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SUBDIRECT PRODUCTS OF M^*-GROUPS

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ABSTRACT. A compact bordered Klein surface X of genus $g \geq 2$ has at most $12(g-1)$ automorphisms. A bordered surface for which the bound is attained is said to have maximal symmetry, and its full automorphism group is called an M^*-group. For M^*-groups G and H, we construct a subdirect product L of G and H that is an M^*-group. We show that there is a normal subgroup of G whose index is the same as the index of L in the direct product $G \times H$. This general result is specialized to give results about the index of the subdirect product L in the direct product $G \times H$ for M^*-groups G and H. Then we give a number of sufficient conditions for L to equal $G \times H$ and to conclude that the direct product is an M^*-group. For example, let G be an M^*-group that acts on a bordered Klein surface X. The elements of G that fix a boundary component of X form a dihedral subgroup of order $2q$. The number q is called an action index of G. If G and H have relatively prime action indices and one of them is perfect, then the direct product of G and H is an M^*-group.

1. Introduction. A compact bordered Klein surface X of genus $g \geq 2$ has at most $12(g-1)$ automorphisms [**10**]. A bordered surface for which the bound is attained is said to have *maximal symmetry* [**8**]. The full automorphism group of a surface with maximal symmetry is called an M^*-group [**11**].

There are infinitely many M^*-groups, and some important groups are known to be M^*-groups. For example, all large alternating groups and all large symmetric groups are M^*-groups [**3**], as well as most of the groups $\mathrm{PSL}\,(2,q)$ [**19**]. In addition, there are constructions that give extensions of abelian groups by a particular M^*-group G; here see [**8**, Section 4]. These constructions do not produce a presentation of the extension, however. On the other hand, there is a construction that forms an M^*-group, with complete presentation, from a 2-generator group that admits an action of D_6, the smallest M^*-group [**14**].

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Here we consider a natural way to construct a larger M^*-group from two M^*-groups. Our approach uses the concept of a subdirect product [**17**]. The basic construction is quite general, with applications possible to several different types of groups. The applications we have in mind are to M^*-groups, however, and we begin with some observations about M^*-groups and the bordered surfaces on which they act. We subsequently describe the construction in the most general way.

Now let G and H be M^*-groups. Then the construction gives a subgroup L of the direct product $G \times H$ that is a subdirect product of G and H, and it is easy to see that the "new" group L is an M^*-group. The construction yields a method to compute $|L|$ if there are complete presentations for the M^*-groups G and H. In general, however, $|L|$ is not immediately apparent. We focus on cases in which the subdirect product L has small index in $G \times H$ and there is a condition that allows the easy determination of $|L|$. We consider two conditions, one involving genus actions of the M^*-groups and one involving abelian quotient groups. First we obtain some results about subdirect products of M^*-groups, including a characterization of the M^*-groups that are subdirect products of two smaller M^*-groups.

Finally we consider direct products of M^*-groups. We obtain necessary conditions for an M^*-group to be the direct product of two M^*-groups. In addition to an easy algebraic necessary condition, there is a necessary condition involving the genus actions of the M^*-groups; one of the two groups must act on a non-orientable surface. We obtain some general sufficient conditions for the direct product of two M^*-groups G and H to be an M^*-group. These conditions can be used to show that if G is a solvable M^*-group and H is a perfect M^*-group, then $G \times H$ is also an M^*-group. Further, if G and H are non-isomorphic simple M^*-groups, then $G \times H$ is an M^*-group.

2. M^*-groups. A finite group G is an M^*-group [**11**] if it is generated by three distinct non-trivial elements t, u, v which satisfy the relations

$$t^2 = u^2 = v^2 = (tu)^2 = (tv)^3 = 1. \tag{1}$$

The fundamental result about M^*-groups is the following; see [**8, 11**].

Theorem A. *The finite group G is an M^*-group with partial presentation* (1) *such that $o(uv) = q$ if and only if G is the automorphism* *group of a bordered Klein surface X with maximal symmetry and k boundary components, where*

$$|G| = 2qk.$$

The connection between the order of uv and the associated action of the M^*-group G on a bordered surface was established in [**8**]. We will say that the group G *acts on* X *with index* q. We will call q *the index of the action* or say that q is an *action index*. In the action of the M^*-group G on X, each component of ∂X is fixed by a dihedral subgroup of G of order $2q$ [**8**, Section 6]. Note that the index of an action of an M^*-group determines the number of boundary components but not the orientability of the surface on which G acts. Indeed, it is possible for an M^*-group to act on two topologically different surfaces with the same index. For an interesting example, see [**20**].

Theorem A was established using NEC groups. Let Δ be the NEC group with signature $(0; +; [\ ]; \{(2, 2, 2, 3)\})$. Group Δ is called the extended quadrilateral group and denoted by $\Gamma[2, 2, 2, 3]$. The finite group G is the automorphism group of a bordered Klein surface X with maximal symmetry if and only if there is a homomorphism $\alpha : \Delta \to G$ onto G such that $\ker(\alpha)$ is a bordered surface group. Further, the surface $X = D/\ker(\alpha)$, where D is the open upper half-plane; X has a non-empty boundary since $\ker(\alpha)$ contains reflections (but no other elements of finite order). For more details, see [**11**, pages 4–6].

An important alternative way to consider M^*-groups is as quotients of the extended modular group Γ. Group Γ has generators t, u and v and defining relations (1) [**6**, pages 85, 86]. It follows that a finite group of order at least 12 is an M^*-group if and only if it is a homomorphic image of Γ. Here we treat M^*-groups mainly as quotients of Γ. A good source for background results on Γ is [**9**].

There is a useful criterion for determining the orientability of the surface X on which an M^*-group acts. Let the M^*-group G have standard generators t, u and v satisfying (1), and let $G^+ = \langle tu, uv \rangle$. Then we have the following basic result.

Proposition 1. *Let the M^*-group G have a particular presentation of form* (1) *with associated action on the bordered surface X. Then X is orientable if and only if G^+ has index two in G.*

For a proof see [**2**, page 100]. If the surface X is orientable, then G^+ is the subgroup of orientation-preserving automorphisms of X. If X is non-orientable, then $G^+ = G$. We will often use this algebraic condition for the orientability of X.

