t \text{ vertices} \end{array} \right\}. \qquad \square$$

Moreover, this bound is at least as good as the result in subsection 2.1 achieved by considering $G(x)$. Indeed, if n is equal to the sum of the cardinality of some facet of $S(x)$ and the total number of loops on its vertices, then these vertices also form a complete subgraph of $G(x)$, and the number of vertices remains unchanged. In fact, by Proposition 4.10, when $S(x)$ is loop free, equivalently x is square-free, then we have the following proposition giving the exact elasticity of the element.

Proposition 4.14 [**9**]. *Let D be a bounded factorization domain. If x is a nonzero nonunit of D which is square-free, then $\rho(x)$ can be computed as the ratio of the largest facet of $S(x)$ to the smallest.*

5. Rings with zero-divisors. Factorizations in commutative rings with zero-divisors have been studied by a number of authors. Perhaps the most difficult issue to handle is that the notions of associate and of irreducible become clouded in these rings. In a domain D, if $a = rb$ and $b = sa$, then $r, s \in U(D)$. However, in a ring with zero-divisors, this does not have to be the case. In [**4**], the authors consider these situations by giving three definitions of associate elements (and hence three definitions of irreducible elements) and provide examples that show these definitions, while equivalent for domains, are not necessarily equivalent for commutative rings with zero-divisors. In [**6**], the authors continued these investigations by generalizing Definition 1.1 in a straightforward fashion to rings with zero-divisors.

However, it quickly becomes apparent that this definition does not always yield results similar to those obtained in the case for domains. For example, in the ring $\mathbf{Z}_6$, it is easy to check that $G(2)$, $G(3)$ and $G(4)$ are isomorphic to K_1 (with infinitely many loops), yet none of 2, 3, or 4 have a unique factorization into irreducibles. Therefore, Theorem 1.4 does not hold in general for commutative rings with zero-divisors. (Recall that a unique factorization ring (UFR) is defined in an analogous fashion as a UFD.) In $\mathbf{Z}_6$, the problem with unique factorization occurs because of the presence of idempotents. In an attempt to alleviate this problem, the authors in [**6**] adopted the U-factorization approach. Recall that, given some $x \in R^*\backslash U(R)$, a *U-factorization* of x is given by $x = a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$, where the following hold:

(i) $a_i, b_j \in R^*\backslash U(R)$ for all i and j.

(ii) $x = a_1a_2\cdots a_m b_1b_2\cdots b_n$.

(iii) $(x) = (b_1b_2\cdots b_n)$.

(iv) $(x) \neq (b_1 b_2 \cdots \widehat{b_j} \cdots b_n)$ for $1 \leq j \leq n$.

(v) $(x) = (a_i b_1 b_2 \cdots b_n)$ for $1 \leq i \leq m$.

The b_j's are called the *essential divisors* of the U-factorization, and the a_is are the *inessential divisors*. As an example, in $\mathbf{Z}_6$ the factorization $3 = 3^n \lceil 3 \rceil$ is a U-factorization for any positive integer n.

A ring R is said to be U-*atomic* if every nonzero nonunit of R has a U-factorization in which all of the essential divisors are irreducible. Clearly, an atomic ring is also U-atomic. A U-atomic ring R is said to be a U-*unique factorization ring* (U-UFR) if the U-factorization of every nonunit whose essential divisors are all irreducible satisfies the uniqueness properties on the essential portion of the U-factorization. Unlike a UFR, in a U-UFR the 0 element needs to have a unique U-factorization into irreducibles.

Definition 5.1. Let R be a U-atomic commutative ring with unity. Let $x \in R \backslash U(R)$. The U-*irreducible divisor graph* of x, denoted $G_u(x)$, is given by (V, E), where $V = \{y \in \overline{\mathrm{Irr}}\,(R) : y$ is an essential divisor of $x\}$ and, given $y_1, y_2 \in V$, $y_1 - y_2 \in E$ if and only if y_1 and y_2 both appear as essential divisors of x in some U-factorization of x.

One immediate difference between the two graphs defined is illustrated by the graph of an irreducible element. It is clear that $G_u(x)$ is K_1, the complete graph on one vertex, whenever $x \in \mathrm{Irr}\,(R)$. However, $G(x)$ need not be K_1 for some $x \in \mathrm{Irr}\,(R)$. Consider $(1, 0) \in \mathbf{Z} \times \mathbf{Z}$. Observe that $(1, 0)$ is irreducible, but $(1, 0)$ is joined by an edge to $(1, 2)$ in $G((1, 0))$.

In [**17**], a third irreducible divisor graph definition was introduced that is essentially a hybrid definition of $G(x)$ and $G_u(x)$.

Definition 5.2 [**17**, Definition 1]. Let R be an atomic commutative ring with unity, and let $x \in R \backslash U(R)$. The *hybrid irreducible divisor graph* of x, denoted $\gamma(x)$, is given by (V, E), where $V = \{y \in \overline{\mathrm{Irr}}\,(R) : y$ is an essential divisor of $x\}$ and, given $y_1, y_2 \in V$, $y_1 - y_2 \in E$ if and only if $y_1 y_2 \mid x$.