The commutator subgroup $G' = \langle tv, tu \cdot tv \cdot tu \rangle$ [**8**, page 278], and $G' \subseteq G^+$. Further, the index $[G^+ : G']$ is at most 2, since $G^+ = \langle G', tu \rangle$ and tu is an element of order 2. The commutator quotient group of an M^*-group is always a quotient of $Z_2 \times Z_2$, and we have

$$G' \subseteq G^+ \subseteq G,$$

where each subgroup has index either 1 or 2 in the next larger group. There are four possibilities, all of which can occur. However, using Proposition 1, we can make the following observation.

Proposition 2. *If an M^*-group G acts on an non-orientable surface with maximal symmetry, then $[G : G'] \leq 2$.*

Proposition 2 may also be easily obtained using the orienting double. Let X be a non-orientable bordered Klein surface of genus g with k boundary components. Associated with X in a natural way is its *orienting double* X_o [**1**, pages 37–41], an orientable bordered surface of genus $g_o = 2g - 1$ with $2k$ boundary components. The surface X_o has an anti-analytic involution $\sigma : X_o \to X_o$ such that $X_o/\sigma = X$. The automorphism groups of X and X_o are intimately connected [**1**, page 79]:

$$A(X) \cong \{f \in A^+(X_o) \mid f\sigma = f\}.$$

Let $G = A(X)$. Then $A(X_o)$ contains a subgroup which is isomorphic to $Z_2 \times G \cong \langle \sigma \rangle \times G$.

In particular, suppose the M^*-group G acts on a non-orientable surface with maximal symmetry, so that $G = A(X)$. Then it follows that X_o has maximal symmetry and $Z_2 \times G$ is also an M^*-group. Thus Proposition 2 holds, since the commutator subgroup of $Z_2 \times G$ has index at most 4.

The M^*-groups are closely related to a family of abstract groups introduced by Coxeter [**5**]. Let $G^{n,q,r}$ be the group with generators A, B and C and defining relations

$$A^n = B^q = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1.$$

If we set $t = BC$, $u = CA$, and $v = BCA$, then we obtain the presentation

$$t^2 = u^2 = v^2 = (tu)^2 = (tv)^n = (uv)^q = (tuv)^r = 1.$$

Thus, G is an M^*-group if and only if G is a quotient of the group $G^{3,q,r}$ for some q and r. The complete table of the known finite groups $G^{n,q,r}$ is in [**6**, pages 139, 140], but also see the recent article [**7**].

The following basic construction first appeared in [**13**, Theorem 2].

Proposition 3. *Suppose an M^*-group G acts on X with odd action index q. Then the group $Z_2 \times G$ is an M^*-group that acts on a surface $Q(X)$ with index $2q$. Further, the surface $Q(X)$ is orientable if and only if the surface X is orientable.*

Proposition 3 has the following two easy algebraic consequences. These two results also follow from the partial presentation (1) and Proposition 1.

Proposition 4. *If an M^*-group G has an odd action index, then $[G : G'] \leq 2$.*

Proposition 5. *If an M^*-group G acts on a non-orientable surface with odd index, then G is perfect.*

Proof. Let G act on the non-orientable surface X with odd index q. Then $Z_2 \times G$ acts on the non-orientable surface $Y = Q(X)$, by Proposition 3. But then the M^*-group $Z_2 \times Z_2 \times G$ acts on Y_o, and hence we must have $G = G'$. □

Corollary 1. *If an M^*-group G is not perfect and G acts on a non-orientable surface with maximal symmetry, then the index of the action must be even.*

There is a simple, but useful, way of obtaining a second action index for an M^*-group.

Proposition 6. *Let G be an M^*-group with generators t, u and v satisfying* (1) *and action index $q = o(uv)$. If $r = o(tuv)$, then r is also the index of an action of G.*

Proposition 6 was established in [**12**, page 24] by noting that t, $u' = tu$ and v form another set of generators for the M^*-group G. Proposition 6 has the following consequence.

Proposition 7. *If an M^*-group G has odd action index q, then G acts on a non-orientable surface with maximal symmetry* (*not necessarily with index q*).

Proof. By Proposition 6, G has a presentation in which $o(tuv) = q$ is odd. But then the involutions t, u and v are in G^+, and the surface on which G acts with this presentation must be non-orientable, by Proposition 1. ◻

For example, $PGL(2,7) \cong G^{3,7,8} \cong G^{3,8,7}$ [**6**, page 139] has action indices 7 and 8. The action with index 8 is on a non-orientable surface [**13**, page 385].

By also including the order of the group element tuv, we have more information about the M^*-group. For example, we have the following result.

Theorem 1. *If an M^*-group G has a presentation in which both the action index $q = o(uv)$ and $r = o(tuv)$ are odd, then G is perfect.*

Proof. Since q is odd, $(uv)^{-1} = (uv)^{q-1}$ is a power of $(uv)^2$, and thus uv is an element of G'. But since r is also odd, we have $t \cdot uv$ in G' as well. Now $t \in G' = \langle tv, tu \cdot tv \cdot tu \rangle$ and clearly $G = G'$. ◻

Corollary 2. *If each action index of the M^*-group G is odd, then G is perfect.*

The converse of the corollary is false. The simple group $PSL(2,19)$, as a quotient of $G^{3,9,10}$ [**6**, page 140], has action indices 9 and 10.

3. The general construction. Here we describe our general construction, which uses the concept of a subdirect product [**17**]. Another good reference for the material on subdirect products is the textbook [**18**]. The basic construction is quite general, with applications possible to several different types of groups. The principal applications we have in mind are to M^*-groups, however.

Definition 1. Let G and H be groups, and let p_G and p_H be the canonical projections of the direct product $G \times H$ onto G and H, respectively. A subdirect product of G and H is a subgroup L of the direct product $G \times H$ such that $p_G(L) = G$ and $p_H(L) = H$.

Suppose that Γ is a group. Let G and H be finite groups that are images of Γ, say $\theta : \Gamma \to G$ and $\phi : \Gamma \to H$, where both θ and ϕ are onto. Define the map $\psi : \Gamma \to G \times H$ by $\psi(x) = (\theta(x), \phi(x))$, and let L be the image of Γ under ψ. Since θ and ϕ are onto G and H, respectively, it is clear that L is a subdirect product of G and H.

Definition 2. Let $\psi : \Gamma \to G \times H$ be a homomorphism such that $L = \mathrm{Im}(\psi)$ is a subdirect product of G and H. Then define epimorphisms $\theta = p_G \circ \psi$ from Γ onto G and $\phi = p_H \circ \psi$ from Γ onto H. Define subgroups $\tau_G(\psi) = \theta(\ker(\phi))$ and $\tau_H(\psi) = \phi(\ker(\theta))$.

Clearly, $\tau_G(\psi) \times \tau_H(\psi) \subset \mathrm{Im}(\psi) \subset G \times H$. It is easy to see that $\tau_G(\psi)$ is a normal subgroup of G, and $\tau_H(\psi)$ is a normal subgroup of H.

Proposition 8. *The group $\tau_G(\psi) \times \{1\}$ is the intersection of groups $\mathrm{Im}(\psi)$ and $G \times \{1\}$. The group $\{1\} \times \tau_H(\psi)$ is the intersection of $\mathrm{Im}(\psi)$ and $\{1\} \times H$.*

Proof. Suppose that w is a word in group Γ such that $\psi(w) \in G \times \{1\}$. Therefore, $\phi(w) = 1$ and $w \in \ker(\phi)$. So $\theta(w) \in \tau_G(\psi)$. The other containment is clear. $\square$

Theorem 2. *Let $L = \mathrm{Im}(\psi)$ be a subdirect product of G and H. Then $|G/\tau_G(\psi)| = [G \times H : L] = |H/\tau_H(\psi)|$.*

Proof. Suppose that $\{g_1, \ldots, g_t\}$ is a complete set of coset representatives of $\tau_G(\psi)$ in G. We will show that $\{(g_1, 1), \ldots, (g_t, 1)\}$ is a complete set of coset representatives of L in $G \times H$.

Suppose that $(g, h) \in (G \times H) - L$. Since ϕ is onto H, there is an element u of Γ such that $\phi(u) = h$. Therefore, $(\theta(u), h) \in L = \mathrm{Im}\,(\psi)$. There exists an $x \in G$ such that $g = x\theta(u)$ and $x = g_j z$, for some j and some element $z \in \tau_G(\psi)$. Now we have

$$(g, h) = (x, 1)(\theta(u), h) = (g_j, 1)(z, 1)(\theta(u), h) \in (g_j, 1)L.$$

Now suppose that $(g_r, 1) \in (g_s, 1)L$. So $(g_r g_s^{-1}, 1) \in L \cap (G \times \{1\})$ and $g_r g_s^{-1} \in \tau_G(\psi)$ by Proposition 8. It follows that $r = s$ and so $\{(g_1, 1), \ldots, (g_t, 1)\}$ is a complete set of coset representatives of L in $G \times H$. The same argument with group H finishes the proof. $\square$

Corollary 3. *$G/\tau_G(\psi) \cong H/\tau_H(\psi) \cong \Gamma/(\ker(\theta) \cdot \ker(\phi))$. In addition, if $n = |G|$ and $m = |H|$, then $[G \times H : L]$ divides the* $\gcd(n, m)$ *and hence $|L|$ is a multiple of lcm(n, m).*

The following corollaries are worth noting. If one of the groups is a simple group, then there are only two possibilities.

Corollary 4. *Suppose that $\theta : \Gamma \to G$ and $\phi : \Gamma \to H$ are maps onto groups G and H respectively. Define $\psi : \Gamma \to G \times H$ as $\psi(x) = (\theta(x), \phi(x))$. If H is a simple group, then either ψ is onto $G \times H$ or H is a quotient of G.*

Proof. If H is a simple group, then $\tau_H(\psi)$ is either trivial or all of H. If $\tau_H(\psi) = H$, then $\mathrm{Im}\,(\psi) = G \times H$. If $\tau_H(\psi) = 1$, then

$$H = H/\tau_H(\psi) \cong G/\tau_G(\psi)$$

by Corollary 3. $\square$

Corollary 5. *Suppose that $\theta : \Gamma \to G$ and $\phi : \Gamma \to H$ are maps onto groups G and H, respectively. Define $\psi : \Gamma \to G \times H$ as $\psi(x) = (\theta(x), \phi(x))$. If no non-trivial quotient of G is isomorphic to any quotient of H, then ψ is onto $G \times H$.*

4. Applications to M^*-groups. In this section we apply the general construction to M^*-groups. Let G be an M^*-group with generators t, u and v satisfying the partial presentation (1), and let Γ be the extended modular group. Then there is a natural homomorphism $\theta : \Gamma \to G$ from Γ onto G. To obtain the action of G on a bordered surface, let Δ be the extended quadrilateral group $\Gamma[2,2,2,3]$. The NEC group Δ has presentation

$$t_1{}^2 = u_1{}^2 = j^2 = v_1{}^2 = (t_1u_1)^2 = (u_1j)^2 = (jv_1)^2 = (t_1v_1)^3 = 1.$$

Then there is a natural homormorphism λ from Δ onto Γ defined by $\lambda(j) = 1$, $\lambda(t_1) = t$, $\lambda(u_1) = u$, $\lambda(v_1) = v$. Then $\alpha = \theta \circ \lambda$ maps Δ onto G and $\ker(\alpha)$ is a bordered surface group. Then G acts on the bordered surface $X = D/\ker(\alpha)$ with action index $q = o(uv)$.

Let H be a second M^*-group with generators a, b and c satisfying the partial presentation (1), where a, b and c correspond to t, u and v, respectively. Then there is a homomorphism $\phi : \Gamma \to H$ from Γ onto H. Then $\beta = \phi \circ \lambda$ maps Δ onto H and H acts on the bordered surface $Y = D/\ker(\beta)$ with action index $r = o(bc)$.