Clearly, $G_u(x) \subseteq \gamma(x) \subseteq G(x)$. To show that the three definitions can yield three distinct graphs, we provide the following example.

Example 5.3 [**17**, Example 5]. Let $R = K[x, y, z]/(xy, yz)$, where K is a finite field, and define $R_1 = \mathbf{Z} \times R$. Consider the element $w = (1, 0)$. Some irreducible factorizations of w are $w = (1, x + y)(1, x)(1, y) = (1, y)(1, z) = (1, x + z)(1, y)$. To simplify our discussion, we only examine subgraphs of $G(w)$, $\gamma(w)$ and $G_u(w)$ that contain the vertices $(1, x + y), (1, x), (1, y), (1, z), (1, x + z)$.

The connections in $G(w)$ are clear. For $\gamma(w)$, observe that $(1, x + y)$ is never an essential divisor of w. Therefore, $(1, x + y)$ is not in the vertex set of $\gamma(w)$, giving $\gamma(w) \subsetneq G(w)$. Further, since $\lceil (1, x+z)(1, y) \rceil$ is a U-factorization, $(1, x + z)$ is an essential divisor of w. However, no two of $(1, x + z)$, $(1, x)$ and $(1, z)$ will ever appear together as essential divisors of w in the same U-factorization. Thus, there are no connections between these vertices in $G_u(w)$, giving $G_u(w) \subsetneq \gamma(w)$. These subgraphs are illustrated in Figure 6.

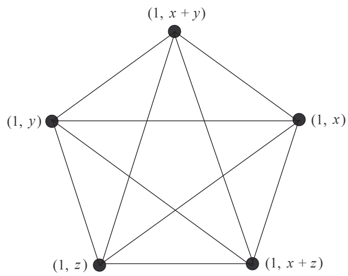
Zero-divisor graphs of commutative rings have received much attention in the literature over the past decade. In rings with zero-divisors, the graphs $G(0)$, $\gamma(0)$ and $G_u(0)$ can be thought of as slight modifications of zero-divisor graphs. Hence, in [**6**, Section 3] and [**17**, Section 2], the authors investigated properties of these two graphs in the spirit of those properties of zero-divisor graphs presented in [**3**].

Proposition 5.4 [**6**, Proposition 3.1], [**17**, Theorem 1]. *Let R be an atomic ring. Then $G(0)$ and $\gamma(0)$ are complete.*

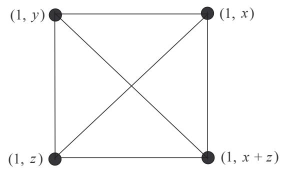
Example 3.2 of [**6**] shows that $G_u(0)$ is not always complete. However, we do obtain the following result.

Proposition 5.5 [**6**, Proposition 3.4, Corollary 3.5]. *Let R be an atomic ring. Then $G_u(0)$ is connected and* $\operatorname{diam}(G_u(0)) \leq 2$.

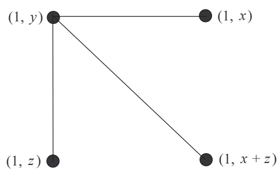
Proof. Let a and d be two distinct irreducible zero-divisors. Then, for some $\overline{a}, \overline{d} \neq 0$, we have $a\overline{a} = 0$ and $d\overline{d} = 0$, where $\overline{a}$ and $\overline{d}$ are not necessarily irreducible. Thus, we have $0 = \lceil a\overline{a} \rceil$ (respectively, $0 = \lceil d\overline{d} \rceil$), and a (respectively d) is a vertex of $G_u(0)$ by [**5**, Lemma 1.4]. By [**3**, Theorem 2.3], the distance between a and d in the zero-divisor graph of R, $\Gamma(R)$, is three or less. If $ad = 0$, then $0 = \lceil ad \rceil$ is clearly a U-factorization, and hence a and d are connected in $G_u(0)$. If $d(a, d) = 3$ in $\Gamma(R)$, say there is a path $a - b - c - d$ with b and c not necessarily irreducible, then $0 = \lceil ad(b + c) \rceil$ is a U-factorization. So,



(a) Subgraph of $G((1,0))$.



(b) Subgraph of $\gamma((1,0))$.



(c) Subgraph of $G_u((1,0))$.

FIGURE 6. Comparison of $G((1,0))$, $\gamma((1,0))$ and $G_u((1,0))$.

by [**5**, Lemma 1.4], a and d are connected in $G_u(0)$.

Finally, consider the case where $d(a,d) = 2$ is a path $\Gamma(R)$, say there is a path $a{-}c{-}d$. If $c \in \operatorname{Irr}(R)$, then $a{-}c{-}d$ in $G_u(0)$, and we are done. Assume $c \notin \operatorname{Irr}(R)$. Then we can factor c as $c = c_1c_2\cdots c_n$, where each c_i is irreducible. If some c_i appears as an essential divisor with a in some U-factorization of 0 and as an essential divisor with d in a U-factorization of 0, then $a - c_i - d$ is a path in $G_u(0)$. Assume otherwise. Then we can write $c = k_1k_2$ with $k_1 \not\sim k_2$, where $0 = k_1\lceil k_2a\rceil$ and $0 = k_2\lceil k_1d\rceil$ are U-factorizations, but $0 = k_2\lceil k_1a\rceil$ and $0 = k_1\lceil k_2d\rceil$ are not U-factorizations. Then we see that $0 = \lceil (k_1 + k_2)ad\rceil$ is a U-factorization. Thus, $a - d$ in $G_u(0)$ by [**5**, Lemma 1.4], and we see that $G_u(0)$ is connected. □

As noted above, Theorem 1.4 does not hold in general for rings with zero-divisors. In [**6**], the interplay between the UFR and U-UFR properties and graphs $G(x)$ and $G_u(x)$ are investigated, and we turn our attention to these results. In this process, it is only natural to seek comparisons between the graph-theoretic aspects of $G_u(x)$ and $G(x)$.

Theorem 5.6 [**6**, Theorem 4.2]. *Let R be an atomic ring and x a nonzero nonunit of R. If $G_u(x)$ is complete, then $G(x)$ is complete.*

Proof. Let $x \in R^*\backslash U(R)$ with a, b irreducible divisors of x. If a, b are both essential divisors of x, then so is ab since $G_u(x)$ is complete. Hence, $ab \mid x$ and a, b are connected in $G(x)$. So, assume that a is an essential irreducible divisor of x while b is an inessential irreducible divisor of x, i.e., $x = c\lceil ad_1 \cdots d_n\rceil = bc'\lceil l_1 \cdots l_n\rceil$. Hence, $b(x) = (x)$, which implies that $(x) = b(x) = b(ad_1 \cdots d_n) = (bad_1 \cdots d_n)$. Thus, $bad_1 \cdots d_n \mid x$, and so $ba \mid x$. The case where both a and b are inessential irreducible divisors of x works similarly. In every case we see that $G(x)$ is complete. □

The converse fails to hold, as the following example demonstrates.

Example 5.7 [**6**, Example 4.3]. Let $R = K[y, z]/(y^2, z^2)$, where K is a finite field. Let $R_1 = R \times \mathbf{Z}$. Then $G((0, 10))$ is complete (in fact, $G(x)$ is complete for all $x \in R\backslash U(R)$), but $G_u((0, 10))$ is not complete.

Theorem 5.8 [**6**, Theorem 4.4]. *Let R be an atomic ring and x a nonzero nonunit of R. If $G_u(x)$ is connected, then $G(x)$ is connected.*

It is an open question as to whether the converse of the previous theorem is true as well as the question as to whether $G(x)$ is complete implies that $G_u(x)$ is connected.

It is clear that if R is a UFR, then $G(x)$ is complete (and hence connected) for every $x \in R^*\backslash U(R)$, and that if R is a U-UFR, then $G_u(x)$ complete (and hence connected) for every $x \in R^*\backslash U(R)$. However, [**6**, Example 4.1] shows that, in the ring $R = \mathbf{Z}_{(2)} \oplus (\mathbf{Z}_2 \times \mathbf{Z}_{2^\infty})$, the idealization of the ring $\mathbf{Z}_{(2)}$ with the module formed by the direct product of $\mathbf{Z}_2$ and $\mathbf{Z}_{2^\infty}$, where the module operation is the natural one, both $G(x)$ and $G_u(x)$ are connected for all $x \in R^*\backslash U(R)$, but in general neither graph is complete and R is neither a UFR nor a U-UFR.

The following results demonstrate other implications that can be made.

Theorem 5.9 [**6**, Theorem 4.6]. *Let R be a* U-UFR*. Then $G(x)$ is complete for every nonzero nonunit $x \in R$.*

Proof. Let $x \in R$ and $x = a_1 a_2 \cdots a_k \lceil b_1 b_2 \cdots b_l \rceil$ be a U-factorization of x into irreducibles. Then the essential portion of the factorization $\lceil b_1 b_2 \cdots b_l \rceil$ is unique up to associates. Let c and d be nonassociate irreducible divisors of x. If $c \sim b_i$ and $d \sim b_j$, then $cd \sim b_i b_j$, giving $cd \mid x$. If $c \sim b_i$ and $d \not\sim b_j$ for any b_j, then $(x) = (b_1 \cdots b_l) = d(b_1 \cdots b_l) = (db_1 \cdots b_l)$, implying $cd \mid x$. If $c \not\sim b_i$ and $d \not\sim b_i$ for any b_i, then $(x) = c(b_1 \cdots b_l) = (b_1 \cdots b_l)$ and $(x) = d(b_1 \cdots b_l) = (b_1 \cdots b_l)$. Thus, $(cdb_1 \cdots b_l) = (c(db_1 \cdots b_l)) = (cb_1 \cdots b_l) = (b_1 \cdots b_l) = (x)$. Therefore, $cd \mid x$. □