Now let L be the subgroup of the direct product $G \times H$ generated by (t,a), (u,b) and (v,c). Then L is clearly a subdirect product of G and H. The group L is $\mathrm{Im}(\psi)$, in the notation of Section 3. We record the following.

Proposition 9. *The subdirect product $L = \mathrm{Im}(\psi)$ is an M^*-group that acts on a bordered surface W with index $\mathrm{lcm}(q,r)$. If either X or Y is orientable, then W is orientable.*

Proof. The natural generators $T = (t,a)$, $U = (u,b)$ and $V = (v,c)$ clearly satisfy (1) with $o(UV) = \mathrm{lcm}(q,r)$. Hence, L is an M^*-group.

Let $\gamma = \psi \circ \lambda$ so that L acts on the surface $W = D/\ker(\gamma)$ with index $\mathrm{lcm}(q,r)$. But

$$\alpha = \theta \circ \lambda = p_G \circ \psi \circ \lambda = p_G \circ \gamma.$$

Thus $\ker(\gamma) \subset \ker(\alpha)$ and $\ker(\alpha)/\ker(\gamma) \cong \ker(p_G)$. Then the quotient group $G = L/\ker(p_G)$ acts on the quotient surface $W/\ker(p_G)$. Now we have

$$W/\ker(p_G) = (D/\ker(\gamma))/(\ker(\alpha)/\ker(\gamma)) = D/\ker(\alpha) = X,$$

and the surface W is a full covering of X. In the same way, of course, W is a full covering of Y. If either X or Y is orientable, then it follows that the covering W is orientable as well [**13**, page 375]. $\square$

If both surfaces X and Y are non-orientable, than the surface W may be orientable or not, as we shall see in the next section.

While this construction always produces an M^*-group, the order of the "new" M^*-group is not immediately apparent. However, Theorem 2 gives a method to compute $|L|$ if there are complete presentations for the M^*-groups G and H. Suppose that $G \cong \langle t, u, v \mid t^2, u^2, v^2, (tu)^2, (tv)^3, \mathscr{S} \rangle$, where $\mathscr{S}$ is a set of additional relators needed to define the finite group G. Then the subgroup $\tau_H(\psi)$ is the normal closure in H of $\phi(\mathscr{S})$. In fact, it is interesting that all that is needed to find $|L|$ is a presentation for one of the two, say G, together with a way to calculate the image of the relators of G under the homomorphism $\phi : \Gamma \to H$.

For example, let $G = G^{3,6,6}$ be the M^*-group of order 108 [**6**, page 139], and let $H = S_4 = G^{3,3,4}$ be the second M^*-group. Group G has presentation (1) with the added relators $(uv)^6$, $(tuv)^6$. The normal closure in H of $\langle (uv)^6, (tuv)^6 \rangle = \langle (tuv)^2 \rangle$ has order 4 so that the index $|H/\tau_H(\psi)| = 6$. Thus, the construction yields the M^*-group $L = \mathrm{Im}\,(\psi)$, which has order 432 by Theorem 2.

If one of the M^*-groups, say H, is perfect, then any quotient group of H is perfect and hence is either trivial or non-solvable. If this quotient is trivial, then the subdirect product is the direct product; we return to this case in the next section. Otherwise, this quotient involves a non-abelian simple group, and the index of L in the direct product is relatively large.

Here we are primarily interested in cases in which the subdirect product L has small index in $G \times H$ and there is a condition that allows the easy determination of $|L|$. We consider two conditions, one involving the action indices and one involving abelian quotient groups.

Theorem 3. *Let G and H be M^*-groups with action indices q and r, respectively. If q and r are relatively prime, then $G^+ \subseteq \tau_G(\psi)$ and consequently $G/\tau_G(\psi)$ is trivial or isomorphic to Z_2. Let $d = \gcd(q, r)$. If $2 \le d \le 5$, then $G/\tau_G(\psi)$ is isomorphic to a quotient of the abstract group $[3, d]$. These groups are $[3, 2] \cong D_6$, $[3, 3] \cong S_4$, $[3, 4] \cong Z_2 \times S_4$, and $[3, 5] \cong Z_2 \times A_5$.*

Proof. We continue to use the notation that G is an M^*-group with generators t, u and v, while H has generators a, b and c. Then the subdirect product $L = \mathrm{Im}\,(\psi)$ is generated by $T = (t, a)$, $U = (u, b)$ and $V = (v, c)$.

Suppose $1 \leq d \leq 5$. Then $((uv)^d, 1)$ and $(1, (bc)^d)$ are in L. Now, using Proposition 8, we see that $G/\tau_G(\psi)$ is an image of the group with presentation (1) and added relation $(uv)^d = 1$. For $d > 1$ these relations define the well-known abstract group $[3, d]$ [**6**, page 37]. Hence, $G/\tau_G(\psi)$ is an image of $[3, d]$.

Finally, suppose q and r are relatively prime, that is, $d = 1$. Define $N = \ker(\theta) \cdot \ker(\phi)$. Then $uv \in N$ and $uN = vN$. Thus, $tuN = tvN$, and since these elements have relatively prime orders, it follows that $tu \in N$. Therefore, by the isomorphism of Corollary 3, $G^+ \subseteq \tau_G(\psi)$. Since $[G : G^+] \leq 2$, $G/\tau_G(\psi)$ must be an image of Z_2. $\square$

Now we consider a natural condition involving abelian quotient groups. The following result allows the determination of $|L|$ in the important case in which the M^*-groups G and H each have abelianization isomorphic to Z_2.

Theorem 4. *Let G and H be M^*-groups with $G/G' \cong H/H' \cong Z_2$. Let $\theta : \Gamma \to G$ and $\phi : \Gamma \to H$ be the epimorphisms of Γ onto G and H, respectively, and construct the subdirect product $L = \mathrm{Im}\,(\psi)$. First, $\ker(\theta) \subseteq \langle x\rangle\Gamma'$ and $\ker(\phi) \subseteq \langle y\rangle\Gamma'$, where x and y are in $\{t, u, tu\}$. If $x = y$, then $[G \times H : L] \geq 2$. If $x \neq y$, then $G/\tau_G(\psi)$ is perfect.*

Furthermore, suppose that, for any normal subgroups N of G and M of H such that $G/N \cong H/M$, the quotient group G/N is abelian. If $x = y$, then $[G \times H : L] = 2$, and if $x \neq y$, then $L = G \times H$.

Proof. It is easy to see that $\Gamma/\Gamma' = \{\Gamma', t\Gamma', u\Gamma', tu\Gamma'\}$. Since

$$Z_2 \cong G/G' \cong \left(\frac{\Gamma}{\ker(\theta)}\right) \Big/ \left(\frac{\ker(\theta)\Gamma'}{\ker(\theta)}\right) \cong \frac{\Gamma}{\ker(\theta) \cdot \Gamma'},$$

it follows that $\ker(\theta) \subseteq \langle x\rangle\Gamma'$ for $x \in \{t, u, tu\}$. Similarly, $\ker(\phi) \subseteq \langle y\rangle\Gamma'$ for $y \in \{t, u, tu\}$. First assume $x = y$. Then $(\ker(\theta) \cdot \ker(\phi)) \subseteq \langle x\rangle\Gamma'$, and, by Corollary 3,

$$(2)\quad [G \times H : L] = [G : \tau_G(\psi)] = [\Gamma : (\ker(\theta) \cdot \ker(\phi))] \geq [\Gamma : \langle x\rangle\Gamma'] = 2.$$

Now assume $x \neq y$. Then $(\ker(\theta) \cdot \ker(\phi))\Gamma' = \Gamma$. Write $J = \ker(\theta) \cdot \ker(\phi)$. Now we have $\Gamma = J\Gamma'$, and by Corollary 3, $G/\tau_G(\psi) \cong \Gamma/J$. But it is easy to see that the quotient group $\Gamma/J = J\Gamma'/J$ is perfect. Thus, $G/\tau_G(\psi)$ is a perfect group.

Finally, if the only quotients of G and of H which are isomorphic are abelian, then $G/\tau_G(\psi) \cong H/\tau_H(\psi)$ is an abelian group. Hence, if $x \neq y$, then $G/\tau_G(\psi)$ is trivial and $L = G \times H$. Also, if $x = y$, then $\Gamma/(\ker(\theta) \cdot \ker(\phi))$ is abelian and therefore, $\Gamma' \subseteq (\ker(\theta) \cdot \ker(\phi))$ and $[G \times H : L] = 2$. $\square$

In the case in which the only isomorphic quotients of G and H are abelian, we develop a condition to determine whether $[G\times H : L]$ is 1 or 2. Suppose $\theta : \Gamma \to G$ is the epimorphism of Γ onto G, so that G has a particular presentation as an M^*-group. Consider the ordered triple of positive integers $(\, o(\theta(uv)), o(\theta(tuv)), o(\theta(t(uv)^2))\,) = (h, k, l)$; we call this ordered triple the *signifier* of the presentation of G. Of course, a finite group may have several different presentations as an M^*-group and, consequently, several different signifiers.

Since $(uv)^h = (tuv)^k = (t(uv)^2)^l = 1$ in G, $(uv)^h$, $(tuv)^k$ and $(t(uv)^2)^l$ are all elements of $\ker(\theta)$. The following observations are easy to check. If h is odd, then the element $(uv)^h$ is in the coset $tu\Gamma'$, and if h is even, it is an element of Γ'. Similarly, if k is odd, then the element $(tuv)^k$ is in the coset $u\Gamma'$, and if k is even, it is an element of Γ'. Finally, if l is odd, then $(t(uv)^2)^l \in t\Gamma'$, and if l is even, then this element is in Γ'.

It follows that if a signifier of G has two or more odd components, then G is a perfect group. This observation includes Theorem 1 as a special case. Indeed, the notion of signifier extends the idea behind Theorem 1 by considering the additional element $t(uv)^2$.

If $G/G' \cong Z_2$, then a signifier of G must have at most one odd component. If $G/G' \cong Z_2$ and a signifier has one odd component, then $K \subseteq \langle x\rangle\Gamma'$, where x is u, tu or t, respectively, depending on whether the odd component is the first, second or third component.

Now suppose $G/G' \cong H/H' \cong Z_2$ and the signifiers of both G and H have one odd component. If the same component is odd, then $[G \times H : L] \geq 2$. If a different component is odd, then the quotient group $G/\tau_G(\psi)$ is perfect. One way to decide what happens is to take the dot product of the signifiers modulo 2. If the result is one, then $[G \times H : L] \geq 2$, and if the result is zero, then $G/\tau_G(\psi)$ is perfect.

Suppose that, further, for any normal subgroups N of G and M of H such that $G/N \cong H/M$, then G/N is abelian. In this case, if the dot product of the signifiers modulo 2 is one, then $[G \times H : L] = 2$, and if the result is zero, then $L = G \times H$.

To illustrate the use of the signifier in applying Theorem 4, let $G = S_4$. Then a simple calculation shows that G has two possible signifiers, $(3, 4, 4)$ and $(4, 3, 4)$. Let $H = \mathrm{PGL}\,(2, 7)$. Then H has three possible signifiers, $(8, 7, 6)$, $(7, 8, 6)$ and $(8, 8, 7)$. The only non-trivial isomorphic quotient that these two groups have is Z_2. Constructing the subdirect product of S_4 and $\mathrm{PGL}\,(2, 7)$ with signifiers $(3, 4, 4)$ and $(8, 7, 6)$, respectively, yields the direct product $S_4 \times PGL(2, 7)$. Thus, $S_4 \times PGL(2, 7)$ is an M^*-group with action index 24; this group also has action indices 8 and 21 (obtained with other choices of the signifiers). On the other hand, a subdirect product of S_4 and $PGL(2, 7)$ with signifiers $(4, 3, 4)$ and $(8, 7, 6)$, respectively, has index two in $S_4 \times PGL(2, 7)$. All of this can also be verified by direct computation.