Corollary 5.10 [**6**, Corollary 4.8]. *If R is a* BFR, FFR, *or* HFR*, x is a nonzero nonunit of R and $G(x)$ is complete, then $G_u(x)$ is complete.*

Corollary 5.11 [**6**, Corollary 4.9]. *If R is a* UFR*, then $G_u(x)$ is complete for every nonzero nonunit $x \in R$.*

The following diagram that appears in [**6**] summarizes the various results. Note that the two arrows with question marks next to them represent open questions.

$$
\begin{array}{ccccccccc}
 & & & G(x)\ \text{complete} & & & G(x)\ \text{connected} & & \\
 & & & & \rightarrow & & & & \\
 & & & \forall x \in R^{*}\backslash U(R) & & & \forall x \in R^{*}\backslash U(R) & & \\
 & \nearrow & & \nearrow & & & & & \\
\text{U-UFR} & & \text{UFR} & \uparrow & & \searrow? & \uparrow & \downarrow? & \\
 & \searrow & & \searrow & & & & & \\
 & & & G_u(x)\ \text{complete} & & & G_u(x)\ \text{connected} & & \\
 & & & & \rightarrow & & & & \\
 & & & \forall x \in R^{*}\backslash U(R) & & & \forall x \in R^{*}\backslash U(R). & &
\end{array}
$$

The authors of [**17**] expand on the work done in [**6**] to the graph $\gamma(x)$. We also find in [**17**] various relationships between the three graphs.

Theorem 5.12 [**17**, Theorem 2]. *Let R be an atomic ring, and let x be a nonzero nonunit of R.*

(i) $\gamma(x)$ *is complete if and only if* $G(x)$ *is complete.*

(ii) *If* $G_u(x)$ *is complete, then* $\gamma(x)$ *is complete.*

Proof. (i) ($\Rightarrow$). Let $x \in R^* \setminus U(R)$ with $a, b \in \overline{\text{Irr}}\,(x) = V[G(x)]$. Suppose $a,b \in V[\gamma(x)]$. Since $\gamma(x)$ is complete, a–b is an edge in $\gamma(x)$. Thus, a–b is an edge in $G(x)$ as well. If, without loss of generality, $a \in V[G(x)] \setminus V[\gamma(x)]$, then a is never an essential divisor of x and thus is always inessential. Thus, by [**17**, Lemma 1], a is connected to every other vertex in $G(x)$. Hence, $G(x)$ is complete.

($\Leftarrow$). For any $a, b \in V[\gamma(x)]$, $a, b \in V[G(x)]$, because $V[\gamma(x)] \subseteq V[G(x)]$. Since $G(x)$ is complete, a–b is an edge in $G(x)$, implying $ab|x$, so a–b is an edge in $\gamma(x)$ as well.

(ii) Let $x \in R^* \setminus U(R)$ with $a, b \in V[\gamma(x)]$. Observe that $V[\gamma(x)] = V[G_u(x)]$. Since $G_u(x)$ is complete, we see that $ab|x$; hence, a–b is an edge in $\gamma(x)$ as well. $\square$

Theorem 5.13 [**17**, Theorem 3]. *Let* R *be an atomic ring, and let* x *be a nonzero nonunit of* R. *If* $G_u(x)$ *is complete, then* $\gamma(x)$ *is complete.*

The following example from [**17**] shows the converse of this theorem is false.

Example 5.14 [**17**, Example 2]. Let $R = \mathbf{Z}_5[x, y, z]/(xy, yz)$, and let $R_1 = \mathbf{Z} \times R$. Then $\gamma((1, 0))$ is complete, but $G_u((1, 0))$ is not complete.

The following result concludes the study that appears in [**17**] of the interplay between these three graphs. It should be noted that the converses of each implication remain open questions.

Theorem 5.15 [**17**, Theorem 4 and Theorem 5]. *Let* R *be an atomic ring, and let* x *be a nonzero nonunit of* R. *Then we have the following implications*: $G_u(x)$ *is connected* $\Rightarrow$ $\gamma(x)$ *is connected* $\Rightarrow$ $G(x)$ *is connected.*

6. Realizations. In this section we provide several examples and results which illustrate how and when certain graphs can be realized as the irreducible divisor graphs of elements in appropriate atomic domains.