Now we consider the cases in which the abelian quotients of G and H are not both of order 2 or less. It is easy to see that the subdirect product will not equal the direct product in these cases. The following theorem gives some results about these cases.

Theorem 5. *Let G and H be M^*-groups with $|G/G'| \geq 2$ and $|H/H'| = 4$. Let $\theta : \Gamma \to G$ and $\phi : \Gamma \to H$ be the epimorphisms of Γ onto G and H, respectively, and construct the subdirect product $L = Im\,(\psi)$. Then $[G \times H : L] \geq 2$. If $|G/G'| = |H/H'| = 4$, then $[G \times H : L] \geq 4$.*

Furthermore, suppose that, for any normal subgroups N of G and M of H such that $G/N \cong H/M$, the quotient group G/N is abelian. If $|G/G'| = 2$ and $|H/H'| = 4$ then $[G \times H : L] = 2$. If $|G/G'| = |H/H'| = 4$, then $[G \times H : L] = 4$.

Proof. If $|G/G'| = 4$, then $\ker\,(\theta) \subseteq \Gamma'$ and if $|H/H'| = 4$, then $\ker\,(\phi) \subseteq \Gamma'$. If $|G/G'| = 2$, then $\ker\,(\theta) \subseteq \langle x \rangle \Gamma'$ for $x \in \{t, u, tu\}$. Then $[G \times H : L] \geq 2$ by equation (2). If $|G/G'| = 4$ and $|H/H'| = 4$, then $[G \times H : L] = [\Gamma : (\ker\,(\theta) \cdot \ker\,(\phi))] \geq [\Gamma : \Gamma'] = 4$.

Finally, suppose that, for any normal subgroups N of G and M of H, if $G/N \cong H/M$, then G/N is abelian. Since $G/\tau_G(\psi) \cong H/\tau_H(\psi)$

by Corollary 3, it follows that $G' \subseteq \tau_G(\psi) \subseteq G$ and a similar equation for H. The result follows. $\square$

For example, let $G = \mathrm{PGL}(2,7)$, and let H be a solvable M^*-group with $|H/H'| = 4$. Numerous choices are possible for H; for any choice, though, any genus action of H is on an orientable surface, by Proposition 2. Then the construction yields an M^*-group of order $168 \cdot |H|$, with the M^*-group acting on an orientable surface.

There is a characterization of the M^*-groups that are subdirect products of two smaller M^*-groups.

Theorem 6. *The M^*-group L is a subdirect product of two smaller M^*-groups if and only if L has normal subgroups J_1 and J_2 such that $[L : J_1] > 6$, $[L : J_2] > 6$ and $J_1 \cap J_2 = 1$.*

Proof. First suppose the M^*-group L has normal subgroups J_1 and J_2 such that $[L : J_1] > 6$, $[L : J_2] > 6$ and $J_1 \cap J_2 = 1$. Then $G = L/J_1$ and $H = L/J_2$ are M^*-groups. Let $\alpha : L \to G$ and $\beta : L \to H$ be the canonical quotient mappings. Then the mapping $\gamma : L \to G \times H$ defined by $\gamma(x) = (\alpha(x), \beta(x))$ is an isomorphism of L onto a subdirect product of G and H (γ is injective since $J_1 \cap J_2 = 1$).

Now suppose that L, G and H are M^*-groups, with $|L| > |G|$ and $|L| > |H|$, such that L is a subdirect product of G and H. Then L is a subgroup of the direct product $G \times H$ such that $p_G(L) = G$ and $p_H(L) = H$. Let δ_G and δ_H denote the restrictions of p_G and p_H, respectively, to the group L. Then let $J_1 = \ker(\delta_G)$ and $J_2 = \ker(\delta_H)$. Then J_1 is normal in L and, since L/J_1 is isomorphic to the M^*-group G, $[L : J_1] > 6$. In the same way, J_2 is normal in L and $[L : J_2] > 6$. It is clear that $J_1 \cap J_2 = 1$, since J_1 and J_2 are subgroups of $G \times H$. $\square$

A special case of Theorem 6 is worth mentioning, since it is easy to check, with the software MAGMA, for example. The Fitting subgroup $\mathrm{Fit}(G)$ of a group G is the maximal normal nilpotent subgroup of G.

Corollary 6. *Let L be an M^*-group with $|L| > 12$ and Fitting subgroup $F = \mathrm{Fit}(L)$. If $|F|$ is divisible by two distinct primes, then L is a subdirect product of two smaller M^*-groups.*

Proof. Suppose that p and q are distinct primes, each of which divides $|F|$. Let P and Q be the Sylow p-subgroup of F and the Sylow q-subgroup of F, respectively. Then P and Q are characteristic subgroups of F and L, and $P \cap Q = 1$.

If $|F|$ is divisible by a third prime (not p or q), then obviously $[L : P] > 6$ and $[L : Q] > 6$. Also, if $[L : F] \geq 4$, then, again, easily, $[L : P] > 6$ and $[L : Q] > 6$.

Suppose that $|F|$ is divisible by exactly two primes and $[L : F] < 4$. Note that an M^*-group cannot have a quotient of order 3, and an M^*-group is not nilpotent, since its Sylow 2-subgroup is not normal. Hence, we must have $[L : F] = 2$, with $p = 2$ and $q = 3$. Write $|L| = 2^i \cdot 3^j$ for some i and j. Now $|Q| = 3^j$ and L/Q is a quotient of the M^*-group L. Thus, i must be 2. Now $L = 4 \cdot 3^j$ and $|P| = 2$. Since L/P is a quotient of L, the only possibility is $j = 1$. Now $L \cong D_6$, the M^*-group of order 12. With $|L| > 12$, then, this case cannot occur.

Now, by the theorem, L is a subdirect product of L/P and L/Q. $\square$

For example, $L = S_3 \times S_4$, the M^*-group of order 144 [**11**, page 7], has Fitting subgroup $F \cong Z_3 \times (Z_2)^2$. The group L is a subdirect product of the smaller M^*-groups $S_3 \times S_3$ and $Z_2 \times S_4$.

5. Direct products of M^*-groups. Now we consider necessary and sufficient conditions for an M^*-group to be the direct product of two smaller M^*-groups. First, there is an easy, algebraic necessary condition.

Proposition 10. *Let $M = G \times H$ be the direct product of G and H, where $|G| > 6$ and $|H| > 6$. If M is an M^*-group, then G and H are M^*-groups such that one of the following holds*:

1) *At least one of the groups G, H is perfect, or*

2) $[G : G'] = [H : H'] = 2$.

Proof. It is basic that a quotient of an M^*-group by a normal subgroup of index larger than 6 is again an M^*-group [**8**, Theorem 6]. Thus G and H are M^*-groups. We know that $M' = G' \times H'$. If neither G nor H is perfect, then condition (2) clearly must hold. $\square$

There is also a necessary condition involving the actions of the factors of the direct product.

Theorem 7. *Let $M = G \times H$ be the direct product of G and H, where $|G| > 6$ and $|H| > 6$. Suppose M is an M^*-group acting on W with index s. Write $X = W/H$ and $Y = W/G$. Let q be the index of the action of the M^*-group $G = M/H$ on X and r the index of the action of the M^*-group $H = M/G$ on Y. Then*

1) *$s = \operatorname{lcm}(q, r)$ and*

2) *at least one of the surfaces X, Y is non-orientable.*

Proof. Let the M^*-group $M = G \times H$ have the partial presentation (1) with generators T, U and V. Then we may write $T = (t, a)$, $U = (u, b)$ and $V = (v, c)$, of course. What is important here is that the elements t, u and v generate the M^*-group G in the natural presentation of the quotient group $G = M/H$ acting on X. In particular, the action index $q = o(uv)$. Further, a, b and c generate the M^*-group $H = M/G$ acting on Y with index $r = o(bc)$. Now, clearly, $s = \operatorname{lcm}(q, r)$.

To prove 2), suppose to the contrary that both X and Y are orientable. Then we must have $[G : G^+] = 2$ and $[H : H^+] = 2$ so that $[M : G^+ \times H^+] = 4$. However, $M^+ = \langle TU, UV \rangle$ is always a subgroup of $G^+ \times H^+$, and $[M : M^+]$ is at most 2. Hence, X and Y cannot both be orientable. $\square$

Thus, if the M^*-group M is the direct product of two M^*-groups G and H, either G or H must act on a non-orientable surface. We assume that it is H and then consider the orientability of the surface on which M acts.

Theorem 8. *Suppose the M^*-group M is the direct product of the M^*-groups G and H. Let M act on the bordered surface W. Write $X = W/H$ and $Y = W/G$, and assume that the surface Y is non-orientable. Then the following hold.*

1) *If X is orientable, then W is orientable.*

2) *Assume that X is non-orientable. If $[G : G'] = [H : H'] = 2$, then W is orientable; otherwise, W is non-orientable.*

Proof. First, 1) holds by Proposition 9.

Assume, then, that surface X is non-orientable. If $[G : G'] = [H : H'] = 2$, then $M' = G' \times H'$ and $[M : M'] = 4$. Now W is orientable, by Proposition 2. If both M^*-groups G and H are perfect, then M is also perfect and the surface W on which M acts is non-orientable. Assume, finally, that one of the M^*-groups, say H, is perfect, while in the other, $[G : G'] = 2$. Suppose to the contrary that W is orientable. Then $[M : M^+] = 2$. But $M' = G' \times H' = G' \times H$ clearly has index 2 in M. Hence $M' = M^+$, and now $1 \times H \subset M^+$. Since $p_G(M^+) = G^+$ always, here we would have $[G : G^+] = 2$ and the surface X would be orientable, contradicting our assumption. Hence W must be non-orientable. □

Next we consider sufficient conditions for the direct product of two M^*-groups G and H to be an M^*-group. First we consider the greatest common divisor of the action indices.

Theorem 9. *Let H be an M^*-group acting on a non-orientable surface with index r. Let G be another M^*-group with action index q. If the action indices q and r are relatively prime, then the direct product $G \times H$ is an M^*-group with action index qr.*

Proof. We continue to use the notation of Proposition 9. Since H acts on a non-orientable surface, $H = H^+ = \langle ab, bc \rangle$. By Theorem 3, $H^+ \subseteq \tau_H(\psi)$ and therefore, by Theorem 2, the subdirect product L is the direct product. □

If the M^*-group H is perfect, then the only surfaces on which H acts are non-orientable. In this case Theorems 3 and 8 yield the following.

Corollary 7. *Let H be a perfect M^*-group with action index r. Let G be another M^*-group acting on X with index q. If* $\gcd(q, r) \leq 4$, *then the direct product $M = G \times H$ is an M^*-group with action index* $\operatorname{lcm}(q, r)$. *Further, the surface W on which M acts is orientable if and only if X is orientable.*

Proof. Let $d = \gcd(q, r)$. If $2 \leq d \leq 4$, then $H/\tau_H(\psi)$ is isomorphic to a quotient of the abstract group $[3, d]$, by Theorem 3. But, in each case, the group $[3, d]$ is solvable, and the perfect group H has no non-trivial solvable quotients. Hence, the subdirect product L is the direct product. If q and r are relatively prime, this follows immediately from Theorem 9, but it is also a consequence of Theorem 3. In any case, the orientability of W is given by Theorem 8. □

For example, let $H = \mathrm{PSL}(2, 23)$, the simple group of order 6072. Then H is an M^*-group [**19**] with action indices 11 and 12; the indices can be found using MAGMA. The M^*-group $G = PGL(2, 7)$ has action indices 7 and 8. Thus $G \times H$ is an M^*-group, with action indices 24, 77, 84 and 88.

Corollary 8. *Let H be a perfect M^*-group. If H has two action indices q and r such that* $\gcd(q, r) \leq 4$*, then the direct product $M = H \times H$ is an M^*-group with action index* $\mathrm{lcm}(q, r)$*. Further, the surface W on which M acts is non-orientable.*

Among the small projective special linear groups [**19**], we can take $H = \mathrm{PSL}(2, 8) \cong G^{3,7,9}$ [**6**, page 140], which has action indices 7 and 9. Hence, $H \times H$ is an M^*-group that acts on a non-orientable surface with index 63. Another possible choice is $H = \mathrm{PSL}(2, 13) \cong G^{3,7,13}$ [**6**, page 140].