6.1. Numerical semigroup rings. We begin with an exploration of irreducible divisor graphs of numerical semigroup rings–rings of the form $\mathbf{F}[x^{n_1}, x^{n_2}, \ldots, x^{n_t}]$ where $\mathbf{F}$ is a field, x is an indeterminate and $n_1 < n_2 < \cdots < n_t$ are positive integers. The purpose of this diversion is that the multiplicative structure of monomials in a numerical semigroup ring $\mathbf{F}[x^{n_1}, x^{n_2}, \ldots, x^{n_t}]$ can often be studied more easily in the additive numerical semigroup $\langle n_1, n_2, \ldots, n_t\rangle$ and that we will take this approach when constructing examples with certain desired properties. Throughout, $\mathbf{N}$ will denote the set of all positive integers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Recall that a numerical semigroup is an additive submonoid of the set of nonnegative integers. More precisely, if $0 < n_1 < n_2 < \cdots < n_t$ are t positive integers such that for all $i \in \{2, \ldots, t\}$, $n_i = a_1 n_1 + \cdots + a_{i-1} n_{i-1}$ has no nonnegative integer solutions $\{a_1, a_2, \ldots, a_{i-1}\}$, then

$$N = \langle n_1, n_2, \ldots, n_t\rangle = \{a_1 n_1 + \cdots + a_t n_t : a_i \in \mathbf{N}_0\} \subseteq \mathbf{N}_0$$

is the *numerical monoid* minimally generated by the set $\{n_1, n_2, \ldots, n_t\}$.

We now give a formal definition of the irreducible divisor graph of an element in a numerical monoid, mimicking the definition of the irreducible divisor graph of a nonzero nonunit of an atomic domain.

Definition 6.1. Let $N = \langle n_1, n_2, \ldots, n_t\rangle$ be a minimally generated numerical monoid. If $x \in \mathbf{N}_0$, the *irreducible divisor graph* of x, denoted $G_N(x)$, is defined by:

(1) $V[G_N(x)] = \{n_i$: there exist $a_1, a_2, \ldots, a_t \in \mathbf{N}_0$ with $x = \sum_{j=1}^{t} a_j n_j$ and $a_i \neq 0\}$ is the *vertex set* of $G(x)$,

(2) $E[G_N(x)] = \{\{n_i, n_j\}$: there exist $a_1, a_2, \ldots, a_t \in \mathbf{N}_0$ with $x = \sum_{k=1}^{t} a_k n_k$, and $a_i, a_j \neq 0\}$ is the *edge set* of $G(x)$, and

(3) We put $A_i - 1 \geq 0$ loops on vertex n_i, where $A_i = \max\{a_i : x = \sum_{k=1}^{t} a_j n_j$ for some $a_1, \ldots, a_t \in \mathbf{N}_0\}$.

If $x \notin N$, we say that $G_N(x)$ is the empty graph which contains no vertices or edges. We write $G(x)$ in place of $G_N(x)$ if N is clear from context.

Remark 6.2. Note that this definition is consistent with Definition 1.1 in that, if R is the semigroup ring $R = \mathbf{F}[y^{n_1}, y^{n_2}, \ldots, y^{n_t}]$ for some

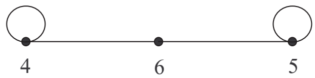


FIGURE 7. $G(14)$ in $N = \langle 4, 5, 6\rangle$.

field $\mathbf{F}$ and some variable y, then as graphs, $G_N(x) \cong G(y^x)$ where $y^x \in R$.

Example 6.3. Let $N = \langle 4, 5, 6\rangle$. In N, the only representations of 14 are $14 = 1\cdot 4+2\cdot 5$ and $14 = 2\cdot 4+1\cdot 6$. Therefore, $V[G(14)] = \{4, 5, 6\}$ since each of 4, 5 and 6 is used in some representation of 14 in N. In the first representation, only 4 and 5 are used, and hence $\{4, 5\} \in E[G(14)]$. In the second representation, only 4 and 6 are used, and hence $\{4, 6\} \in E[G(14)]$. Therefore, $G(14)$ is the line graph $4 - 6 - 5$ with a single loop on each of vertices 4 and 5.

We now give an example to illustrate how it is possible to construct numerical semigroups (and hence numerical semigroup rings) and elements in that numerical semigroup with given irreducible divisor graphs. In particular, in Example 6.4, a complete k-partite graph is realized as the irreducible divisor graph in a numerical semigroup.

Let $m_1, m_2, \dots, m_n$ denote a set of $n > 1$ positive integers. Let

$$p_{1,1}, p_{1,2}, \dots, p_{1,m_1}, p_{2,1}, \dots, p_{n,m_n}, q_1, q_2, \dots, q_n$$

be a set of distinct primes. Now set $P = \prod_{i=1}^{n}\prod_{j=1}^{m_i} p_{i,j}$ and $Q = \prod_{i=1}^{n} q_i$. Finally, for each $i \in \{1, 2, \dots, n\}$ and each $j \in \{1, 2, \dots, m_i\}$, set $a_{i,j} = PQ/p_{i,j}q_i$, and let $N = \langle a_{1,1}, a_{1,2}, \dots, a_{1,m_1}, a_{2,1}, \dots, a_{n,m_n}\rangle$. Then $G_N(P\sum_{i=1}^{n}(Q/q_i))$ is a complete n-partite graph.

To see this, first note that $h, k \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m_h\}$, and $l \in \{1, 2, \dots, m_k\}$. We have that

$$p_{h,j}a_{h,j} + p_{k,l}a_{k,l} + \sum_{i=1, i\neq h,k}^{n} p_{i,1}a_{i,1} = \sum_{i=1}^{n} \frac{PQ}{q_i} = P\sum_{i=1}^{n}\frac{Q}{q_i},$$

and hence there is an edge connecting vertices $a_{h,j}$ and $a_{k,l}$. Moreover, these $m_1 \cdot m_2 \cdots m_n$ edges are the only edges in $G_N(P\sum_{i=1}^{n}(Q/q_i))$.

If $\{x_{1,1}, \ldots, x_{n,m_n}\}$ is a set of $m_1 + m_2 + \cdots m_n$ nonnegative integers with $\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{i,j}a_{i,j} = P\sum_{i=1}^{n}(Q/q_i)$, then for each $b \in \{1, 2, \ldots, n\}$, $\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{i,j}a_{i,j} \equiv P\sum_{i=1}^{n}(Q/q_i) \mod q_b$, and hence $\sum_{j=1}^{m_b} x_{b,j}a_{b,j} \equiv P(Q/q_b) \mod q_b$. Therefore, for each $b \in \{1, 2, \ldots, n\}$, there exists a $z_b \in \{1, 2, \ldots, m_b\}$ such that $x_{b,z_b} > 0$.

Furthermore, $\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{i,j}a_{i,j} \equiv P\sum_{i=1}^{n}(Q/q_i) \mod p_{b,z_b}$, and hence $x_{b,z_b} \equiv 0 \mod p_{b,z_b}$. Therefore, for each $b \in \{1, 2, \ldots, n\}$, there is some $z_b \in \{1, 2, \ldots, m_b\}$ with $x_{b,z_b} \geq p_{b,z_b}$. We now have that

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{i,j}a_{i,j} &= \sum_{i=1}^{n}\left(x_{i,z_i}a_{i,z_i} + \sum_{\substack{j=1\\ j\neq z_i}}^{m_i} x_{i,j}a_{i,j}\right)\\
&= \sum_{i=1}^{n} x_{i,z_i}a_{i,z_i} + \sum_{i=1}^{n}\left(\sum_{j=1, j\neq z_i}^{m_i} x_{i,j}a_{i,j}\right)\\
&\geq \sum_{i=1}^{n}\frac{PQ}{q_i} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq z_i}}^{m_i} x_{i,j}a_{i,j}\\
&= P\sum_{i=1}^{n}\frac{Q}{q_i}.
\end{aligned}$$

Thus, $\sum_{i=1}^{n}\sum_{j=1, j\neq z_i}^{m_i} x_{i,j}a_{i,j} = 0$, and thus $x_{i,j} = 0$ whenever $j \neq z_i$. This proves that the previously discovered $m_1 \cdot m_2 \cdots m_n$ edges are the only edges in $G_N(P\sum_{i=1}^{n}(Q/q_i))$.

Similarly, the following example illustrates how to construct irreducible divisor graphs with arbitrarily many connected components.

Example 6.4 [**13**]. Let k be a positive integer, $b_1 = 1$ and $b_n = 2b_{n-1} + 2$ for $n = 2, \ldots, k$. Let N be a multiple of 3 with $N > 9(2^{k+1}) - 24$, and let

$$D = \mathbf{F}[x^{(x/3)-b_1}, x^{(x/3)-(b_1-1)}, x^{(x/3)+2b_1-1}, \ldots, x^{(x/3)-b_k}, x^{(x/3)-(b_k-1)}, x^{(x/3)+2b_k-1}]$$

where $\mathbf{F}$ is a field and where x is an indeterminate. Then $G_D(x^N)$ is a graph comprising k disconnected 3-cliques.

In [**8**, Section 3], all 31 connected graphs on at most five vertices are realized as irreducible divisor graphs. In addition, it is shown which of these graphs can be realized for numerical monoids generated by an interval, that is, numerical monoids of the form $\langle n, n+1, \ldots, n+t\rangle$. Moreover, that paper provides necessary and sufficient conditions for the following properties when N is a numerical monoid generated by an interval: $G_N(x)$ has the maximum number of vertices, $G_N(x)$ is a complete graph and $G_N(x)$ is connected. The cleaner (and much easier) result is when $N = \langle n, n+1, \ldots, 2n-1\rangle$ is generated by a full interval, an interval of maximum size, and we provide the proof from [**8**].

Proposition 6.5 [**8**, Corollary 2.8]. *Let* $n > 2$, *set* $N = \langle n, n+1, \ldots, 2n-1\rangle$, *and let* $x \in N$.

(1) $G(x)$ *has* n *vertices if and only if* $x \geq 3n-1$.