In the case $[G : G'] = [H : H'] = 2$, Theorems 8 and 9 yield the following.

Corollary 9. *Let H be an M^*-group with $[H : H'] = 2$ acting on the non-orientable surface Y with index r. Let G be another M^*-group with $[G : G'] = 2$ acting on X with index q. If the action indices q and r are relatively prime, then the direct product $M = G \times H$ is an M^*-group with action index qr. Further, the surface W on which M acts is orientable.*

In particular Corollary 9 can be applied to any M^*-group H with $[H : H'] = 2$ that has two relatively prime action indices, as long as one of the actions is on a non-orientable surface. For example, $S_4 \times S_4$ and $\mathrm{PGL}(2, 7) \times \mathrm{PGL}(2, 7)$ are M^*-groups.

The next sufficient condition is more general and depends on the construction of Section 3; see Corollary 5.

Theorem 10. *Let H be an M^*-group that acts on a non-orientable surface with index r. Let G be another M^*-group with action index q. If no non-trivial quotient of G is isomorphic to a quotient of H, then the direct product $G \times H$ is an M^*-group with action index* $\operatorname{lcm}(q, r)$.

We note two interesting special cases. The first, in particular, provides a wealth of examples of relatively small order.

Corollary 10. *Let G be a solvable M^*-group with action index q, and let H be a perfect M^*-group with action index r. Then $G \times H$ is an M^*-group with action index* $\operatorname{lcm}(q, r)$.

For example, let $H = A_5$, the smallest simple M^*-group. The group A_5 has the unique action index $r = 5$ [**13**, Lemma 3]. The M^*-group $G = S_4$ acts on a sphere with four holes with index 3 and a real projective plane with three holes with index 4. Thus we see that $S_4 \times A_5$ is an M^*-group. This group acts on an orientable bordered surface with index 15 and a non-orientable one with index 20. Of course, many similar examples are possible.

Corollary 11. *Let G and H be non-isomorphic simple M^*-groups with action indices q and r, respectively. Then $G \times H$ is an M^*-group with action index* $\operatorname{lcm}(q, r)$.

For some final examples, we use $\mathrm{PSL}(2, 19)$, which is an M^*-group [**19**] with action indices 9 and 10. Then $A_5 \times \mathrm{PSL}(2, 19)$ is an M^*-group with action indices 10 and 45. Also $\mathrm{PSL}(2, 19) \times \mathrm{PSL}(2, 23)$ is an M^*-group with action indices 36, 60, 99 and 110.

6. Other applications. In connection with group actions on surfaces of a fixed genus, there are many instances in which the groups of maximum possible order have a particular partial presentation. In this case the construction of Section 3 can be applied. There are, in fact, numerous possibilities. The surfaces could be Klein surfaces, with or without boundary, as well as Riemann surfaces. In some instances, the group actions could be restricted to actions that preserve orientation.

Best known, perhaps, are the classical Hurwitz groups that act on Riemann surfaces. Each Hurwitz group acts on a group of $84(g-1)$ automorphisms on a Riemann surface of genus $g \geq 2$ and has a partial presentation $x^2 = y^3 = (xy)^7$. See the survey article [**4**] for a nice discussion of these groups. Indeed, it is pointed out in [**4**, Section 4] that the subdirect product can be used to construct larger Hurwitz groups.

Another possibility would be to consider 2-groups acting on bordered Klein surfaces; here see the recent article [**15**]. A third possibility would be to consider groups of odd order acting on Riemann surfaces; these groups were considered in [**16**]. These are instances where the subdirect product construction of Section 3 can be applied to construct a larger group of the same type, at least. However, we do not claim here that the construction will necessarily lead to interesting general results. We conclude with some examples with groups of odd order.

In [**21**], Zomorrodian gave a p-group analog of the Hurwitz theorem. He showed, among other things, that the largest 3-group acting on a Riemann surface of a certain genus was an image of the Fuchsian group $\Gamma(3,3,9)$. It is easy to see that there are two images of $\Gamma(3,3,9)$ of order 81. These are groups 7 and 9 in the Magma Library of Small Groups; we use the notation $SG(n,k)$ to refer to group k of order n in this MAGMA library. If $G = SG(81,7)$ and $H = SG(81,9)$, then $[G : \tau_G(\psi)] = [H : \tau_H(\psi)] = 27$. It follows that the subdirect product of G and H has index 27 in the direct product and so is of order 243. It is isomorphic to $SG(243,3)$ and is a $\Gamma(3,3,9)$ group. Similarly, if $G = SG(81,7)$ and $H = SG(243,26)$, both $\Gamma(3,3,9)$ groups, then $[G : \tau_G(\psi)] = [H : \tau_H(\psi)] = 27$. It follows that the subdirect product of G and H has index 27 in the direct product and so is of order 729. It is isomorphic to $SG(729,40)$ and is a $\Gamma(3,3,9)$ group. This procedure is not too difficult and can be done with very large groups.

A final example is with $\Gamma(3,3,9)$ groups that are not 3-groups. Let $G = \langle x,y \mid x^3 = y^3 = (xy)^9 = [x,y]^{21} = (x*y*x)^{21} = [[x,y],[x,y]^x] = [[x,y],[x,y]^y] = 1.\rangle$. This is clearly a $\Gamma(3,3,9)$ group and its order is 3969. Let $H = \langle x,y \mid x^3 = y^3 = (xy)^9 = [x,y]^{39} = (x*y*x)^{39} =$

$[[x,y],[x,y]^x] = [[x,y],[x,y]^y] = 1.\rangle$. This is clearly a $\Gamma(3,3,9)$ group and its order is 13689. Finally, $[G:\tau_G(T)] = [H:\tau_H(S)] = 81$ and so the subdirect product is a $\Gamma(3,3,9)$ group, and its order is 670761.

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