(2) *The following statements are equivalent.*

(a) $G(x)$ *is connected with* n *vertices.*

(b) $\deg(n) = n-1$.

(c) $x \geq 4n-1$.

(3) $G(x)$ *is complete on* n *vertices if and only if* $x \geq 5n-3$.

Proof. Notice that $N = \{0\} \cup [n, \infty)$.

For (1), we require that $[x-(2n-1), x-n] \subset N$, which is true precisely when $x-(2n-1) \geq n$.

For (2), we note that $\deg(n) = n-1$ (omitting loops), when $[x-n-(2n-1), x-n-(n-1)] \subset N$, which is true precisely when $x-3n+1 \geq n$, so conditions (b) and (c) are equivalent. It is clear that, in this case, $G(x)$ is connected. Conversely, if $G(x)$ is connected, then vertex $2n-1$ is adjacent to at least one other vertex, i.e., $x-(2n-1)-(n+j) \in N$ for some $j \in [0,n]$, so $x-(3n-1) \geq x-(3n-1)-j \geq n$, and the inequality (c) is established.

For (3) we note that vertices $2n-1$ and $2n-2$ must be adjacent, so $x-4n+3 = x-(2n-1)-(2n-2) > n$, which produces the inequality. Moreover, if the inequality is satisfied all pairs of vertices are adjacent since $x-(n+i)-(n+j) > x-4n+3$ if i and j are distinct integers in $[0, n-1]$. ◻

We now state without proof a summary of the more general results from [**8**]. Although Proposition 6.5 follows as an immediate corollary to this proposition, the proof of Proposition 6.5 is much more straightforward and much less technical. Note that $\mathcal{F}(N)$ denotes the *Frobenius number* of $N = \langle n, n+1, \dots, n+t\rangle$, the largest nonnegative integer not in N, which by [**14**] is $\lceil (n-1)/t\rceil n - 1$.

Proposition 6.6 [**8**, Propositions 2.3, 2.4, 2.7]. *Let $N = \langle n, \dots, n+t\rangle$, where $n > 1$ and $0 < t < n$.*

(1) *$G(x)$ has $t+1$ vertices if and only if $(p+1)n+t \leq x \leq (p+1)n+pt$ with $p > 0$. Moreover, if $x > \mathcal{F}(N)+n+t$, then $G(x)$ has $t+1$ vertices.*

(2) *$G(x)$ is connected on $t+1$ vertices if and only if at least one of the following conditions holds*:

(a) *$(p+2)n+t \leq x \leq (p+2)n+(p+1)t$.*

(b) *$x > C(N)$, where $C(N) = \mathcal{F}(N)+2n+t+1$ if $t \mid n-1$, and $C(N) = \mathcal{F}(N)+n+t+1$ otherwise.*

(3) *$G(x)$ is complete on $t+1$ vertices if and only if $(p+2)n+2t-1 \leq x \leq (p+2)n+pt+1$. Moreover, if $x > \mathcal{F}(N)+2n+2t+1$, then $G(x)$ is complete on $t+1$ vertices.*

6.2. Rings with zero-divisors. The presence of nontrivial zero-divisors in a commutative ring added a level of complication to the process of factoring into irreducibles, which necessitated the consideration of alternate factorization forms (U-factorizations) and alternate irreducible divisor graph constructions as presented earlier. Fortunately, in an atomic domain, the only inessential divisors in a U-factorization are units and all non-unit divisors are essential; hence, $G(x)$ and $\gamma(x)$ will be identical for any nonzero, nonunit $x \in D$. It is also straightforward to see that, if the product of two non-unit (essential) divisors, y and z, divides x, then a U-factorization of x exists containing y and z as essential divisors. Thus, $G_u(x)$ is identical to $\gamma(x)$, and hence to $G(x)$ in an atomic domain.

In the situation where the vertices of $G(x)$ allow no loops, we get the following realizability result.

Theorem 6.7 [**17**, Theorems 6 and 7]. *Let D be an atomic domain, and let x be a nonzero nonunit of D such that, for every $y \in V(G(x))$,* *we have y^2 does not divide x. If $|V(G(x))| > 2$, then $G(x)$ is not realizable as a non-trivial complete bipartite graph.*

Proof. Suppose that $G(x)$ is a non-trivial complete bipartite graph with $|V(G(x))| > 2$. Thus, $V(G(x))$ may be partitioned into two disjoint sets, A_1 and A_2, with (without loss of generality) $|A_2| \geq 2$. Let $A_1 = \{\alpha_1, \alpha_2, \dots\}$, $A_2 = \{\varepsilon_1, \varepsilon_2, \dots\}$.

Since $G(x)$ is complete bipartite and loop-free by hypothesis, we have that, for any ε_j and ε_k there exist two distinct U-factorizations of x of the following form: $x = u_1\lceil \alpha_1 \varepsilon_j \rceil$ and $x = u_2 \lceil \alpha_1 \varepsilon_k \rceil$, with $j \neq k$ and $u_1, u_2 \in U(D)$. Thus, we have that $(x/\alpha_1) = u_1\varepsilon_j = u_2\varepsilon_k$. Hence, ε_j and ε_k are associates, which implies that ε_j and ε_k are not distinct vertices in $G(x)$. Thus, $|A_2| = 1$, a contradiction. □

Note that this result also shows that non-trivial star graphs are not realizable in such situations. If the above conditions were to be relaxed so as to allow looped irreducible divisors of a given nonzero nonunit $x \in D$, the situation changes completely.

6.3. Diameter and girth. As mentioned in the Introduction, the study of irreducible divisor graphs stemmed from the study of zero-divisor graphs of commutative rings. In the setting of zero divisor graphs there are significant restrictions on the diameter and the girth. In fact, the diameter of a zero-divisor graph is always at most 3 and the girth, if a cycle exists, is either 3 or 4.

In [**6**], the authors posed the following two questions pertaining to the girth and diameter of irreducible divisor graphs.

(1) Given $n \in \mathbf{N}$, does there exist a commutative ring R and a nonzero nonunit $x \in R$ with $\text{diam}\,(G(x)) = n$?

(2) Given $n \in \mathbf{N}$, does there exist a commutative ring R and a nonzero nonunit $x \in R$ with $g(G(x)) = n$?

These questions have both been answered affirmatively in [**14**], again utilizing the simpler structure in numerical semigroups, as the following examples illustrate.

Example 6.8 [**14**]. Given any positive integer n, let $b = 2^{n+2}$, and let $e_k = b + (-2)^{k-1}$ for each $k \in \{1, 2, \dots, n\}$. Set $N = \langle e_1, \dots, e_n \rangle$, and consider $G(3b)$. For each $i \in \{1, 2, \dots, n-1\}$, $3b = 2e_i + e_{i+1}$. Moreover, by construction, these are the only representations of $3b$ in N, and hence $G(3b)$ is a path graph with n vertices. Using the correspondence in Remark 6.2, $G_R(x^{3b})$, where $\mathbf{F}$ is a field, x an indeterminate, and $R = \mathbf{F}[x^{e_1}, \dots, x^{e_n}]$, is a path graph with n vertices, having diameter $n-1$.

Example 6.9 [**14**]. In a construction similar to the one in Example 6.8, we now see that it is possible to construct an irreducible divisor graph with arbitrarily large girth. Given any positive integer n, let $b = 3 \cdot 2^{n+1} + (-1)^{n-1}$, and let $R = \mathbf{F}[x^{e_1}, x^{e_2}, \dots, x^{e_n}]$ where $\mathbf{F}$ is a field, x is an indeterminate, and where $e_k = b + (-1)^n(-2)^{k-1}$, for each k. Then $G(x^{3b})$ is a polygon with n vertices, a graph with girth n.

If we do not restrict ourselves to domains and adopt the hybrid irreducible divisor graph, then in many cases our bounds on diameter and girth become much more restrictive.

Theorem 6.10 [**18**, Theorem 1]. *Suppose that R is an atomic ring that can be decomposed as the direct product of at least two rings, and let x be decomposable in R.*

(1) $\gamma(x)$ *is connected.*

(2) $\operatorname{diam}(\gamma(x)) \leq 2$.

(3) $g(\gamma(x)) \in \{3, \infty\}$.

Theorem 6.11 [**18**, Theorem 2]. *Let R be an atomic ring with x a nonzero nonunit of R. If $G_u(x) \subsetneq \gamma(x)$, then* $\operatorname{diam}(\gamma(x)) \leq 2$.

Proof. We have $V[\gamma(x)] = V[G_u(x)]$ and $E[G_u(x)] \subseteq E[\gamma(x)]$. Thus, if $G_u(x) \subsetneq \gamma(x)$, then $E[G_u(x)] \subsetneq E[\gamma(x)]$. Let $a, b \in V[G_u(x)] = V[\gamma(x)]$ such that a–b is an edge in $\gamma(x)$ but not in $G_u(x)$. Hence, a and b never appear as essential divisors in the same U-factorization of x. However, since a–b in $\gamma(x)$, we must have that $ab|x$. Therefore, we have a factorization of x of the form $abc_1c_2 \cdots c_n$. Considering the corresponding U-factorization, either a or b (or both) must appear as an inessential divisor. If, without loss of generality, a appears as an inessential divisor in this U-factorization, then a–c for all $c \in V[\gamma(x)]$ by [**17**, Lemma 1]. Therefore, $\operatorname{diam}[\gamma(x)] \leq 2$. □

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Department of Mathematics, University of St. Thomas, St. Paul, MN 55105
Email address: axte2004@stthomas.edu

Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, MO 64093
Email address: baeth@ucmo.edu

Department of Mathematics, Millikin University, Decatur, IL 62522
Email address: jstickles@millikin.edu